Liao and Wen *Advances in Difference Equations* (2017) 2017:142
DOI 10.1186/s13662-017-1196-0

Advances in Difference Equations
a SpringerOpen Journal

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Dissipativity of Runge-Kutta methods for a class of nonlinear functional-integro-differential equations

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Abstract

This paper is concerned with the dissipativity of Runge-Kutta methods for a class of nonlinear functional-integro-differential equations (FIDEs). The dissipativity results of Runge-Kutta methods for the FIDEs are given. It is shown under a suitable condition that an algebraically stable Runge-Kutta method is dissipative when applied to the FIDEs. Numerical examples are given to illustrate the correctness of our theoretical results.

Keywords: functional-integro-differential equation; Runge-Kutta method; dissipativity; algebraic stability; dynamical systems

1 Introduction

Among various properties of dynamical systems, dissipativity is one of important characteristics. A dissipative dynamical system is characterized by possessing a bounded absorbing set that all trajectories enter in a finite time and thereafter remain inside [1]. In the study of numerical methods for these systems, one natural wish is for the numerical solution to preserve the dissipativity of the analytic solution.

Over the past few decades, the dissipativity of the analytic solution and numerical methods for some special class dynamical systems of VFDEs have been studied widely. One can refer to the following works and corresponding authors: [2–5] for ordinary differential equations (ODEs), [6–10] for delay differential equations (DDEs), and [11–20] for other kinds of Volterra functional differential equations, such as delay integro-differential equations (DIDEs), neutral delay differential equations (NDDEs), neutral delay integro-differential equations (NDIDEs) and so on.

In this paper, we investigate numerical dissipativity of a class of nonlinear functional-integro-differential equations (FIDEs) (see (2.1) in the next section). In 2014 and 2015, Zhang and Qin studied the stability of Runge-Kutta methods [21] and one-leg methods [22] for this kind of problems, respectively. Recently, we also studied the dissipativity of systems (2.1) and of one-leg methods for FIDEs (2.1) [23]. In addition we do not find more dissipativity results for this kind of nonlinear FIDEs. The aim of this paper is to investigate the dissipativity of Runge-Kutta methods for (2.1).

This paper is organized as follows. In Section 2, the descriptions of the nonlinear FIDEs and their Runge-Kutta methods are given. In Section 3, the results on the dissipativity of

Runge-Kutta methods are deduced. In Section 4, some numerical experiments are given to illustrate the theoretical results which we stated in previous sections.

2 The descriptions of problem class and numerical methods

Let $\mathbb{C}^d$ be a d-dimensional complex Euclidean space with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. For any nonnegative diagonal matrix $B = \mathrm{diag}(b_1, b_2, \ldots, b_s)$, we define a pseudo inner product on $\mathbb{C}^{ds} := (\mathbb{C}^d)^s$ by

$$\langle Y, Z\rangle_B = \sum_{j=1}^{s} b_j \langle Y_j, Z_j\rangle, \quad Y = (Y_1, Y_2, \ldots, Y_s) \in \mathbb{C}^{ds}, Z = (Z_1, Z_2, \ldots, Z_s) \in \mathbb{C}^{ds},$$

and the corresponding pseudo norm on $\mathbb{C}^{ds}$ by

$$\|Y\|_B = \sqrt{\langle Y, Y\rangle_B}.$$

It is obvious that when B is positive definite, they are the inner product and the norm on $\mathbb{C}^{ds}$, respectively.

Consider nonlinear functional integro-differential equations (FIDEs) of the form (cf. [21, 22])

$$\begin{cases} \frac{d}{dt}\big[x(t) - \int_{t-\tau}^{t} g(t, \xi, x(\xi))\, d\xi\big] = f(t, x(t), x(t-\tau)), & t \in [t_0, +\infty), \\ x(t) = \varphi(t), & t_0 - \tau \le t \le t_0, \end{cases} \tag{2.1}$$

where $\tau > 0$ is a given constant delay, the functions $f : [t_0, +\infty) \times \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}^d$, $g : \mathbb{D} \times \mathbb{C}^d \to \mathbb{C}^d$, and $\varphi : [t_0 - \tau, t_0] \to \mathbb{C}^d$ are assumed to be continuous so that system (2.1) has a unique solution $x(t)$, and f and g satisfy also the conditions

$$\begin{aligned} &\mathrm{Re}\big\langle f(t, u, v), u - w\big\rangle \le \gamma + \alpha\|u\|^2 + \beta\|v\|^2 + \eta\|w\|^2, \\ &\quad t \ge t_0, u, v, w \in \mathbb{C}^d \end{aligned} \tag{2.2}$$

and

$$\big\|g(t, \xi, u)\big\| \le \lambda\|u\|, \quad (t, \xi) \in \mathbb{D}, u \in \mathbb{C}^d, \tag{2.3}$$

where $\gamma, \alpha, \beta, \eta, \lambda$ are given real constants and $\gamma, -\alpha, \beta, \eta$ are nonnegative, and $\lambda > 0$ with $2\lambda\tau < 1$,

$$\mathbb{D} := \big\{(t, \xi) : t \in [t_0, +\infty), \xi \in [t - \tau, t]\big\}.$$

In order to investigate the numerical dissipativity of (2.1), we assume further that f satisfies the condition: for any constant $M > 0$, there exists $L > 0$ which is only dependent on M such that $\|f(t, u, v)\| \le L$ holds for any $t \ge t_0$, $\|u\| \le M$ and $\|v\| \le M$.

Definition 2.1 (cf. [13]) Problem (2.1) in FIDEs is said to be dissipative in $\mathbb{C}^d$ if there exists a bounded set $B \subset \mathbb{C}^d$ such that for any given bounded set $\Phi \subset \mathbb{C}^d$, there is a time $t^* = t^*(\Phi)$

such that for any given continuous initial function $\varphi : [t_0 - \tau, t_0] \to \mathbb{C}^d$ with $\varphi(t)$ contained in Φ for all $t \in [t_0 - \tau, t_0]$, the corresponding solution $x(t)$ of the problem is contained in B for all $t \geq t^*$. Here B is called an absorbing set in $\mathbb{C}^d$.

In our recent paper [23], we studied the dissipativity of (2.1) and gave the following results.

Theorem 2.2 *Suppose that $x(t)$ is a solution of problem* (2.1) *where f and g satisfy* (2.2) *with $\alpha < 0$ and* (2.3), *respectively, and there exists constant $0 < \delta < 1$ such that*

$$\frac{4}{1-2\lambda^2\tau^2}\frac{\beta + (\eta - \alpha)\lambda^2\tau^2}{|\alpha|} \leq \delta. \tag{2.4}$$

Then,

(i) *for any $t \geq t_0$, we have*

$$\|x(t)\|^2 \leq \frac{4}{1-2\lambda^2\tau^2}\frac{-\gamma}{(1-\delta)\alpha} + \frac{1-2\lambda^2\tau^2}{1-2\lambda^2\tau^2 e^{\bar{\mu}\tau}}\phi e^{-\bar{\mu}(t-t_0)},$$

where $\phi = \sup_{t_0-\tau\leq\xi\leq t_0}\|\varphi(\xi)\|^2$, and $\bar{\mu} > 0$ is defined as

$$\bar{\mu} = \inf_{t\geq t_0}\left\{\mu(t) : \mu(t) + \alpha + \left(\beta + (\eta - \alpha)\lambda^2\tau^2\right)\frac{4e^{\mu(t)\tau}}{1-2\lambda^2\tau^2 e^{\mu(t)\tau}} = 0\right\};$$

here and later, the symbols γ, α, β, η, λ are given by (2.2) *and* (2.3);

(ii) *for any given $\varepsilon > 0$, there exists $t^* = t^*(\phi, \varepsilon)$ such that*

$$\|x(t)\|^2 \leq \frac{4}{1-2\lambda^2\tau^2}\frac{-\gamma}{(1-\delta)\alpha} + \varepsilon, \quad t \geq t^*.$$

Hence system (2.1) *is dissipative with an absorbing set*

$$B = B\left(0, \sqrt{\frac{4}{1-2\lambda^2\tau^2}\frac{-\gamma}{(1-\delta)\alpha} + \varepsilon}\right).$$

Remark 2.3 In [21] and [22], the authors studied the stability of Runge-Kutta methods and one-leg methods for FIDEs (2.1) on a limited closed interval $[0, T]$, but the monotonicity condition

$$\begin{aligned} &\operatorname{Re}\langle f(t, u_1, v_1) - f(t, u_2, v_2), u_1 - u_2 - (w_1 - w_2)\rangle \\ &\quad \leq \alpha\|u_1 - u_2\|^2 + \beta\|v_1 - v_2\|^2 + \eta\|w_1 - w_2\|^2, \\ &\qquad t \geq t_0, u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{C}^d \end{aligned} \tag{2.5}$$

is required. There exist some important differences between conditions (2.2) and (2.5). In fact, as an example without delay and integral terms, Humphries and Stuart [2] proved that after translation of the origin, the Lorenz equations are dissipative, but do not satisfy condition (2.5). In addition, the dissipativity is a long time characteristic of a system rather than the stability on a limited closed interval.

The aim of this paper is to investigate whether the Runge-Kutta methods for system (2.1) preserve the dissipativity of the system itself.

It is well known that an s-stage Runge-Kutta method for ODEs can be expressed as

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}, \tag{2.6}$$

where $A = (a_{ij}) \in \mathbb{R}^{s\times s}$, $b = (b_1, b_2, \dots, b_s)^T \in \mathbb{R}^s$ and $c = (c_1, c_2, \dots, c_s)^T \in \mathbb{R}^s$ with $0 \le c_i \le 1$ $(i = 1, 2, \dots, s)$ and $\sum_{j=1}^{s} b_j = 1$.

The following algebraic stability concept is the basis for studying the dissipativity of Runge-Kutta methods.

Definition 2.4 (see [5, 6]) Runge-Kutta method (2.6) is said to be algebraically stable if

$$B = \operatorname{diag}(b_1, b_2, \dots, b_s) \quad \text{and} \quad M = BA + A^T B - bb^T$$

are nonnegative definite.

Let the step size $h = \frac{\tau}{m}$ with some positive integer m and $t_n = t_0 + nh$. An adaptation of method (2.6) for solving problem (2.1) leads to (see [22])

$$\begin{cases} X_i^{(n)} - Z_i^{(n)} = x_n - z_n + h\sum_{j=1}^{s} a_{ij} f(t_n + c_j h, X_j^{(n)}, X_j^{(n-m)}), \quad i = 1, 2, \dots, s, \\ x_{n+1} - z_{n+1} = x_n - z_n + h\sum_{j=1}^{s} b_j f(t_n + c_j h, X_j^{(n)}, X_j^{(n-m)}), \end{cases} \tag{2.7}$$

where x_n, $X_i^{(n)}$ denote approximations to $x(t_n)$, $x(t_n + c_i h)$ and z_n and $Z_i^{(n)}$ approximations to $z(t_n)$ and $z(t_n + c_i h)$, respectively. Here and later, we put that

$$z(t) = \int_{t-\tau}^{t} g\bigl(t, \xi, x(\xi)\bigr)\, d\xi. \tag{2.8}$$

In addition,

$$\begin{cases} x_n = \varphi(t_n), & n \le 0, \\ X_i^{(n)} = \varphi(t_n + c_i h), & t_n + c_i h \le t_0. \end{cases} \tag{2.9}$$

As to the computation of integral terms z_n, $Z_i^{(n)}$, we apply the compound quadrature formulas

$$z_n = h\sum_{i=0}^{m} v_i g(t_n, t_{n-i}, x_{n-i}), \tag{2.10}$$

$$Z_j^{(n)} = h\sum_{i=0}^{m} v_i g\bigl(t_n + c_j h, t_{n-i} + c_j h, X_j^{(n-i)}\bigr), \quad j = 1, 2, \dots, s, \tag{2.11}$$

where the quadrature formulas (2.10) and (2.11) can be derived from a uniform repeated rule (cf. [13, 21, 22]). For the numerical dissipativity analysis, we assume (2.10) or (2.11) to satisfy the following condition:

$$h\sqrt{(m+1)\sum_{i=0}^{m}|\nu_i|^2} < \nu \quad \text{with } mh=\tau \text{ and a positive constant } \nu. \tag{2.12}$$

Definition 2.5 Method (2.7) with a quadrature formula is said to be dissipative if, whenever the method is applied with a step size h to a dynamical system of the form (2.1) subject to (2.2)-(2.3), there exists a constant r such that, for any initial function $\varphi(t)$, there exists $n_0(\bar{\varphi},h)$, $\bar{\varphi}=\sup_{t_0-\tau\le t\le t_0}\|\varphi(t)\|$ such that

$$\|x_n\|\le r, \quad n>n_0 \tag{2.13}$$

holds.

3 Dissipativity of Runge-Kutta methods

In this section we focus on the dissipativity analysis of algebraically stable Runge-Kutta methods with respect to nonlinear FIDEs.

Theorem 3.1 *Assume that Rung-Kutta method* (2.6) *is algebraically stable and* $b_j>0$ *for* $j=1,2,\ldots,s$, *and that problem* (2.1) *satisfies* (2.2) *and* (2.3) *with* (2.4) *and* $\alpha+\beta+\eta\nu^2\lambda^2<0$. *Then method* (2.7) *with* (2.10)-(2.11) *and* (2.12) *for FIDEs* (2.1) *is dissipative.*

Proof For simplicity, we let

$$t_i^{(n)}=t_n+c_ih, \qquad Q_i=hf\bigl(t_i^{(n)},X_i^{(n)},X_i^{(n-m)}\bigr), \quad i=1,2,\ldots,s.$$

It is well known (see, for example, [2]) that

$$\|x_{n+1}-z_{n+1}\|^2-\|x_n-z_n\|^2-2\sum_{i=1}^{s}b_i\operatorname{Re}\bigl\langle X_i^{(n)}-Z_i^{(n)},Q_i\bigr\rangle=-\sum_{i=1}^{s}\sum_{j=1}^{s}m_{ij}\langle Q_i,Q_j\rangle, \tag{3.1}$$

where $m_{ij}=b_ia_{ij}+b_ja_{ji}-b_ib_j$.

By means of algebraic stability of the method, (3.1) leads to

$$\|x_{n+1}-z_{n+1}\|^2\le\|x_n-z_n\|^2+2\sum_{i=1}^{s}b_i\operatorname{Re}\bigl\langle X_i^{(n)}-Z_i^{(n)},Q_i\bigr\rangle. \tag{3.2}$$

Using conditions (2.2) and (2.3) , then (3.2) gives

$$\|x_{n+1}-z_{n+1}\|^2\le\|x_n-z_n\|^2+2h\sum_{i=1}^{s}b_i\bigl[\gamma+\alpha\bigl\|X_i^{(n)}\bigr\|^2+\beta\bigl\|X_i^{(n-m)}\bigr\|^2+\eta\bigl\|Z_i^{(n)}\bigr\|^2\bigr]. \tag{3.3}$$

We let

$$X^{(n)}=\bigl(X_1^{(n)},X_2^{(n)},\ldots,X_s^{(n)}\bigr), \qquad Z^{(n)}=\bigl(Z_1^{(n)},Z_2^{(n)},\ldots,Z_s^{(n)}\bigr), \quad n=0,1,\ldots.$$

Hence by induction, from (3.3) we can obtain that

$$\begin{aligned}\|x_n - z_n\|^2 &\leq \|x_{n-1} - z_{n-1}\|^2 + 2h\gamma \\ &\quad + 2h\alpha \|X^{(n-1)}\|_B^2 + 2h\beta \|X^{(n-m-1)}\|_B^2 + 2h\eta \|Z^{(n-1)}\|_B^2 \\ &\leq \|x_0 - z_0\|^2 + 2hn\gamma + 2h\alpha \sum_{j=0}^{n-1} \|X^{(j)}\|_B^2 \\ &\quad + 2h\beta \sum_{j=0}^{n-1} \|X^{(j-m)}\|_B^2 + 2h\eta \sum_{j=0}^{n-1} \|Z^{(j)}\|_B^2, \end{aligned} \tag{3.4}$$

where the condition $\sum_{j=0}^{s} b_j = 1$ has been used.

Now we estimate the quadrature terms $\|z_n\|$ and $\|Z^{(n)}\|_B$. From (2.10) and condition (2.3) we obtain that

$$\|z_n\| \leq h\lambda \sum_{k=0}^{m} |\nu_k| \|x_{n-k}\|. \tag{3.5}$$

Making the square of the both sides and using (2.12) and the Cauchy-Schwarz inequality, we get

$$\|z_n\|^2 \leq \frac{\nu^2\lambda^2}{m+1} \sum_{k=0}^{m} \|x_{n-k}\|^2. \tag{3.6}$$

Similarly, from (2.11), (2.3) and (2.12) we can also obtain

$$\|Z_i^{(n)}\| \leq h\lambda \sum_{k=0}^{m} |\nu_k| \|X_i^{(n-k)}\| \tag{3.7}$$

and

$$\|Z_i^{(n)}\|^2 \leq \frac{\nu^2\lambda^2}{m+1} \sum_{k=0}^{m} \|X_i^{(n-k)}\|^2, \quad i = 1, 2, \ldots, s,$$

which gives

$$\|Z^{(n)}\|_B^2 \leq \frac{\nu^2\lambda^2}{m+1} \sum_{k=0}^{m} \|X^{(n-k)}\|_B^2. \tag{3.8}$$

Hence it can be deduced that

$$\begin{aligned}\sum_{j=0}^{n-1} \|Z^{(j)}\|_B^2 &\leq \frac{\nu^2\lambda^2}{m+1} \sum_{j=0}^{n-1} \sum_{k=0}^{m} \|X^{(j-k)}\|_B^2 \\ &\leq \nu^2\lambda^2 \left(\sum_{j=0}^{n-1} \|X^{(j)}\|_B^2 + \frac{m}{2} \min_{-m \leq j \leq -1} \|X^{(j)}\|_B^2 \right). \end{aligned} \tag{3.9}$$

Therefore, substituting (3.9) into (3.4) shows

$$\begin{aligned}\|x_n - z_n\|^2 &\le 2hn\gamma + 2h\bigl(\alpha + \beta + \eta\nu^2\lambda^2\bigr)\sum_{j=0}^{n-1}\bigl\|X^{(j)}\bigr\|_B^2 \\ &\quad + \bigl[(1+\tau\lambda)^2 + \tau\bigl(2\beta + \eta\nu^2\lambda^2\bigr)\bigr]\max_{-\tau\le\xi\le 0}\bigl\|\varphi(\xi)\bigr\|^2. \end{aligned} \tag{3.10}$$

When $\gamma = 0$, it follows from (3.10) and $\alpha + \beta + \eta\nu^2\lambda^2 < 0$ that

$$\lim_{n\to\infty}\bigl\|X^{(n)}\bigr\|_B = 0,$$

which shows that for any $\varepsilon > 0$, there exists $n_1 = n_1(\bar{\varphi}, h) > 0$ such that

$$\bigl\|X_j^{(n)}\bigr\| < \varepsilon, \qquad \bigl\|X_j^{(n-m)}\bigr\| < \varepsilon, \quad j = 1, \ldots, s, n \ge n_1 \tag{3.11}$$

and

$$\bigl\|Z_j^{(n)}\bigr\| < \nu\lambda\varepsilon, \quad j = 1, \ldots, s, n \ge n_1. \tag{3.12}$$

On the other hand, from (2.7) we have

$$\begin{aligned}\|x_n - z_n\| &= \left\|X_i^{(n)} - Z_i^{(n)} - h\sum_{j=1}^{s} a_{ij} f\bigl(t_j^{(n)}, X_j^{(n)}, X_j^{(n-m)}\bigr)\right\| \\ &\le \bigl\|X_i^{(n)}\bigr\| + \bigl\|Z_i^{(n)}\bigr\| + h\sum_{j=1}^{s}|a_{ij}|\bigl\|f\bigl(t_j^{(n)}, X_j^{(n)}, X_j^{(n-m)}\bigr)\bigr\|, \quad i = 1, 2, \ldots, s. \end{aligned} \tag{3.13}$$

Therefore, we can obtain that

$$\|x_n - z_n\| \le hL\sum_{j=1}^{s}|a_{ij}| + (1+\nu\lambda)\varepsilon, \quad n \ge n_1, \tag{3.14}$$

where

$$L = \sup_{\substack{\|u\|\le\varepsilon \\ \|v\|\le\varepsilon}}\bigl\|f(t,u,v)\bigr\|, \quad t \in [0, +\infty), u, v \in \mathbb{C}^d.$$

When $\gamma > 0$, using techniques similar to those presented in [6], we can conclude that there exist $r_2 > 0$ and a positive integer $n_2(\bar{\varphi}, h)$ such that

$$\|x_n - z_n\| \le r_2 \quad \text{for } n \ge n_2, \tag{3.15}$$

where

$$r_2 = \sqrt{2\bigl[1 + \tau\bigl(2\beta + \eta\nu^2\lambda^2\bigr)\bigr]R_0 + 4(m+1)h\gamma},$$

$$n_2 = \frac{[(1+\tau\lambda)^2 + \tau(2\beta + \eta\nu^2\lambda^2)]\bar{\varphi}^2}{2h\gamma} + 2(m+1)$$

with

$$\begin{cases} R_0 = \frac{4(m+1)h\gamma}{\sigma} + h|C|, \\ \sigma = -(\alpha + \beta + \eta\nu^2\lambda^2), \\ \bar{\varphi} = \sup_{t_0-\tau\le t\le t_0} \|\varphi(t)\|, \end{cases}$$

$$C = \sup_{\substack{\|u\|_B^2\le 4(m+1)h\gamma/\sigma \\ \|v\|_B^2\le 4(m+1)h\gamma/\sigma \\ \|w\|_B^2\le 4\nu^2\lambda^2(m+1)h\gamma/\sigma}} \sum_{i=1}^{s} b_i\left[\sum_{j=1}^{s}(b_j - a_{ij})\operatorname{Re}\langle u_i - w_i, hf(t_j,u_j,v_j)\rangle \right.$$
$$\left. + h\left\|\sum_{j=1}^{s}(b_j - a_{ij})f(t_j,u_j,v_j)\right\|^2\right],$$
$$u = (u_1,u_2,\dots,u_s)\in\mathbb{C}^{ds}, \qquad v = (v_1,v_2,\dots,v_s)\in\mathbb{C}^{ds}, \qquad w = (w_1,w_2,\dots,w_s)\in\mathbb{C}^{ds}.$$

A combination of (3.14) and (3.15) shows that there exist a constant R_1 and $n_0(\bar{\varphi},h)$ such that

$$\|x_n - z_n\| \le R_1, \quad n \ge n_0. \tag{3.16}$$

The next thing to do in the proof is estimating $\|x_n\|$. Because of the fact that

$$\|x_n\| \le \|x_n - z_n\| + \|z_n\|,$$

therefore, for $n \ge n_0$, from (3.5), (2.12) and (3.16) we have

$$\begin{aligned} \|x_n\| &\le R_1 + h\lambda\sum_{k=0}^{m}|\nu_k|\|x_{n-k}\| \\ &\le R_1 + \nu\lambda \max_{0\le k\le m}\|x_{n-k}\| \\ &\le R_1 + \nu\lambda \max_{1\le k\le m}\|x_{n-k}\| + \nu\lambda\|x_n\|. \end{aligned} \tag{3.17}$$

Since we have assumed $2\lambda\nu < 1$, then $1 - \lambda\nu > 0$, and (3.17) leads to

$$\|x_n\| \le \frac{R_1}{1-\nu\lambda} + \frac{\nu\lambda}{1-\nu\lambda}\max_{1\le k\le m}\|x_{n-k}\|, \quad n \ge n_0. \tag{3.18}$$

Let

$$\theta = \frac{\nu\lambda}{1-\nu\lambda}, \qquad \mu = \frac{R_1}{1-\nu\lambda}, \qquad \varphi_0 = \max_{n_0-m\le k\le n_0-1}\|x_k\|.$$

Thus $0 < \theta < 1$ and (3.18) can be written as follows:

$$\|x_n\| \le \mu + \theta\max_{1\le k\le m}\|x_{n-k}\|, \quad n \ge n_0. \tag{3.19}$$

When $n = n_0$, we have

$$\|x_{n_0}\| \le \mu + \theta\varphi_0. \tag{3.20}$$

In the following, we consider two cases. First, when $\mu + \theta\varphi_0 \geq \varphi_0$, we can obtain by induction that

$$\|x_{n_0+j}\| \leq \mu \sum_{k=0}^{j} \theta^k + \theta^{j+1}\varphi_0, \quad j = 0, 1, 2, \ldots. \tag{3.21}$$

In fact, (3.20) implies (3.21) satisfied for $j = 0$. It is easy to see that

$$\begin{aligned} \mu \sum_{k=0}^{j} \theta^k + \theta^{j+1}\varphi_0 &= \mu \sum_{k=0}^{j-1} \theta^k + \theta^j(\mu + \theta\varphi_0) \\ &\geq \mu \sum_{k=0}^{j-1} \theta^k + \theta^j\varphi_0, \quad j \geq 1. \end{aligned} \tag{3.22}$$

If (3.21) holds for $j < l$, where l is a positive integer, then it follows from (3.19) and (3.22) that

$$\begin{aligned} \|x_{n_0+l}\| &\leq \mu + \theta\left(\mu \sum_{k=0}^{l-1} \theta^k + \theta^l\varphi_0\right) \\ &= \mu \sum_{k=0}^{l} \theta^k + \theta^{l+1}\varphi_0, \end{aligned}$$

which shows that (3.21) holds for any $j \geq 0$.

Second, when $\mu + \theta\varphi_0 < \varphi_0$, then for $l = 0, 1, \ldots$, it can be given by induction that

$$\|x_{n_0+ml+j}\| \leq \mu \sum_{k=0}^{l} \theta^k + \theta^{l+1}\varphi_0 \quad \text{for any } j \in \{0, 1, \ldots, m-1\}. \tag{3.23}$$

In order to prove this conclusion, we first consider the case of $l = 0$.

As a matter of fact, when $l = 0$, (3.20) implies that (3.23) holds for $j = 0$. If here (3.23) holds for $j < q < m - 1$, then from (3.19) we have

$$\|x_{n_0+q}\| \leq \mu + \theta \max\{\mu + \theta\varphi_0, \varphi_0\} \leq \mu + \theta\varphi_0,$$

which shows (3.23) holds for $l = 0$.

Suppose that (3.23) holds for $l < p$, where p is a positive integer. When $j = 0$, (3.19) reads

$$\begin{aligned} \|x_{n_0+pm}\| &\leq \mu + \theta \max_{1 \leq k \leq m} \|x_{n_0+pm-k}\| \\ &\leq \mu + \theta\left(\mu \sum_{k=0}^{p-1} \theta^k + \theta^p\varphi_0\right) \\ &= \mu \sum_{k=0}^{p} \theta^k + \theta^{p+1}\varphi_0. \end{aligned}$$

If it holds for $j < q < m-1$ that

$$\|x_{n_0+mp+j}\| \le \mu \sum_{k=0}^{p} \theta^k + \theta^{p+1}\varphi_0,$$

then

$$\begin{aligned} \|x_{n_0+mp+q}\| &\le \mu + \theta \max\left\{\mu \sum_{k=0}^{p-1} \theta^k + \theta^{p}\varphi_0, \mu \sum_{k=0}^{p} \theta^k + \theta^{p+1}\varphi_0\right\} \\ &\le \mu \sum_{k=0}^{p} \theta^k + \theta^{p+1}\varphi_0, \end{aligned}$$

where we have used that

$$\mu \sum_{k=0}^{p-1} \theta^k + \theta^{p}\varphi_0 > \mu \sum_{k=0}^{p} \theta^k + \theta^{p+1}\varphi_0.$$

This shows that (3.23) holds for any integer $l \ge 0$.

Noting that $0 < \theta < 1$, a combination of (3.21) and (3.23) leads to the fact that, for any given $\varepsilon > 0$, there exists $n_3 > n_0$ such that

$$\|x_n\| \le \frac{R_1}{1 - 2\lambda\nu} + \varepsilon, \quad n \ge n_3.$$

This completes the proof of Theorem 3.1. □

Remark 3.2 It is well known that the s stage Gauss, Radau IA, Radau IIA and Lobatto IIIC Runge-Kutta methods are all algebraically stable [24], then from Theorem 3.1 they can preserve the dissipativity of the system when applied to FIDEs (2.1).

4 Numerical experiments

As an example, we consider the following two-dimensional system:

$$\begin{cases} \frac{d}{dt}(x_1(t) - \int_{t-\frac{\pi}{12}}^{t} \frac{1}{4\pi} e^{\xi - t}(7x_1(\xi) + 3x_2(\xi))\, d\xi) \\ \quad = -x_1(t) + \frac{1}{96}(\bar{x}_1(t - \frac{\pi}{12}) + \sqrt{5}\bar{x}_2(t - \frac{\pi}{12})) + f_1(t), \\ \frac{d}{dt}(x_2(t) - \int_{t-\frac{\pi}{12}}^{t} \frac{1}{4\pi} e^{\xi - t}(3x_1(\xi) - x_2(\xi))\, d\xi) \\ \quad = -x_2(t) + \frac{1}{96}(\sqrt{5}\bar{x}_1(t - \frac{\pi}{12}) - 3\bar{x}_2(t - \frac{\pi}{12})) + f_2(t), \end{cases} \quad t \ge 0, \tag{4.1}$$

where

$$\begin{aligned} f_1(t) &= \cos(at) - a\sin(at), \\ f_2(t) &= \sin(bt) + b\cos(bt), \\ \bar{x}_1\left(t - \frac{\pi}{12}\right) &= \frac{x_1(t - \frac{\pi}{12})}{1 + x_1^2(t - \frac{\pi}{12})}, \\ \bar{x}_2\left(t - \frac{\pi}{12}\right) &= \frac{x_2(t - \frac{\pi}{12})}{1 + x_2^2(t - \frac{\pi}{12})}. \end{aligned}$$

For this system, we choose

$$\alpha = -\frac{23}{48}, \qquad \beta = \frac{1}{24}, \qquad \eta = \frac{25}{48}, \qquad \lambda = \frac{2}{\pi},$$
$$\delta = \frac{16}{25}, \qquad \gamma = 2\sqrt{(1-a)^2 + (1+b)^2}, \qquad \tau = \frac{\pi}{12},$$

which ensures all the conditions of Theorem 2.2 hold. System (4.1) is dissipative and possesses an absorbing set $B = B(0, 7\sqrt{(1-a)^2 + (1+b)^2} + \varepsilon)$ for any given $\varepsilon > 0$.

In order to solve system (4.1), we use the third order Radau IIA method where

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cc} \frac{1}{3} & \frac{5}{12} & -\frac{1}{12} \\ 1 & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}. \tag{4.2}$$

Method (4.2) is algebraically stable and order 3. We let $\tau = mh$ with a given positive integer m and apply the composite Simpson's rule to approach the integral terms

$$z_n = \int_{t_n-\tau}^{t_n} g\bigl(t_n, \xi, x(\xi)\bigr)\, d\xi \quad \text{and} \quad Z_i^{(n)} = \int_{t_n+c_ih-\tau}^{t_n+c_ih} g\bigl(t_n + c_ih, \xi, x(\xi)\bigr)\, d\xi.$$

Here we can let $\nu = \frac{4}{3}$ in (2.10) and have $\alpha + \beta + \eta\nu^2\lambda^2 < 0$. According to Theorem 3.1, the numerical solution is dissipative.

Now we let the step size $h = 0.004\pi/12$ and consider different initial functions for $t \in [\frac{\pi}{12}, 0]$ as follows:

(I) $y_1(t) = \sin(t)e^t$, $y_2(t) = 2t^2$;
(II) $y_1(t) = \cos(2t)$, $y_2(t) = 3\sin(2t)$;
(III) $y_1(t) = 3\sin(4t)$, $y_2(t) = \cos(3t)$,

respectively. The numerical results are shown in Figures 1, 2, 3, 4, 5 and 6.

These numerical examples prove that problem (4.1) is dissipative. Therefore, the numerical examples illustrate the correctness of our theoretical results.

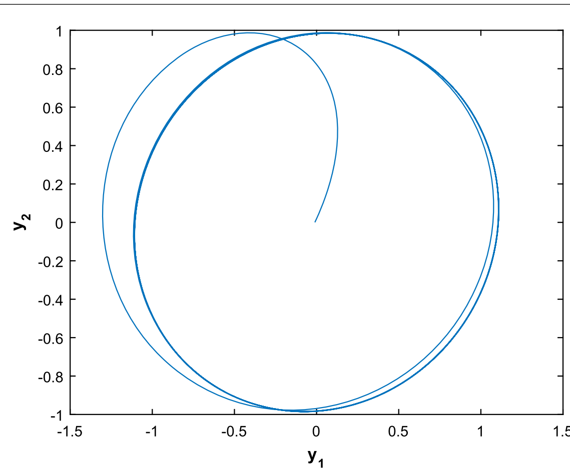

Figure 1 The numerical solution of (4.1) with initial function (I) and $a = 2$, $b = 2$ in the interval $[0, 10\pi]$.

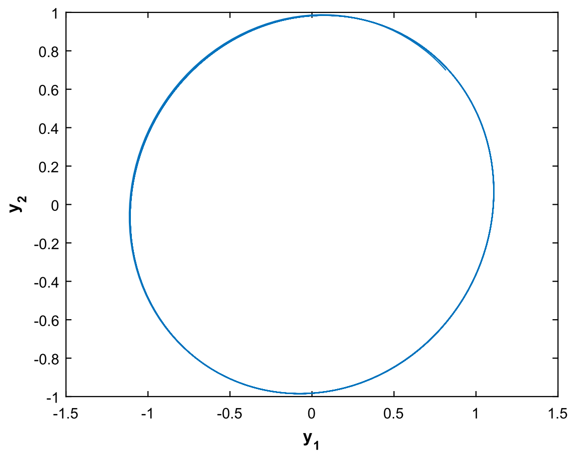

Figure 2 **The numerical solution of (4.1) with initial function (I) and $a = 2$, $b = 2$ in the interval $[\frac{7\pi}{6}, 10\pi]$.**

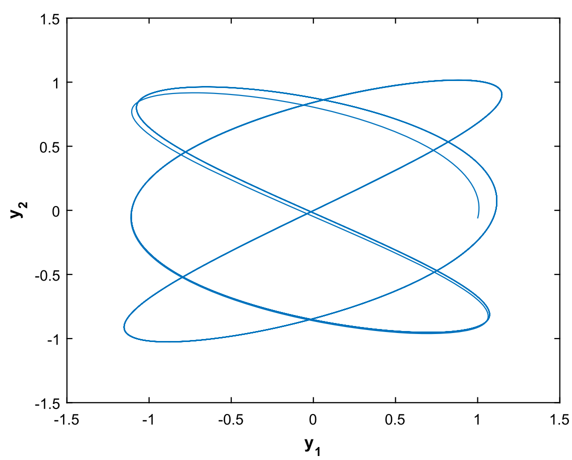

Figure 3 **The numerical solution of (4.1) with initial function (II) and $a = 3$, $b = 2$ in the interval $[1, 10\pi]$.**

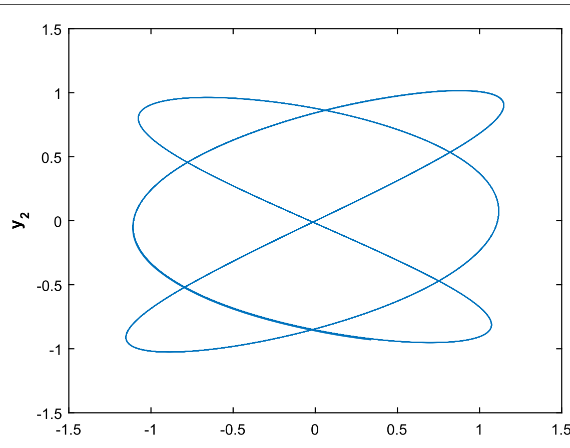

Figure 4 **The numerical solution of (4.1) with initial function (II) and $a = 3$, $b = 2$ in the interval $[\frac{5\pi}{6}, 10\pi]$.**

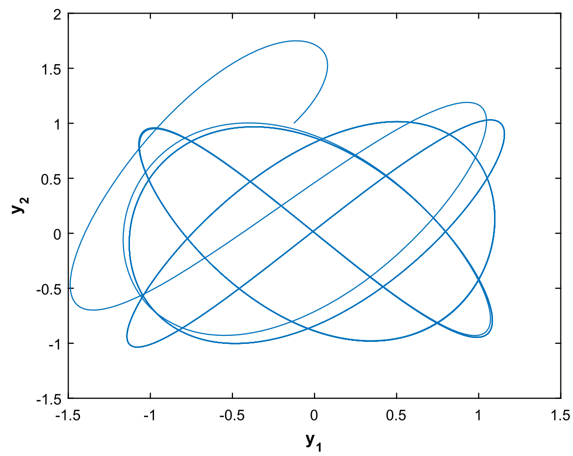

Figure 5 The numerical solution of (4.1) with initial function (III) and $a = 3$, $b = 4$ in the interval $[1, 10\pi]$.

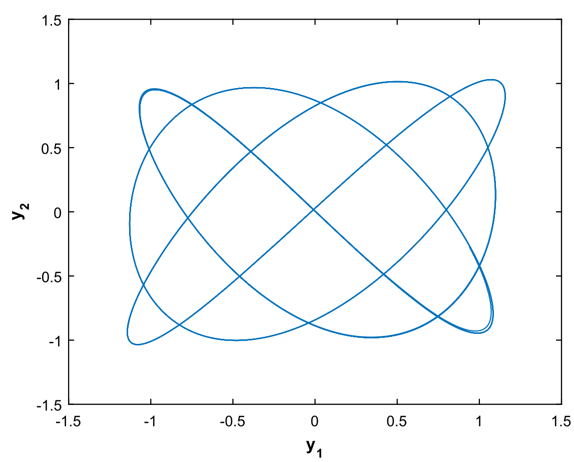

Figure 6 The numerical solution of (4.1) with initial function (III) and $a = 3$, $b = 4$ in the interval $[\frac{4\pi}{3}, 10\pi]$.

Acknowledgements
This work is supported by the National Natural Science Foundation of China (11371302, 11571291).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 31 October 2016 Accepted: 4 May 2017 Published online: 23 May 2017

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