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New *a posteriori* error estimates of mixed finite element methods for quadratic optimal control problems governed by semilinear parabolic equations with integral constraint

Zuliang Lu[1,2*], Shaohong Du[3] and Yuelong Tang[4]

*Correspondence: zulianglux@126.com
[1]School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing, 404000, P.R. China
[2]Laboratory for Applied Mathematics, Beijing Computational Science Research Center, Beijing, 100084, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate new $L^\infty(L^2)$ and $L^2(L^2)$-*posteriori* error estimates of mixed finite element solutions for quadratic optimal control problems governed by semilinear parabolic equations. The state and the co-state are discretized by the order one Raviart-Thomas mixed finite element spaces and the control is approximated by piecewise constant functions. We derive *a posteriori* error estimates in $L^\infty(J;L^2(\Omega))$-norm and $L^2(J;L^2(\Omega))$-norm for both the state and the control approximation. Such estimates, which are apparently not available in the literature, are an important step towards developing reliable adaptive mixed finite element approximation schemes for the optimal control problem.

MSC: 49J20; 65N30

Keywords: *a posteriori* error estimates; quadratic optimal control problems; semilinear parabolic equations; mixed finite element methods; integral constraint

1 Introduction

In this paper we consider quadratic optimal control problems governed by the semilinear parabolic equations

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_0^T\left(\|\mathbf{p}-\mathbf{p}_d\|^2+\|y-y_d\|^2+\|u\|^2\right)dt\right\}, \tag{1.1}$$

$$y_t(x,t)+\operatorname{div}\mathbf{p}(x,t)+\phi\bigl(y(x,t)\bigr)=f(x,t)+u(x,t),\quad x\in\Omega,t\in J, \tag{1.2}$$

$$\mathbf{p}(x,t)=-A(x)\nabla y(x,t),\quad x\in\Omega,t\in J, \tag{1.3}$$

$$y(x,t)=0,\quad x\in\partial\Omega,t\in J,\qquad y(x,0)=y_0(x),\quad x\in\Omega, \tag{1.4}$$

where the bounded open set $\Omega\subset\mathbf{R}^2$ is a convex polygon with the boundary $\partial\Omega$. $J=[0,T]$. Let K be a closed convex set in the control space $U=L^2(J;L^2(\Omega))$, $\mathbf{p},\mathbf{p}_d\in(L^2(J;H^1(\Omega)))^2$, $y,y_d\in L^2(J;H^1(\Omega))$, $f,u\in L^2(J;L^2(\Omega))$, $y_0(x)\in H_0^1(\Omega)$. For any $R>0$, the function $\phi(\cdot)\in W^{1,\infty}(-R,R)$, $\phi'(y)\in L^2(\Omega)$ for any $y\in H^1(\Omega)$, and $\phi'(y)\geq 0$. Assume that the coefficient matrix $A(x)=(a_{ij}(x))_{2\times 2}\in C^\infty(\bar{\Omega};\mathbf{R}^{2\times 2})$ is a symmetric 2×2-matrix and there are constants $c_1,c_2>0$ satisfying for any vector $\mathbf{X}\in\mathbf{R}^2$, $c_1\|\mathbf{X}\|_{\mathbf{R}^2}^2\leq\mathbf{X}^tA\mathbf{X}\leq c_2\|\mathbf{X}\|_{\mathbf{R}^2}^2$. We assume that the constraint on the control is an obstacle such that

$$K = \left\{ u \in L^2\bigl(J; L^2(\Omega)\bigr) : \int_\Omega u(x,t)\,dx \geq 0, \text{ a.e. in } \Omega \times J \right\}.$$

Optimal control problems have been successfully utilized in scientific and engineering numerical simulation. Thus they must be solved by using some efficient numerical methods. Among these numerical methods, the finite element method was a good choice for solving partial differential equations. There have been extensive studies in convergence for finite element approximation of optimal control problems. A systematic introduction of the finite element method for optimal control problems can be found in [1–7].

Recently, an adaptive finite element method has been investigated extensively. It has become one of the most popular methods in the scientific computation and numerical modeling. Adaptive finite element approximation ensures a higher density of nodes in a certain area of the given domain, where the solution is more difficult to approximate, indicated by *a posteriori* error estimators. Hence it is an important approach to boost the accuracy and efficiency of finite element discretizations. There are lots of works concentrating on the adaptivity of many optimal control problems, for example, [8–12]. Note that all the above works aimed at the standard finite element method.

In many control problems, the objective functional contains the gradient of state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods. When the objective functional contains the gradient of the state variable, mixed finite element methods should be used for discretization of the state equation with which both the scalar variable and its flux variable can be approximated in the same accuracy.

Recently, in [13, 14] we did some primary work on *a priori* error estimates for nonlinear parabolic optimal control problems by mixed finite element methods. In [15], we considered *a posteriori* error estimates of triangular mixed finite element methods for semilinear elliptic optimal control problems. The state and the co-state were discretized by the Raviart-Thomas mixed finite element spaces and the control was approximated by piecewise constant functions. In [16], we derived *a posteriori* error estimates for linear parabolic optimal control problems by the lowest order Raviart-Thomas mixed finite element methods.

This paper is motivated by the idea of the article [17]. We shall use the order one Raviart-Thomas mixed finite element to discretize the state and the co-state. Due to the limited regularity of the optimal control u in general, we therefore only consider a piecewise constant space. Then we derive *a posteriori* error estimates for the mixed finite element approximation of the optimal control problem. The estimators for the control, the state and the co-state variables are derived in the sense of $L^\infty(J; L^2(\Omega))$-norm or $L^2(J; L^2(\Omega))$-norm, which are different from the ones in [16].

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\leq m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we define $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. We denote by $L^s(0,T; W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with the norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = (\int_0^T \|v\|^s_{W^{m,p}(\Omega)}\,dt)^{\frac{1}{s}}$ for $s \in [1,\infty)$, and the standard modification for $s = \infty$. Similarly, one can define the spaces $H^1(J; W^{m,p}(\Omega))$ and $C^k(J; W^{m,p}(\Omega))$. The details can be found in [18].

The plan of this paper is as follows. In the next section, we shall construct the mixed finite element approximation and the backward Euler discretization for quadratic optimal control problems governed by semilinear parabolic equations (1.1)-(1.4). Then, we derive *a posteriori* error estimates for both the state and the control approximation in Section 3. Finally, we give a conclusion and some future work.

2 Mixed methods of optimal control problems

In this section we shall now discuss the mixed finite element approximation and the backward Euler discretization of quadratic semilinear parabolic optimal control problems (1.1)-(1.4). To fix the idea, we shall take the state spaces $L^2(\mathbf{V}) = L^2(J;\mathbf{V})$ and $H^1(W) = H^1(J;W)$, where $\mathbf{V}$ and W are defined as follows:

$$\mathbf{V} = H(\mathrm{div};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^2, \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}, \qquad W = L^2(\Omega).$$

The Hilbert space $\mathbf{V}$ is equipped with the following norm:

$$\|\mathbf{v}\|_{H(\mathrm{div};\Omega)} = \left(\|\mathbf{v}\|^2_{0,\Omega} + \|\,\mathrm{div}\,\mathbf{v}\|^2_{0,\Omega}\right)^{1/2}.$$

Let $\alpha = A^{-1}$, we recast (1.1)-(1.4) as the following weak form: find $(\mathbf{p},y,u) \in L^2(\mathbf{V}) \times H^1(W) \times K$ such that

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_0^T \left(\|\mathbf{p}-\mathbf{p}_d\|^2 + \|y-y_d\|^2 + \|u\|^2\right)dt\right\}, \tag{2.1}$$

$$(\alpha\mathbf{p},\mathbf{v}) - (y,\mathrm{div}\,\mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.2}$$

$$(y_t,w) + (\mathrm{div}\,\mathbf{p},w) + \big(\phi(y),w\big) = (f+u,w), \quad \forall w \in W, \tag{2.3}$$

$$y(x,0) = y_0(x), \quad \forall x \in \Omega. \tag{2.4}$$

It follows from [17] that optimal control problem (2.1)-(2.4) has a solution $(\mathbf{p},y,u)$, and that if a triplet $(\mathbf{p},y,u)$ is the solution of (2.1)-(2.4), then there is a co-state $(\mathbf{q},z) \in L^2(\mathbf{V}) \times H^1(W)$ such that $(\mathbf{p},y,\mathbf{q},z,u)$ satisfies the following optimality conditions:

$$(\alpha\mathbf{p},\mathbf{v}) - (y,\mathrm{div}\,\mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.5}$$

$$(y_t,w) + (\mathrm{div}\,\mathbf{p},w) + \big(\phi(y),w\big) = (f+u,w), \quad \forall w \in W, \tag{2.6}$$

$$y(x,0) = y_0(x), \quad \forall x \in \Omega, \tag{2.7}$$

$$(\alpha\mathbf{q},\mathbf{v}) - (z,\mathrm{div}\,\mathbf{v}) = -(\mathbf{p}-\mathbf{p}_d,\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.8}$$

$$-(z_t,w) + (\mathrm{div}\,\mathbf{q},w) + \big(\phi'(y)z,w\big) = (y-y_d,w), \quad \forall w \in W, \tag{2.9}$$

$$z(x,T) = 0, \quad \forall x \in \Omega, \tag{2.10}$$

$$\int_0^T (u+z,\tilde{u}-u)\,dt \geq 0, \quad \forall \tilde{u} \in K, \tag{2.11}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

In [19], the expression of the control variable was given. Here, we adopt the same method to derive the following operator:

$$u = \max\{0, \bar{z}\} - z, \tag{2.12}$$

where $\bar{z} = \int_\Omega z / \int_\Omega 1\, dx$ denotes the integral average on Ω of the function z.

Let $\mathcal{T}_h$ be regular triangulations of Ω. h_τ is the diameter of τ and $h = \max h_\tau$. Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote the order one Raviart-Thomas space associated with the triangulations $\mathcal{T}_h$ of Ω. P_k denotes the space of polynomials of total degree at most k. Let $\mathbf{V}(\tau) = \{\mathbf{v} \in P_1^2(\tau) + x \cdot P_1(\tau)\}$, $W(\tau) = P_1(\tau)$. We define

$$\begin{aligned}
\mathbf{V}_h &:= \bigl\{\mathbf{v}_h \in \mathbf{V} : \forall \tau \in \mathcal{T}_h, \mathbf{v}_h|_\tau \in \mathbf{V}(\tau)\bigr\}, \\
W_h &:= \bigl\{w_h \in W : \forall \tau \in \mathcal{T}_h, w_h|_\tau \in W(\tau)\bigr\}, \\
K_h &:= \bigl\{\tilde{u}_h \in K : \forall \tau \in \mathcal{T}_h, \tilde{u}_h|_\tau \in P_0(\tau)\bigr\}.
\end{aligned}$$

Let $L^2(\mathbf{V}_h) = L^2(J; \mathbf{V}_h)$ and $H^1(W_h) = H^1(J; W_h)$. The mixed finite element discretization of (2.1)-(2.4) is as follows: compute $(\mathbf{p}_h, y_h, u_h) \in L^2(\mathbf{V}_h) \times H^1(W_h) \times K_h$ such that

$$\min_{u_h \in K_h} \left\{ \frac{1}{2} \int_0^T \bigl(\|\mathbf{p}_h - \mathbf{p}_d\|^2 + \|y_h - y_d\|^2 + \|u_h\|^2 \bigr)\, dt \right\}, \tag{2.13}$$

$$(\alpha \mathbf{p}_h, \mathbf{v}_h) - (y_h, \operatorname{div} \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.14}$$

$$(y_{ht}, w_h) + (\operatorname{div} \mathbf{p}_h, w_h) + \bigl(\phi(y_h), w_h\bigr) = (f + u_h, w_h), \quad \forall w_h \in W_h, \tag{2.15}$$

$$y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.16}$$

where $y_0^h(x) \in W_h$ is an approximation of y_0. Optimal control problem (2.13)-(2.16) again has a solution $(\mathbf{p}_h, y_h, u_h)$, and that if a triplet $(\mathbf{p}_h, y_h, u_h)$ is the solution of (2.13)-(2.16), then there is a co-state $(\mathbf{q}_h, z_h) \in L^2(\mathbf{V}_h) \times H^1(W_h)$ such that $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(\alpha \mathbf{p}_h, \mathbf{v}_h) - (y_h, \operatorname{div} \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.17}$$

$$(y_{ht}, w_h) + (\operatorname{div} \mathbf{p}_h, w_h) + \bigl(\phi(y_h), w_h\bigr) = (f + u_h, w_h), \quad \forall w_h \in W_h, \tag{2.18}$$

$$y_h(x, 0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.19}$$

$$(\alpha \mathbf{q}_h, \mathbf{v}_h) - (z_h, \operatorname{div} \mathbf{v}_h) = -(\mathbf{p}_h - \mathbf{p}_d, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.20}$$

$$-(z_{ht}, w_h) + (\operatorname{div} \mathbf{q}_h, w_h) + \bigl(\phi'(y_h) z_h, w_h\bigr) = (y_h - y_d, w_h), \quad \forall w_h \in W_h, \tag{2.21}$$

$$z_h(x, T) = 0, \quad \forall x \in \Omega, \tag{2.22}$$

$$\int_0^T (u_h + z_h, \tilde{u}_h - u_h)\, dt \geq 0, \quad \forall \tilde{u}_h \in K_h. \tag{2.23}$$

Next, we define the standard $L^2(\Omega)$-orthogonal projection $\mathcal{Q}_h : K \to K_h$, which satisfies: for any $\tilde{u} \in K$,

$$(\tilde{u} - \mathcal{Q}_h \tilde{u}, \tilde{u}_h) = 0, \quad \forall \tilde{u}_h \in K_h, \tag{2.24}$$

$$\|\tilde{u}-\mathcal{Q}_h\tilde{u}\|_{-s,r}\leq C|\tilde{u}|_{1,r}h^{1+s},\quad s=0,1\text{ for }\tilde{u}\in W^{1,s}(\Omega). \tag{2.25}$$

Similar to (2.12), for variational inequality (2.23), we have the following conclusion [19]. Assume that z_h is known in variational inequality (2.23). The solution of the variational inequality is

$$u_h=\mathcal{Q}_h\bigl(\max\{0,\bar{z}_h\}-z_h\bigr),\qquad \bar{z}_h=\frac{\int_\Omega z_h\,dx}{\int_\Omega 1\,dx}. \tag{2.26}$$

Now we consider the fully discrete approximation for the above semidiscrete problem. Let $\Delta t>0$, $N=\frac{T}{\Delta t}\in\mathbb{Z}$, and $t_i=i\Delta t$, $i\in\mathbb{Z}$. Also, let

$$\psi^i=\psi^i(x)=\psi(x,t_i),\qquad d_t\psi^i=\frac{\psi^i-\psi^{i-1}}{\Delta t}.$$

The following fully discrete approximation scheme is to find $(\mathbf{p}_h^i,y_h^i,u_h^i)\in\mathbf{V}_h\times W_h\times K_h$, $i=1,2,\ldots,N$, such that

$$\min_{u_h^i\in K_h}\left\{\frac{1}{2}\sum_{i=1}^{N}\Delta t\bigl(\|\mathbf{p}_h^i-\mathbf{p}_d^i\|^2+\|y_h^i-y_d^i\|^2+\|u_h^i\|^2\bigr)\right\}, \tag{2.27}$$

$$\bigl(\alpha\mathbf{p}_h^i,\mathbf{v}_h\bigr)-\bigl(y_h^i,\operatorname{div}\mathbf{v}_h\bigr)=0,\quad\forall\mathbf{v}_h\in\mathbf{V}_h, \tag{2.28}$$

$$\bigl(d_ty_h^i,w_h\bigr)+\bigl(\operatorname{div}\mathbf{p}_h^i,w_h\bigr)+\bigl(\phi\bigl(y_h^i\bigr),w_h\bigr)=\bigl(f^i+u_h^i,w_h\bigr),\quad\forall w_h\in W_h, \tag{2.29}$$

$$y_h^0(x)=y_0^h(x),\quad\forall x\in\Omega. \tag{2.30}$$

It follows that optimal control problem (2.27)-(2.30) has a solution $(\mathbf{p}_h^i,y_h^i,u_h^i)$, $i=1,2,\ldots,N$, and that if a triplet $(\mathbf{p}_h^i,y_h^i,u_h^i)\in\mathbf{V}_h\times W_h\times K_h$, $i=1,2,\ldots,N$, is the solution of (2.27)-(2.30), then there is a co-state $(\mathbf{q}_h^{i-1},z_h^{i-1})\in\mathbf{V}_h\times W_h$ such that $(\mathbf{p}_h^i,y_h^i,\mathbf{q}_h^{i-1},z_h^{i-1},u_h^i)\in(\mathbf{V}_h\times W_h)^2\times K_h$ satisfies the following optimality conditions:

$$\bigl(\alpha\mathbf{p}_h^i,\mathbf{v}_h\bigr)-\bigl(y_h^i,\operatorname{div}\mathbf{v}_h\bigr)=0,\quad\forall\mathbf{v}_h\in\mathbf{V}_h, \tag{2.31}$$

$$\bigl(d_ty_h^i,w_h\bigr)+\bigl(\operatorname{div}\mathbf{p}_h^i,w_h\bigr)+\bigl(\phi\bigl(y_h^i\bigr),w_h\bigr)=\bigl(f^i+u_h^i,w_h\bigr),\quad\forall w_h\in W_h, \tag{2.32}$$

$$y_h^0(x)=y_0^h(x),\quad\forall x\in\Omega, \tag{2.33}$$

$$\bigl(\alpha\mathbf{q}_h^{i-1},\mathbf{v}_h\bigr)-\bigl(z_h^{i-1},\operatorname{div}\mathbf{v}_h\bigr)=-\bigl(\mathbf{p}_h^{i-1}-\mathbf{p}_d^{i-1},\mathbf{v}_h\bigr),\quad\forall\mathbf{v}_h\in\mathbf{V}_h, \tag{2.34}$$

$$-\bigl(d_tz_h^i,w_h\bigr)+\bigl(\operatorname{div}\mathbf{q}_h^{i-1},w_h\bigr)+\bigl(\phi'\bigl(y_h^i\bigr)z_h^{i-1},w_h\bigr)=\bigl(y_h^{i-1}-y_d^{i-1},w_h\bigr),\quad\forall w_h\in W_h, \tag{2.35}$$

$$z_h^N(x)=0,\quad\forall x\in\Omega, \tag{2.36}$$

$$\bigl(u_h^i+z_h^{i-1},\tilde{u}_h-u_h^i\bigr)\geq 0,\quad\forall\tilde{u}_h\in K_h. \tag{2.37}$$

For $i=1,2,\ldots,N$, let

$$Y_h|_{(t_{i-1},t_i]}=\bigl((t_i-t)y_h^{i-1}+(t-t_{i-1})y_h^i\bigr)/\Delta t,$$
$$Z_h|_{(t_{i-1},t_i]}=\bigl((t_i-t)z_h^{i-1}+(t-t_{i-1})z_h^i\bigr)/\Delta t,$$
$$P_h|_{(t_{i-1},t_i]}=\bigl((t_i-t)\mathbf{p}_h^{i-1}+(t-t_{i-1})\mathbf{p}_h^i\bigr)/\Delta t,$$

$$Q_h|_{(t_{i-1},t_i]} = \big((t_i - t)\mathbf{q}_h^{i-1} + (t - t_{i-1})\mathbf{q}_h^i\big)/\Delta t,$$
$$U_h|_{(t_{i-1},t_i]} = u_h^i.$$

For any function $w \in C(J; L^2(\Omega))$, let

$$\hat{w}(x,t)|_{t\in(t_{i-1},t_i]} = w(x,t_i), \qquad \tilde{w}(x,t)|_{t\in(t_{i-1},t_i]} = w(x,t_{i-1}).$$

Then optimality conditions (2.31)-(2.37) satisfy

$$(\alpha\hat{P}_h, \mathbf{v}_h) - (\hat{Y}_h, \operatorname{div}\mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.38}$$
$$(Y_{ht}, w_h) + (\operatorname{div}\hat{P}_h, w_h) + \big(\phi(\hat{Y}_h), w_h\big) = (\hat{f} + U_h, w_h), \quad \forall w_h \in W_h, \tag{2.39}$$
$$Y_h(x,0) = y_0^h(x), \quad \forall x \in \Omega, \tag{2.40}$$
$$(\alpha\widetilde{Q}_h, \mathbf{v}_h) - (\widetilde{Z}_h, \operatorname{div}\mathbf{v}_h) = -(\widetilde{P}_h - \tilde{\mathbf{p}}_d, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.41}$$
$$-(Z_{ht}, w_h) + (\operatorname{div}\widetilde{Q}_h, w_h) + \big(\phi'(\hat{Y}_h)\widetilde{Z}_h, w_h\big) = (\widetilde{Y}_h - \tilde{y}_d, w_h), \quad \forall w_h \in W_h, \tag{2.42}$$
$$Z_h(x,T) = 0, \quad \forall x \in \Omega, \tag{2.43}$$
$$(U_h + \widetilde{Z}_h, \tilde{u}_h - U_h) \geq 0, \quad \forall \tilde{u}_h \in K_h. \tag{2.44}$$

Similar to (2.26), the solution of variational inequality (2.44) is

$$U_h = \mathcal{Q}_h\big(\max\{0, \overline{\widetilde{Z}}_h\} - \widetilde{Z}_h\big), \qquad \overline{\widetilde{Z}}_h = \frac{\int_\Omega \widetilde{Z}_h\,dx}{\int_\Omega 1\,dx}. \tag{2.45}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $U_h \in K_h$, we first define the state solution $(\mathbf{p}(U_h), y(U_h), \mathbf{q}(U_h), z(U_h))$ satisfying

$$\big(\alpha\mathbf{p}(U_h), \mathbf{v}\big) - \big(y(U_h), \operatorname{div}\mathbf{v}\big) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.46}$$
$$\big(y_t(U_h), w\big) + \big(\operatorname{div}\mathbf{p}(U_h), w\big) + \big(\phi\big(y(U_h)\big), w\big) = (f + U_h, w), \quad \forall w \in W, \tag{2.47}$$
$$y(U_h)(x,0) = y_0(x), \quad \forall x \in \Omega, \tag{2.48}$$
$$\big(\alpha\mathbf{q}(U_h), \mathbf{v}\big) - \big(z(U_h), \operatorname{div}\mathbf{v}\big) = -\big(\mathbf{p}(U_h) - \mathbf{p}_d, \mathbf{v}\big), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.49}$$
$$\begin{aligned} &-\big(z_t(U_h), w\big) + \big(\operatorname{div}\mathbf{q}(U_h), w\big) + \big(\phi'\big(y(U_h)\big)z(U_h), w\big) \\ &\quad = \big(y(U_h) - y_d, w\big), \quad \forall w \in W, \end{aligned} \tag{2.50}$$
$$z(U_h)(x,T) = 0, \quad \forall x \in \Omega. \tag{2.51}$$

For $\varphi \in W_h$, we shall write

$$\phi(\varphi) - \phi(\rho) = -\tilde{\phi}'(\varphi)(\rho - \varphi) = -\phi'(\rho)(\rho - \varphi) + \tilde{\phi}''(\varphi)(\rho - \varphi)^2, \tag{2.52}$$

where

$$\tilde{\phi}'(\varphi) = \int_0^1 \phi'\big(\varphi + s(\rho - \varphi)\big)\,ds, \qquad \tilde{\phi}''(\varphi) = \int_0^1 (1 - s)\phi''\big(\rho + s(\varphi - \rho)\big)\,ds$$

are bounded functions in $\bar{\Omega}$.

Let $R_h : W \to W_h$ be the orthogonal $L^2(\Omega)$-projection into W_h [20] which satisfies:

$$(R_h w - w, \chi) = 0, \quad w \in W, \chi \in W_h, \tag{2.53}$$

$$\|R_h w - w\|_{0,q} \le C\|w\|_{t,q} h^t, \quad 0 \le t \le k+1, \text{ if } w \in W \cap W^{t,q}(\Omega), \tag{2.54}$$

$$\|R_h w - w\|_{-r} \le C\|w\|_t h^{r+t}, \quad 0 \le r, t \le k+1, \text{ if } w \in H^t(\Omega). \tag{2.55}$$

Let $\Pi_h : \mathbf{V} \to \mathbf{V}_h$ be the Raviart-Thomas projection operator [21] which satisfies: for any $\mathbf{v} \in \mathbf{V}$,

$$\int_E w_h(\mathbf{v} - \Pi_h \mathbf{v}) \cdot \boldsymbol{\nu}_E \, ds = 0, \quad w_h \in W_h, E \in \mathcal{E}_h, \tag{2.56}$$

$$\int_T (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \mathbf{v}_h \, dx \, dy = 0, \quad \mathbf{v}_h \in \mathbf{V}_h, T \in \mathcal{T}_h, \tag{2.57}$$

where $\mathcal{E}_h$ denote the set of element sides in $\mathcal{T}_h$. We have the commuting diagram property

$$\operatorname{div} \circ \Pi_h = R_h \circ \operatorname{div} : \mathbf{V} \to W_h \quad \text{and} \quad \operatorname{div}(I - \Pi_h)\mathbf{V} \perp W_h, \tag{2.58}$$

where and after, I denotes an identity matrix.

Further, the interpolation operator Π_h satisfies a local error estimate

$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_{0,\Omega} \le Ch|\mathbf{v}|_{1,\mathcal{T}_h}, \quad \mathbf{v} \in \mathbf{V} \cap H^1(\mathcal{T}_h). \tag{2.59}$$

The following lemmas are important in deriving *a posteriori* error estimates of residual type.

Lemma 2.1 *Let $\hat{\pi}_h$ be the average interpolation operator defined in* [22]. *For $m = 0$ or 1, $1 \le q \le \infty$ and $\forall v \in W^{1,q}(\Omega^h)$,*

$$|v - \hat{\pi}_h v|_{W^{m,q}(\tau)} \le \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} Ch_\tau^{1-m} |v|_{W^{1,q}(\tau')}. \tag{2.60}$$

Lemma 2.2 *Let π_h be the standard Lagrange interpolation operator* [23]. *Then, for $m = 0$ or 1, $1 < q \le \infty$ and $\forall v \in W^{2,q}(\Omega^h)$,*

$$|v - \pi_h v|_{W^{m,q}(\tau)} \le Ch_\tau^{2-m} |v|_{W^{2,q}(\tau)}. \tag{2.61}$$

3 *A posteriori* error estimates

In this section we study new $L^\infty(L^2)$ and $L^2(L^2)$-*posteriori* error estimates for the mixed finite element approximation to the semilinear parabolic optimal control problems. Let

$$S(u) = \frac{1}{2}\left(\|\mathbf{p} - \mathbf{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2\right), \tag{3.1}$$

$$S_h(U_h) = \frac{1}{2}\left(\|P_h - \mathbf{p}_d\|^2 + \|Y_h - y_d\|^2 + \|U_h\|^2\right). \tag{3.2}$$

It can be shown that (see [13] for some detail discussions)

$$\left(S'(u), v\right) = (u + z, v), \tag{3.3}$$

$$\big(S'(U_h), v\big) = \big(U_h + z(U_h), v\big), \tag{3.4}$$

$$\big(S_h'(U_h), v\big) = (U_h + \widetilde{Z}_h, v). \tag{3.5}$$

It is clear that S and S_h are well defined and continuous on K and K_h. Also, the functional S_h can be naturally extended on K. Then (2.1) and (2.27) can be represented as

$$\min_{u\in K}\left\{\int_0^T S(u)\,dt\right\} \tag{3.6}$$

and

$$\min_{U_h\in K_h}\left\{\int_0^T S_h(U_h)\,dt\right\}. \tag{3.7}$$

In many applications, $S(\cdot)$ is uniform convex near the solution u. The convexity of $S(\cdot)$ is closely related to the second-order sufficient conditions of the optimal control problems, which are assumed in many studies on numerical methods of the problem. For instance, in many applications, there is $c > 0$, independent of h, such that

$$\int_0^T \big(S'(u) - S'(U_h), u - U_h\big)_U \geq c\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \tag{3.8}$$

Firstly, let us derive *a posteriori* error estimates for the control u.

Theorem 3.1 *Let u and U_h be the solutions of* (3.6) *and* (3.7), *respectively. Assume that* $(S_h'(U_h))|_\tau \in H^s(\tau)$, $\forall \tau \in \mathcal{T}_h$ $(s = 0, 1)$, *and there is $v_h \in K_h$ such that*

$$\big|\big(S_h'(U_h), v_h - u\big)\big| \leq C\sum_{\tau\in\mathcal{T}_h} h_\tau \big\|S_h'(U_h)\big\|_{H^s(\tau)}\|u - U_h\|^s_{L^2(\tau)}. \tag{3.9}$$

Then we have

$$\|u - U_h\|^2_{L^2(J;L^2(\Omega))} \leq C\eta_1^2 + C\big\|z(U_h) - \widetilde{Z}_h\big\|^2_{L^2(J;L^2(\Omega))}, \tag{3.10}$$

where

$$\eta_1^2 = \int_0^T \sum_{\tau\in\mathcal{T}_h} h_\tau^{1+s}\|U_h + \widetilde{Z}_h\|^{1+s}_{H^1(\tau)}\,dt.$$

Proof It follows from (3.6) and (3.7) that

$$\int_0^T \big(S'(u), u - v\big) \leq 0, \quad \forall v \in K, \tag{3.11}$$

$$\int_0^T \big(S_h'(U_h), U_h - v_h\big) \leq 0, \quad \forall v_h \in K_h \subset K. \tag{3.12}$$

Then it follows from assumptions (3.8), (3.9), and the Schwarz inequality that

$$\begin{aligned} c\|u-U_h\|^2_{L^2(J;L^2(\Omega))} &\le \int_0^T \bigl(S'(u)-S'(U_h),u-U_h\bigr) \\ &\le \int_0^T \bigl\{\bigl(S'_h(U_h),v_h-u\bigr)+\bigl(S'_h(U_h)-S'(U_h),u-U_h\bigr)\bigr\} \\ &\le C\int_0^T \biggl\{\sum_{\tau\in\mathcal{T}_h} h_\tau^{1+s}\bigl\|S'_h(U_h)\bigr\|^{1+s}_{H^s(\tau)} \\ &\quad + \bigl\|S'_h(U_h)-S'(U_h)\bigr\|^2_{L^2(\Omega)}\biggr\} + \frac{c}{2}\|u-U_h\|^2_{L^2(J;L^2(\Omega))}. \end{aligned} \tag{3.13}$$

It is not difficult to show

$$S'_h(U_h)=U_h+\widetilde{Z}_h, \qquad S'(U_h)=U_h+z(U_h), \tag{3.14}$$

where $z(U_h)$ is defined in (2.46)-(2.51). Thanks to (3.14), it is easy to derive

$$\bigl\|S'_h(U_h)-S'(U_h)\bigr\|_{L^2(\Omega)}=\bigl\|\widetilde{Z}_h-z(U_h)\bigr\|_{L^2(\Omega)}. \tag{3.15}$$

Then, by estimates (3.13) and (3.15), we can prove the requested result (3.10). □

To estimate the error $\|\widetilde{Z}_h-z(U_h)\|^2_{L^2(J;L^2(\Omega))}$, we need the following well-known stability results for the following dual equations:

$$\begin{cases} -\kappa_t-\operatorname{div}(A\nabla\kappa)+\lambda\kappa=0, & x\in\Omega, t\in[0,t^*], \\ \kappa|_{\partial\Omega}=0, & t\in[0,t^*], \\ \kappa(x,t^*)=\kappa_0(x), & x\in\Omega, \end{cases} \tag{3.16}$$

and

$$\begin{cases} \varpi_t-\operatorname{div}(A^*\nabla\varpi)+\phi'(y(U_h))\varpi=0, & x\in\Omega, t\in[t^*,T], \\ \varpi|_{\partial\Omega}=0, & t\in[t^*,T], \\ \varpi(x,t^*)=\varpi_0(x), & x\in\Omega, \end{cases} \tag{3.17}$$

where

$$\lambda=\begin{cases} \frac{\phi(y(U_h))-\phi(Y_h)}{y(U_h)-Y_h}, & y(U_h)\neq Y_h, \qquad (3.18) \\ \phi'(Y_h), & y(U_h)=Y_h. \qquad (3.18') \end{cases}$$

Lemma 3.1 *Let κ and ϖ be the solutions of* (3.16) *and* (3.17), *respectively* [24, 25]. *Let Ω be a convex domain. Then*

$$\begin{aligned}
&\int_\Omega \bigl|\kappa(x,t)\bigr|^2\,dx \le C\|\kappa_0\|^2_{L^2(\Omega)}, \quad \forall t\in\bigl[0,t^*\bigr],\\
&\int_0^{t^*}\int_\Omega |\nabla\kappa|^2\,dx\,dt \le C\|\kappa_0\|^2_{L^2(\Omega)},\\
&\int_0^{t^*}\int_\Omega \bigl|t-t^*\bigr|\bigl|D^2\kappa\bigr|^2\,dx\,dt \le C\|\kappa_0\|^2_{L^2(\Omega)},\\
&\int_0^{t^*}\int_\Omega \bigl|t-t^*\bigr||\kappa_t|^2\,dx\,dt \le C\|\kappa_0\|^2_{L^2(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \bigl|\varpi(x,t)\bigr|^2\,dx \le C\|\varpi_0\|^2_{L^2(\Omega)}, \quad \forall t\in\bigl[t^*,T\bigr],\\
&\int_{t^*}^{T}\int_\Omega |\nabla\varpi|^2\,dx\,dt \le C\|\varpi_0\|^2_{L^2(\Omega)},\\
&\int_{t^*}^{T}\int_\Omega \bigl|t-t^*\bigr|\bigl|D^2\varpi\bigr|^2\,dx\,dt \le C\|\varpi_0\|^2_{L^2(\Omega)},\\
&\int_{t^*}^{T}\int_\Omega \bigl|t-t^*\bigr||\varpi_t|^2\,dx\,dt \le C\|\varpi_0\|^2_{L^2(\Omega)},
\end{aligned}$$

where $|D^2v| = \max\{|\partial^2 v/\partial x_i\partial x_j|, 1\le i,j\le 2\}$.

Next, we estimate the errors $Y_h - y(U_h)$ and $P_h - \mathbf{p}(U_h)$.

Theorem 3.2 *Let* $(P_h, Y_h, Q_h, Z_h, U_h)$ *and* $(\mathbf{p}(U_h), y(U_h), \mathbf{q}(U_h), z(U_h), U_h)$ *be the solutions of* (2.38)-(2.44) *and* (2.46)-(2.51), *respectively. Then we have*

$$\bigl\|Y_h - y(U_h)\bigr\|^2_{L^\infty(J;L^2(\Omega))} + \bigl\|P_h - \mathbf{p}(U_h)\bigr\|_{L^2(J;L^2(\Omega))} \le C\sum_{i=2}^{7}\eta_i^2, \tag{3.19}$$

where

$$\begin{aligned}
&\eta_2^2 = \max_{i\in[1,N-1]}\biggl\{\int_{t_{i-1}}^{t_{i+1}}\sum_\tau h_\tau^2\int_\tau \bigl(Y_{ht} + \operatorname{div}\hat{P}_h + \phi(\hat{Y}_h) - \hat{f} - U_h\bigr)^2\,dx\,dt\biggr\};\\
&\eta_3^2 = \max_{i\in[1,N-1]}\biggl\{\int_{t_{i-1}}^{t_{i+1}}\sum_\tau \min_{w_h\in W_h}\int_\tau (\alpha P_h - \nabla w_h)^2\,dx\,dt\biggr\};\\
&\eta_4^2 = |\ln\Delta t|\max_{t\in[0,T]}\biggl\{\sum_\tau h_\tau^4\int_\tau \bigl(Y_{ht} + \operatorname{div}\hat{P}_h + \phi(\hat{Y}_h) - \hat{f} - U_h\bigr)^2\,dx\biggr\};\\
&\eta_5^2 = |\ln\Delta t|\max_{t\in[0,T]}\biggl\{\sum_\tau h_\tau^2\cdot\min_{w_h\in W_h}\int_\tau (\alpha P_h - \nabla w_h)^2\,dx\biggr\};\\
&\eta_6^2 = \|\hat{f} - f\|^2_{L^1(0,t^*;L^2(\Omega))} + \bigl\|y_0^h(x) - y_0(x)\bigr\|^2_{L^2(\Omega)};\\
&\eta_7^2 = \|\hat{P}_h - P_h\|^2_{L^2(0,t^*;L^2(\Omega))} + \|\hat{Y}_h - Y_h\|^2_{L^2(0,t^*;L^2(\Omega))}.
\end{aligned}$$

Proof We define $\mathbf{p}_h^N$ as follows:

$$\big(\alpha\mathbf{p}_h^N,\mathbf{v}_h\big)-\big(y_h^N,\operatorname{div}\mathbf{v}_h\big)=0,\quad\forall\mathbf{v}_h\in\mathbf{V}_h.\tag{3.20}$$

Then from (3.20) we deduce that

$$\big(\alpha\mathbf{p}_h^{N-1},\mathbf{v}_h\big)-\big(y_h^{N-1},\operatorname{div}\mathbf{v}_h\big)=0,\quad\forall\mathbf{v}_h\in\mathbf{V}_h.\tag{3.21}$$

Combining (3.20)-(3.21) and the definitions of Y_h and P_h, we can get the following equality:

$$(\alpha P_h,\mathbf{v}_h)-(Y_h,\operatorname{div}\mathbf{v}_h)=0,\quad\forall\mathbf{v}_h\in\mathbf{V}_h.\tag{3.22}$$

Let κ be the solution of (3.16) with $\kappa_0(x)=(Y_h-y(U_h))(x,t^*)$, we infer that

$$\begin{aligned}
&\|Y_h-y(U_h)\|_{L^2(\Omega)}^2\\
&\quad=\big(\big(Y_h-y(U_h)\big)\big(x,t^*\big),\kappa\big(x,t^*\big)\big)\\
&\quad=\int_0^{t^*}\big(\big(\big(Y_h-y(U_h)\big)_t,\kappa\big)-\big(Y_h-y(U_h),\operatorname{div}(A\nabla\kappa)\big)\big)\,dt\\
&\qquad+\int_0^{t^*}\big(\phi(Y_h)-\phi\big(y(U_h)\big),\kappa\big)\,dt+\big(\big(Y_h-y(U_h)\big)(x,0),\kappa(x,0)\big)\\
&\quad=\int_0^{t^*}\big(\big(\big(Y_h-y(U_h)\big)_t,\kappa\big)+\big(p(U_h),\nabla\kappa\big)\big)\,dt\\
&\qquad-\int_0^{t^*}\big(Y_h,\operatorname{div}\big(\Pi_h(A\nabla\kappa)\big)\big)\,dt-\int_0^{t^*}\big(\phi\big(y(U_h)\big),\kappa\big)\,dt\\
&\qquad+\int_0^{t^*}\big(\phi(Y_h),\kappa\big)\,dt+\big(\big(Y_h-y(U_h)\big)(x,0),\kappa(x,0)\big).
\end{aligned}$$

Furthermore, using (2.38)-(2.40), (2.46)-(2.48) and (2.56)-(2.58), we can obtain that

$$\begin{aligned}
&\|Y_h-y(U_h)\|_{L^2(\Omega)}^2\\
&\quad=\int_0^{t^*}\big(\big(\big(Y_h-y(U_h)\big)_t,\kappa\big)+\big(\operatorname{div}\big(\hat{P}_h-p(U_h)\big),\kappa\big)\big)\,dt\\
&\qquad+\int_0^{t^*}\big(\big(\alpha P_h,\Pi_h(A\nabla\kappa)\big)-(\operatorname{div}\hat{P}_h,\kappa)\big)\,dt-\int_0^{t^*}\big(\phi\big(y(U_h)\big),\kappa\big)\,dt\\
&\qquad+\int_0^{t^*}\big(\phi(\hat{Y}_h),\kappa\big)\,dt+\big(\big(Y_h-y(U_h)\big)(x,0),\kappa(x,0)\big)\\
&\qquad+\int_0^{t^*}\big(\phi(Y_h)-\phi(\hat{Y}_h),\kappa\big)\,dt\\
&\quad=\int_0^{t^*}\big(Y_{ht}+\operatorname{div}\hat{P}_h+\phi(\hat{Y}_h)-\hat{f}-U_h,\kappa\big)\,dt\\
&\qquad+\int_0^{t^*}\big((\hat{f}-f,\kappa)+(\hat{P}_h-P_h,\nabla\kappa)\big)\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{t^*} \big(\nabla w_h - \alpha P_h, A\nabla\kappa - \Pi_h(A\nabla\kappa)\big)\,dt \\
&+ \int_0^{t^*} \big(\phi(Y_h) - \phi(\hat{Y}_h), \kappa\big)\,dt + \big(\big(Y_h - y(U_h)\big)(x,0), \kappa(x,0)\big).
\end{aligned} \tag{3.23}
$$

When $t^* \in (t_{i-1}, t_i]$, $i \leq 2$,

$$
\begin{aligned}
&\big\|\big(Y_h - y(U_h)\big)(x,t^*)\big\|_{L^2(\Omega)}^2 \\
&\quad\leq C\int_{t_0}^{t_2} \sum_\tau h_\tau^2 \int_\tau (Y_{ht} + \operatorname{div}\hat{P}_h - \hat{f} - U_h)^2\,dx\,dt \\
&\qquad + C\int_{t_0}^{t_2} \sum_\tau \min_{w_h\in W_h} \int_\tau (\alpha P_h - \nabla w_h)^2\,dx\,dt \\
&\qquad + C\|\hat{f} - f\|_{L^1(0,t^*;L^2(\Omega))}^2 + C\|\hat{P}_h - P_h\|_{L^2(0,t^*;L^2(\Omega))}^2 \\
&\qquad + C\|\hat{Y}_h - Y_h\|_{L^2(0,t^*;L^2(\Omega))}^2 + C\big\|y_0^h(x) - y_0(x)\big\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.24}
$$

When $i > 2$,

$$
\begin{aligned}
&\big\|\big(Y_h - y(U_h)\big)(x,t^*)\big\|_{L^2(\Omega)}^2 \\
&\quad\leq C\int_{t_{i-2}}^{t_i} \sum_\tau h_\tau^2 \int_\tau (Y_{ht} + \operatorname{div}\hat{P}_h - \hat{f} - U_h)^2\,dx\,dt \\
&\qquad + C\left|\ln\frac{\Delta t}{t^*}\right| \max_{t\in[0,t_{i-2}]} \left\{\sum_\tau h_\tau^4 \int_\tau (Y_{ht} + \operatorname{div}\hat{P}_h - \hat{f} - U_h)^2\,dx\right\} \\
&\qquad + C\int_{t_{i-2}}^{t_i} \sum_\tau \min_{w_h\in W_h} \int_\tau (\alpha P_h - \nabla w_h)^2\,dx\,dt \\
&\qquad + C\left|\ln\frac{\Delta t}{t^*}\right| \max_{t\in[0,t_{i-2}]} \left\{\sum_\tau h_\tau^2 \cdot \min_{w_h\in W_h} \int_\tau (\alpha P_h - \nabla w_h)^2\,dx\right\} \\
&\qquad + C\|\hat{f} - f\|_{L^1(0,t^*;L^2(\Omega))}^2 + C\|\hat{P}_h - P_h\|_{L^2(0,t^*;L^2(\Omega))}^2 \\
&\qquad + \big(\phi(Y_h) - \phi(\hat{Y}_h), \kappa\big) + C\big\|y_0^h(x) - y_0(x)\big\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.25}
$$

Hence

$$
\big\|Y_h - y(U_h)\big\|_{L^\infty(J;L^2(\Omega))}^2 \leq C\sum_{i=2}^{7} \eta_i^2. \tag{3.26}
$$

Similar to Theorem 3.2 of reference [16], we have derived the following estimate:

$$
\begin{aligned}
&\big\|P_h - \mathbf{p}(U_h)\big\|_{L^2(J;L^2(\Omega))} \\
&\quad\leq C\big(\|\hat{f} - f\|_{L^2(J;L^2(\Omega))} + \big\|(\hat{Y}_h - Y_h)_t\big\|_{L^2(J;L^2(\Omega))} \\
&\qquad + \|\hat{P}_h - P_h\|_{L^2(J;L^2(\Omega))} + \big\|y_0^h(x) - y_0(x)\big\|_{L^2(\Omega)}^2\big).
\end{aligned} \tag{3.27}
$$

This proves (3.19). □

Now, we are in a position to estimate the errors $Z_h - z(U_h)$ and $Q_h - \mathbf{q}(U_h)$.

Theorem 3.3 *Let* $(P_h, Y_h, Q_h, Z_h, U_h)$ *and* $(\mathbf{p}(U_h), y(U_h), \mathbf{q}(U_h), z(U_h), U_h)$ *be the solutions of* (2.38)-(2.44) *and* (2.46)-(2.51), *respectively. Then we have the following error estimate*:

$$\|Z_h - z(U_h)\|^2_{L^\infty(J;L^2(\Omega))} + \|Q_h - \mathbf{q}(U_h)\|_{L^2(J;L^2(\Omega))} \le C \sum_{i=2}^{15} \eta_i^2, \tag{3.28}$$

where η_2-η_7 *are defined in Theorem* 3.2, *and*

$$\eta_8^2 = \max_{i\in[1,N-1]} \left\{ \int_{t_{i-1}}^{t_{i+1}} \sum_\tau h_\tau^2 \int_\tau \left(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\right)^2 dx\,dt \right\};$$

$$\eta_9^2 = \max_{i\in[1,N-1]} \left\{ \int_{t_{i-1}}^{t_{i+1}} \sum_\tau \min_{w_h\in W_h} \int_\tau (\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h)^2 dx\,dt \right\};$$

$$\eta_{10}^2 = |\ln \Delta t| \max_{t\in[0,T]} \left\{ \sum_\tau h_\tau^4 \int_\tau \left(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\right)^2 dx \right\};$$

$$\eta_{11}^2 = |\ln \Delta t| \max_{t\in[0,T]} \left\{ \sum_\tau h_\tau^2 \cdot \min_{w_h\in W_h} \int_\tau (\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h)^2 dx \right\};$$

$$\eta_{12}^2 = \|\widetilde{Q}_h - Q_h\|^2_{L^2(J;L^2(\Omega))} + \|\widetilde{P}_h - P_h\|^2_{L^2(J;L^2(\Omega))};$$

$$\eta_{13}^2 = \|\mathbf{p}_d - \tilde{\mathbf{p}}_d\|^2_{L^2(J;L^2(\Omega))} + \|\bar{\mathbf{p}}_d - \mathbf{p}_d\|^2_{L^2(J;L^2(\Omega))};$$

$$\eta_{14}^2 = \|\widetilde{Y}_h - Y_h\|^2_{L^2(J;L^2(\Omega))} + \|\tilde{y}_d - y_d\|^2_{L^2(J;L^2(\Omega))};$$

$$\eta_{15}^2 = \|\widetilde{Z}_h - Z_h\|^2_{L^2(J;L^2(\Omega))} + \|(\widetilde{Z}_h - Z_h)_t\|^2_{L^2(J;L^2(\Omega))}.$$

Proof We first define $\mathbf{q}_h^N$ as follows:

$$\left(\alpha \mathbf{q}_h^N, \mathbf{v}_h\right) - \left(z_h^N, \operatorname{div} \mathbf{v}_h\right) = -\left(\mathbf{p}_h^N - \mathbf{p}_d^N, \mathbf{v}_h\right), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.29}$$

Then from (2.34) and (3.29) we deduce that

$$(\alpha \hat{Q}_h, \mathbf{v}_h) - (\hat{Z}_h, \operatorname{div} \mathbf{v}_h) = -(\hat{P}_h - \hat{\mathbf{p}}_d, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.30}$$

Now, we let

$$\bar{\mathbf{p}}_d|_{(t_{i-1},t_i]} = \left((t_i - t)\mathbf{p}_d^{i-1} + (t - t_{i-1})\mathbf{p}_d^i\right)/\Delta t.$$

Combining (2.41), (3.30) and the definitions of Z_h, Q_h, P_h and $\bar{\mathbf{p}}_d$, we get

$$(\alpha Q_h, \mathbf{v}_h) - (Z_h, \operatorname{div} \mathbf{v}_h) = -(P_h - \bar{\mathbf{p}}_d, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{3.31}$$

Let ϖ be the solution of (3.17) with $\varpi_0(x) = (Z_h - z(U_h))(x, t^*)$. Then it follows from (2.41)-(2.43), (2.49)-(2.51) and (2.56)-(2.58) that

$$
\begin{aligned}
&\big\|\big(Z_h - z(U_h)\big)\big(x,t^*\big)\big\|^2_{L^2(\Omega)} \\
&\quad= \big(\big(Z_h - z(U_h)\big)\big(x,t^*\big), \varpi\big(x,t^*\big)\big) \\
&\quad= \int_{t^*}^{T} \big(-\big(\big(Z_h - z(U_h)\big)_t, \varpi\big) - \big(Z_h - z(U_h), \operatorname{div}\big(A^*\nabla\varpi\big)\big) \\
&\qquad+ \big(\phi'\big(y(U_h)\big)\big(Z_h - z(U_h)\big), \varpi\big)\big)\,dt \\
&\quad= \int_{t^*}^{T} \big(-\big(\big(Z_h - z(U_h)\big)_t, \varpi\big) + \big(\mathbf{q}(U_h), \nabla\varpi\big) + \big(\phi'(\hat{Y}_h)\widetilde{Z}_h, \varpi\big)\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\big(\mathbf{p}(U_h) - \mathbf{p}_d, \nabla\varpi\big) - \big(Z_h, \operatorname{div}(A\nabla\varpi)\big)\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\big(\phi'\big(y(U_h)\big)(Z_h - \widetilde{Z}_h), \varpi\big) + \big(\big(\phi'\big(y(U_h)\big) - \phi'(\hat{Y}_h)\big)\widetilde{Z}_h, \varpi\big)\big)\,dt \\
&\quad= \int_{t^*}^{T} \big(-\big(\big(Z_h - z(U_h)\big)_t, \varpi\big) + \big(\operatorname{div}\big(\widetilde{Q}_h - \mathbf{q}(U_h)\big), \varpi\big) + \big(\phi'(\hat{Y}_h)\widetilde{Z}_h, \varpi\big)\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\big(\mathbf{p}(U_h) - \mathbf{p}_d, \nabla\varpi\big) - (\operatorname{div}\widetilde{Q}_h, \varpi)\big)\,dt \\
&\qquad- \int_{t^*}^{T} \big(Z_h, \operatorname{div}\big(\Pi_h(A\nabla\varpi)\big)\big)\,dt + \int_{t^*}^{T} \big(\phi'\big(y(U_h)\big)(Z_h - \widetilde{Z}_h), \varpi\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\tilde{\phi}''\big(y(U_h)\big)\big(y(U_h) - \hat{Y}_h\big)\widetilde{Z}_h, \varpi\big)\,dt \\
&\quad= \int_{t^*}^{T} \big(-Z_{ht} + \operatorname{div}\widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d, \varpi\big)\,dt \\
&\qquad- \int_{t^*}^{T} \big(y(U_h) - y_d - \widetilde{Y}_h + \tilde{y}_d, \varpi\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\big(\mathbf{p}(U_h) - \mathbf{p}_d, \nabla\varpi\big) + (\widetilde{Q}_h, \nabla\varpi)\big)\,dt \\
&\qquad- \int_{t^*}^{T} \big(\alpha Q_h + P_h - \bar{\mathbf{p}}_d, \Pi_h(A\nabla\varpi)\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\big(\phi'\big(y(U_h)\big)(Z_h - \widetilde{Z}_h), \varpi\big) + \big(\tilde{\phi}''\big(y(U_h)\big)\big(y(U_h) - \hat{Y}_h\big)\widetilde{Z}_h, \varpi\big)\big)\,dt \\
&\quad= \int_{t^*}^{T} \big(-Z_{ht} + \operatorname{div}\widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d, \varpi\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h, A\nabla\varpi - \Pi_h(A\nabla\varpi)\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\mathbf{p}(U_h) - P_h + \bar{\mathbf{p}}_d - \mathbf{p}_d + \widetilde{Q}_h - Q_h, \nabla\varpi\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\tilde{y}_d - y_d + \widetilde{Y}_h - y(U_h), \varpi\big)\,dt \\
&\qquad+ \int_{t^*}^{T} \big(\phi'\big(y(U_h)\big)(Z_h - \widetilde{Z}_h), \varpi\big)\,dt + \int_{t^*}^{T} \big(\tilde{\phi}''\big(y(U_h)\big)\big(y(U_h) - \hat{Y}_h\big)\widetilde{Z}_h, \varpi\big)\,dt \\
&\quad\equiv E_1 + E_2 + E_3 + E_4 + E_5 + E_6.
\end{aligned}
\tag{3.32}
$$

To prove (3.28), the first step is to estimate E_1. Let $t^* \in (t_{i-1}, t_i]$, when $i \geq N-1$, by Lemmas 2.1, 2.2 and 3.1, we have

$$\begin{aligned}
E_1 &= \int_{t^*}^{T} \big(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d, \varpi - \hat{\pi}_h \varpi\big)\, dt \\
&\leq C \int_{t^*}^{T} \sum_{\tau} \big\| -Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d \big\|_{L^2(\tau)} h_\tau |\varpi|_{H^1(\tau)}\, dt \\
&\leq C(\delta) \int_{t^*}^{T} \sum_{\tau} h_\tau^2 \int_\tau \big(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\big)^2\, dx\, dt \\
&\quad + C\delta \int_{t^*}^{T} \int_\Omega |\nabla \varpi|^2\, dx\, dt \\
&\leq C(\delta) \int_{t_{N-2}}^{t_N} \sum_{\tau} h_\tau^2 \int_\tau \big(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\big)^2\, dx\, dt \\
&\quad + C\delta \big\| \big(Z_h - z(U_h)\big)(x, t^*) \big\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.33}$$

When $i < N-1$,

$$\begin{aligned}
E_1 &= \int_{t^*}^{t_{i+1}} \big(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d, \varpi - \hat{\pi}_h \varpi\big)\, dt \\
&\quad + \int_{t_{i+1}}^{T} \big(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d, \varpi - \pi_h \varpi\big)\, dt \\
&\leq C \int_{t^*}^{t_{i+1}} \sum_{\tau} \big\| -Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d \big\|_{L^2(\tau)} h_\tau |\varpi|_{H^1(\tau)}\, dt \\
&\quad + C \int_{t_{i+1}}^{T} \sum_{\tau} \big\| -Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d \big\|_{L^2(\tau)} h_\tau^2 |\varpi|_{H^2(\tau)}\, dt \\
&\leq C(\delta) \int_{t^*}^{t_{i+1}} \sum_{\tau} h_\tau^2 \int_\tau \big(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\big)^2\, dx\, dt \\
&\quad + C(\delta) \int_{t_{i+1}}^{T} \big|t - t^*\big|^{-1} \sum_{\tau} h_\tau^4 \int_\tau \big(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\big)^2\, dx\, dt \\
&\quad + C\delta \int_{t^*}^{t_{i+1}} \int_\Omega |\nabla \varpi|^2\, dx\, dt + C\delta \int_{t_{i+1}}^{T} \big|t - t^*\big| \int_\Omega \big|D^2 \varpi\big|^2\, dx\, dt \\
&\leq C(\delta) \int_{t_{i-1}}^{t_{i+1}} \sum_{\tau} h_\tau^2 \int_\tau \big(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\big)^2\, dx\, dt \\
&\quad + C(\delta) \left| \ln \frac{\Delta t}{T - t^*} \right| \\
&\quad \times \max_{t \in [t_{i+1}, T]} \left\{ \sum_{\tau} h_\tau^4 \int_\tau \big(-Z_{ht} + \operatorname{div} \widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\big)^2\, dx \right\} \\
&\quad + C\delta \big\| \big(Z_h - z(U_h)\big)(x, t^*) \big\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.34}$$

Now we estimate E_2. Let $t^* \in (t_{i-1}, t_i]$ again. Similarly, when $i \geq N-1$,

$$\begin{aligned} E_2 &= \int_{t_*}^{T} \big(\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h, A\nabla\varpi - \Pi_h(A\nabla\varpi)\big)\,dt \\ &\leq C(\delta)\int_{t_{N-2}}^{t_N}\sum_\tau \min_{w_h\in W_h}\int_\tau (\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h)^2\,dx\,dt \\ &\quad + C\delta\big\|\big(Z_h - z(U_h)\big)(x,t^*)\big\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.35}$$

When $i < N-1$,

$$\begin{aligned} E_2 &\leq C(\delta)\int_{t_{i-1}}^{t_{i+1}}\sum_\tau \min_{w_h\in W_h}\int_\tau (\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h)^2\,dx\,dt \\ &\quad + C(\delta)\left|\ln\frac{\Delta t}{T-t^*}\right| \max_{t\in[t_{i+1},T]}\left\{\sum_\tau h_\tau^2\cdot \min_{w_h\in W_h}\int_\tau (\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h)^2\,dx\right\} \\ &\quad + C\delta\big\|\big(Z_h - z(U_h)\big)(x,t^*)\big\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.36}$$

Next we estimate E_3, E_4. It follows from Lemma 3.1 that

$$\begin{aligned} E_3 &= \int_{t^*}^{T}\big(\mathbf{p}(U_h) - P_h + \bar{\mathbf{p}}_d - \mathbf{p}_d + \widetilde{Q}_h - Q_h, \nabla\varpi\big)\,dt \\ &\leq C(\delta)\big\|P_h - \mathbf{p}(U_h)\big\|^2_{L^2(t^*,T;L^2(\Omega))} + C(\delta)\|\bar{\mathbf{p}}_d - \mathbf{p}_d\|^2_{L^2(t^*,T;L^2(\Omega))} \\ &\quad + C(\delta)\|\widetilde{Q}_h - Q_h\|^2_{L^2(t^*,T;L^2(\Omega))} + C\delta\int_{t^*}^{T}\int_\Omega|\nabla\varpi|^2\,dx\,dt \\ &\leq C(\delta)\big\|P_h - \mathbf{p}(U_h)\big\|^2_{L^2(t^*,T;L^2(\Omega))} + C(\delta)\|\bar{\mathbf{p}}_d - \mathbf{p}_d\|^2_{L^2(t^*,T;L^2(\Omega))} \\ &\quad + C(\delta)\|\widetilde{Q}_h - Q_h\|^2_{L^2(t^*,T;L^2(\Omega))} + C\delta\big\|\big(Z_h - z(U_h)\big)(x,t^*)\big\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} E_4 &= \int_{t^*}^{T}\big(\tilde{y}_d - y_d + \widetilde{Y}_h - y(U_h), \varpi\big)\,dt \\ &\leq C(\delta)\|\tilde{y}_d - y_d\|^2_{L^1(t^*,T;L^2(\Omega))} + C(\delta)\big\|\widetilde{Y}_h - y(U_h)\big\|^2_{L^1(t^*,T;L^2(\Omega))} \\ &\quad + C\delta\max_{t\in[t^*,T]}\big\{\big\|\varpi(x,t)\big\|^2_{L^2(\Omega)}\big\} \\ &\leq C(\delta)\|\tilde{y}_d - y_d\|^2_{L^1(t^*,T;L^2(\Omega))} + C(\delta)\|\widetilde{Y}_h - Y_h\|^2_{L^1(t^*,T;L^2(\Omega))} \\ &\quad + C(\delta)\big\|Y_h - y(U_h)\big\|^2_{L^1(t^*,T;L^2(\Omega))} + C\delta\big\|\big(Z_h - z(U_h)\big)(x,t^*)\big\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.38}$$

Furthermore, we estimate E_5, E_6. It follows from Lemma 3.1 that

$$\begin{aligned} E_5 &= \int_{t^*}^{T}\big(\phi'\big(y(U_h)\big)(Z_h - \widetilde{Z}_h), \varpi\big)\,dt \\ &\leq C(\delta)\|\widetilde{Z}_h - Z_h\|^2_{L^2(t^*,T;L^2(\Omega))} + C\delta\max_{t\in[t^*,T]}\big\{\big\|\varpi(x,t)\big\|^2_{L^2(\Omega)}\big\} \\ &\leq C(\delta)\|\widetilde{Z}_h - Z_h\|^2_{L^2(t^*,T;L^2(\Omega))} + C\delta\big\|\big(Z_h - z(U_h)\big)(x,t^*)\big\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.39}$$

and

$$
\begin{aligned}
E_6 &= \int_{t^*}^{T} \big(\tilde{\phi}''\big(y(U_h)\big)\big(y(U_h) - \hat{Y}_h\big)\widetilde{Z}_h, \varpi\big)\,dt \\
&\le C(\delta)\big\|\hat{Y}_h - y(U_h)\big\|^2_{L^1(t^*,T;L^2(\Omega))} + C\delta \max_{t\in[t^*,T]}\big\{\big\|\varpi(x,t)\big\|^2_{L^2(\Omega)}\big\} \\
&\le C(\delta)\big\|Y_h - y(U_h)\big\|^2_{L^1(t^*,T;L^2(\Omega))} + C(\delta)\|\widetilde{Y}_h - Y_h\|^2_{L^1(t^*,T;L^2(\Omega))} \\
&\quad + C\delta\big\|\big(Z_h - z(U_h)\big)(x,t^*)\big\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.40}
$$

Hence, from (3.33)-(3.40) we have that when $t^* \in (t_{i-1}, t_i]$, $i \ge N-1$,

$$
\begin{aligned}
&\big\|\big(Z_h - z(U_h)\big)(x,t^*)\big\|^2_{L^2(\Omega)} \\
&\quad \le C\int_{t_{N-2}}^{t_N}\sum_\tau h_\tau^2\int_\tau\big(-Z_{ht} + \operatorname{div}\widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\big)^2\,dx\,dt \\
&\qquad + C\int_{t_{N-2}}^{t_N}\sum_\tau \min_{w_h\in W_h}\int_\tau(\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h)^2\,dx\,dt \\
&\qquad + C\big\|Y_h - y(U_h)\big\|^2_{L^1(t^*,T;L^2(\Omega))} + C\big\|P_h - \mathbf{p}(U_h)\big\|^2_{L^2(t^*,T;L^2(\Omega))} \\
&\qquad + C\|\widetilde{Q}_h - Q_h\|^2_{L^2(t^*,T;L^2(\Omega))} + C\|\widetilde{Y}_h - Y_h\|^2_{L^1(t^*,T;L^2(\Omega))} \\
&\qquad + C\|\widetilde{Z}_h - Z_h\|^2_{L^2(t^*,T;L^2(\Omega))} + C\|\tilde{y}_d - y_d\|^2_{L^1(t^*,T;L^2(\Omega))} \\
&\qquad + C\|\bar{\mathbf{p}}_d - \mathbf{p}_d\|^2_{L^2(t^*,T;L^2(\Omega))}.
\end{aligned}
\tag{3.41}
$$

When $i < N-1$,

$$
\begin{aligned}
&\big\|\big(Z_h - z(U_h)\big)(x,t^*)\big\|^2_{L^2(\Omega)} \\
&\quad \le C\int_{t_{i-1}}^{t_{i+1}}\sum_\tau h_\tau^2\int_\tau(-Z_{ht} + \operatorname{div}\widetilde{Q}_h - \widetilde{Y}_h + \tilde{y}_d)^2\,dx\,dt \\
&\qquad + C\left|\ln\frac{\Delta t}{T-t^*}\right|\max_{t\in[t_{i+1},T]}\left\{\sum_\tau h_\tau^4\int_\tau\big(-Z_{ht} + \operatorname{div}\widetilde{Q}_h + \phi'(\hat{Y}_h)\widetilde{Z}_h - \widetilde{Y}_h + \tilde{y}_d\big)^2\,dx\right\} \\
&\qquad + C\int_{t_{i-1}}^{t_{i+1}}\sum_\tau \min_{w_h\in W_h}\int_\tau(\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h)^2\,dx\,dt \\
&\qquad + C\left|\ln\frac{\Delta t}{T-t^*}\right|\max_{t\in[t_{i+1},T]}\left\{\sum_\tau h_\tau^2\cdot\min_{w_h\in W_h}\int_\tau(\alpha Q_h + P_h - \bar{\mathbf{p}}_d - \nabla w_h)^2\,dx\right\} \\
&\qquad + C\big\|Y_h - y(U_h)\big\|^2_{L^1(t^*,T;L^2(\Omega))} + C\big\|P_h - \mathbf{p}(U_h)\big\|^2_{L^2(t^*,T;L^2(\Omega))} \\
&\qquad + C\|\widetilde{Q}_h - Q_h\|^2_{L^2(t^*,T;L^2(\Omega))} + C\|\widetilde{Y}_h - Y_h\|^2_{L^1(t^*,T;L^2(\Omega))} \\
&\qquad + C\|\widetilde{Z}_h - Z_h\|^2_{L^2(t^*,T;L^2(\Omega))} + C\|\tilde{y}_d - y_d\|^2_{L^1(t^*,T;L^2(\Omega))} \\
&\qquad + C\|\bar{\mathbf{p}}_d - \mathbf{p}_d\|^2_{L^2(t^*,T;L^2(\Omega))}.
\end{aligned}
\tag{3.42}
$$

Then it follows from (3.41)-(3.42) that

$$\begin{aligned}\|Z_h - z(U_h)\|^2_{L^\infty(J;L^2(\Omega))} \le{}& C\sum_{i=8}^{15}\eta_i^2 + C\|P_h - \mathbf{p}(U_h)\|^2_{L^2(J;L^2(\Omega))} \\ &+ C\|Y_h - y(U_h)\|^2_{L^2(J;L^2(\Omega))}.\end{aligned} \tag{3.43}$$

Similar to (3.27), we can prove that

$$\begin{aligned}&\|Q_h - \mathbf{q}(U_h)\|_{L^2(J;L^2(\Omega))} \\ &\quad\le C\big(\|Y_h - y(U_h)\|_{L^2(J;L^2(\Omega))} + \|P_h - \mathbf{p}(U_h)\|_{L^2(J;L^2(\Omega))} + \|\tilde{\mathbf{p}}_d - \mathbf{p}_d\|_{L^2(J;L^2(\Omega))} \\ &\qquad + \|\widetilde{Q}_h - Q_h\|_{L^2(J;L^2(\Omega))} + \|\tilde{y}_d - y_d\|_{L^2(J;L^2(\Omega))} \\ &\qquad + \|\widetilde{Y}_h - Y_h\|_{L^2(J;L^2(\Omega))} + \|\widetilde{Z}_h - Z_h\|_{L^2(J;L^2(\Omega))} \\ &\qquad + \|(\widetilde{Z}_h - Z_h)_t\|_{L^2(J;L^2(\Omega))} + \|\widetilde{P}_h - P_h\|_{L^2(J;L^2(\Omega))}\big).\end{aligned} \tag{3.44}$$

The triangle inequality and (3.43) yield (3.28). □

Let $(\mathbf{p}, y, \mathbf{q}, z, u)$ and $(P_h, Y_h, Q_h, Z_h, U_h)$ be the solutions of (2.5)-(2.11) and (2.38)-(2.44), respectively. We decompose the errors as follows:

$$\begin{aligned}&\mathbf{p} - P_h = \mathbf{p} - \mathbf{p}(U_h) + \mathbf{p}(U_h) - P_h := \epsilon_1 + \varepsilon_1, \\ &y - Y_h = y - y(U_h) + y(U_h) - Y_h := r_1 + e_1, \\ &\mathbf{q} - Q_h = \mathbf{q} - \mathbf{q}(U_h) + \mathbf{q}(U_h) - Q_h := \epsilon_2 + \varepsilon_2, \\ &z - Z_h = z - z(U_h) + z(U_h) - Z_h := r_2 + e_2.\end{aligned}$$

From (2.5)-(2.11) and (2.38)-(2.44), we derive the error equations:

$$(\alpha\epsilon_1, \mathbf{v}) - (r_1, \operatorname{div}\mathbf{v}) = 0, \tag{3.45}$$

$$(r_{1t}, w) + (\operatorname{div}\epsilon_1, w) + \big(\phi(y) - \phi(y(U_h)), w\big) = (u - U_h, w), \tag{3.46}$$

$$(\alpha\epsilon_2, \mathbf{v}) - (r_2, \operatorname{div}\mathbf{v}) = -(\epsilon_1, \mathbf{v}), \tag{3.47}$$

$$(r_{2t}, w) + (\operatorname{div}\epsilon_2, w) + \big(\phi'(y)z - \phi'(y(U_h))z(U_h), w\big) = (r_1, w) \tag{3.48}$$

for any $\mathbf{v} \in \mathbf{V}$, $w \in W$.

Theorem 3.4 *Let* $(\mathbf{p}, y, \mathbf{q}, z, u)$ *and* $(\mathbf{p}(U_h), y(U_h), \mathbf{q}(U_h), z(U_h), U_h)$ *be the solutions of* (2.5)-(2.11) *and* (2.46)-(2.51), *respectively. There is a constant* $C > 0$, *independent of* h, *such that*

$$\|\epsilon_1\|_{L^2(J;L^2(\Omega))} + \|r_1\|_{L^\infty(J;L^2(\Omega))} \le C\|u - U_h\|_{L^2(J;L^2(\Omega))}, \tag{3.49}$$

$$\|\epsilon_2\|_{L^2(J;L^2(\Omega))} + \|r_2\|_{L^\infty(J;L^2(\Omega))} \le C\|u - U_h\|_{L^2(J;L^2(\Omega))}. \tag{3.50}$$

Proof Part I. Choosing $\mathbf{v} = \epsilon_1$ and $w = r_1$ as the test functions and adding the two relations of (3.45)-(3.46), we have

$$\begin{aligned}(\alpha\epsilon_1, \epsilon_1) + (r_{1t}, r_1) &= (u - U_h, r_1) - \big(\phi(y) - \phi\big(y(U_h)\big), r_1\big) \\ &= (u - U_h, r_1) - \big(\tilde{\phi}'(y)\big(y - y(U_h)\big), r_1\big). \end{aligned} \tag{3.51}$$

Then, using the ϵ-Cauchy inequality, we find an estimate as follows:

$$(\alpha\epsilon_1, \epsilon_1) + (r_{1t}, r_1) \leq C\big(\|r_1\|^2_{L^2(\Omega)} + \|u - U_h\|^2_{L^2(\Omega)}\big). \tag{3.52}$$

Note that

$$(r_{1t}, r_1) = \frac{1}{2}\frac{\partial}{\partial t}\|r_1\|^2_{L^2(\Omega)},$$

then, using the assumption on A, we obtain that

$$\|\epsilon_1\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|r_1\|^2_{L^2(\Omega)} \leq C\big(\|r_1\|^2_{L^2(\Omega)} + \|u - U_h\|^2_{L^2(\Omega)}\big). \tag{3.53}$$

Integrating (3.53) in time and since $r_1(0) = 0$, applying Gronwall's lemma, we easily obtain the following error estimate:

$$\|\epsilon_1\|^2_{L^2(J;L^2(\Omega))} + \|r_1\|^2_{L^\infty(J;L^2(\Omega))} \leq C\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \tag{3.54}$$

This implies (3.49).

Part II. Similarly, choosing $v = \epsilon_2$ and $w = r_2$ as the test functions and adding the two relations of (3.47)-(3.48), we obtain that

$$\begin{aligned}(\alpha\epsilon_2, \epsilon_2) - (r_{2t}, r_2) &= \big(g_2'(y) - g_2'\big(y(U_h)\big), r_2\big) - \big(g_1'(\mathbf{p}) - g_1'\big(\mathbf{p}(U_h)\big), \epsilon_2\big) \\ &\quad - \big(\phi'(y)z - \phi'\big(y(U_h)\big)z(U_h), r_2\big). \end{aligned} \tag{3.55}$$

Then, using the ϵ-Cauchy inequality, we find an estimate as follows:

$$(\alpha\epsilon_2, \epsilon_2) + (r_{2t}, r_2) \leq C\big(\|r_1\|^2_{L^2(\Omega)} + \|r_2\|^2_{L^2(\Omega)} + \|\epsilon_1\|^2_{L^2(\Omega)}\big) + \frac{c}{2}\|\epsilon_2\|^2_{L^2(\Omega)}. \tag{3.56}$$

Note that

$$(r_{2t}, r_2) = \frac{1}{2}\frac{\partial}{\partial t}\|r_2\|^2_{L^2(\Omega)},$$

then, using the assumption on A, we verify that

$$\|\epsilon_2\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|r_2\|^2_{L^2(\Omega)} \leq C\big(\|r_1\|^2_{L^2(\Omega)} + \|r_2\|^2_{L^2(\Omega)} + \|\epsilon_1\|^2_{L^2(\Omega)}\big). \tag{3.57}$$

Integrating (3.57) in time and since $r_2(T) = 0$, applying Gronwall's lemma, we easily obtain the following error estimate:

$$\|\epsilon_2\|^2_{L^2(J;L^2(\Omega))} + \|r_2\|^2_{L^\infty(J;L^2(\Omega))} \leq C\|u - U_h\|^2_{L^2(J;L^2(\Omega))}. \tag{3.58}$$

Then (3.50) follows from (3.58) and the previous statements immediately. □

Collecting Theorems 3.1-3.4, we can derive the following result.

Theorem 3.5 *Let* $(\mathbf{p}, y, \mathbf{q}, z, u)$ *and* $(P_h, Y_h, Q_h, Z_h, U_h)$ *be the solutions of* (2.5)-(2.11) *and* (2.38)-(2.44), *respectively. Assume that* $(U_h + \widetilde{Z}_h)|_\tau \in H^s(\tau)$, $\forall \tau \in \mathcal{T}_h$ $(s = 0, 1)$, *and that there is* $v_h \in K_h$ *such that*

$$\left|(U_h + \widetilde{Z}_h, v_h - u)\right| \le C \sum_{\tau \in \mathcal{T}_h} h_\tau \|U_h + \widetilde{Z}_h\|_{H^s(\tau)} \|u - U_h\|^s_{L^2(\tau)}. \tag{3.59}$$

Then we have that

$$\begin{aligned} &\|u - U_h\|^2_{L^2(J;L^2(\Omega))} + \|y - Y_h\|^2_{L^\infty(J;L^2(\Omega))} + \|\mathbf{p} - P_h\|^2_{L^2(J;L^2(\Omega))} \\ &\quad + \|z - Z_h\|^2_{L^\infty(J;L^2(\Omega))} + \|\mathbf{q} - Q_h\|^2_{L^2(J;L^2(\Omega))} \le C \sum_{i=1}^{15} \eta_i^2, \end{aligned} \tag{3.60}$$

where η_1 *is defined in Theorem* 3.1, $\eta_2, \ldots, \eta_7$ *are defined in Theorem* 3.2, *and* $\eta_8, \ldots, \eta_{15}$ *are defined in Theorem* 3.3.

4 Conclusion and future work

In this paper, we derive new $L^\infty(L^2)$ and $L^2(L^2)$-*posteriori* error estimates of the mixed finite element solutions for quadratic optimal control problems governed by semilinear parabolic equations. The *a posteriori* error estimates for the semilinear parabolic optimal control problems by mixed finite element methods seem to be new.

In our future work, we shall use the mixed finite element method to deal with nonlinear parabolic integro-differential optimal control problems. Furthermore, we shall consider *a posteriori* error estimates and superconvergence of a mixed finite element solution for nonlinear parabolic integro-differential optimal control problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing, 404000, P.R. China. [2]Laboratory for Applied Mathematics, Beijing Computational Science Research Center, Beijing, 100084, P.R. China. [3]School of Science, Chongqing Jiaotong University, Chongqing, 400074, P.R. China. [4]Department of Mathematics and Computational Science, Hunan University of Science and Engineering, Yongzhou, 425100, P.R. China.

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