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Energy decay for a viscoelastic Kirchhoff plate equation with a delay term

Baowei Feng[1] and Haiyan Li[2*]

*Correspondence:
lihaiyanmath@163.com
[2]College of Mathematics and Information Science, Beifang University of Nationalities, Yinchuan, 750021, P.R. China
Full list of author information is available at the end of the article

Abstract

A nonlinear viscoelastic Kirchhoff plate equation with a time delay term in the internal feedback is considered. Under suitable assumptions, we establish the general rates of energy decay of the initial and boundary value problem by using the energy perturbation method.

MSC: 35L75; 35B40; 35B35

Keywords: general decay; Kirchhoff plate; delay feedbacks

1 Introduction

In this paper, we are concerned with the following nonlinear viscoelastic Kirchhoff plate equation with a time delay term in the internal feedback:

$$\begin{aligned} & u_{tt}(x,t) + \Delta^2 u(x,t) - \operatorname{div} F\bigl(\nabla u(x,t)\bigr) - \sigma(t)\int_0^t g(t-s)\Delta^2 u(s)\,ds + \mu_1|u_t|^{m-1}u_t(x,t) \\ & \quad + \mu_2\bigl|u_t(x,t-\tau)\bigr|^{m-1}u_t(x,t-\tau) = 0, \end{aligned} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The function $u = u(x,t)$ is the transverse displacement of a plate filament, and $\sigma(t)$ and $g(t)$ are positive functions defined on $\mathbb{R}^+$. μ_1, μ_2 are positive constants and $\tau > 0$ represents the time delay.

To equation (1.1), we add the following initial conditions:

$$\begin{cases} u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega, t \in (0,\tau), \end{cases} \tag{1.2}$$

and the support boundary conditions

$$u = \Delta u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+. \tag{1.3}$$

In 1950, Woinowsky-Krieger [1] introduced the one-dimensional nonlinear equation of vibration of beams

$$u_{tt} + \alpha u_{xxxx} - \left(\beta + \gamma \int_0^L |u_x|^2\,dx\right)u_{xx} = 0,$$

where L is the length of the beam and α, β, γ are positive physical constants. Since then many mathematicians studied the related model in one dimension and higher dimensions. The main results are mainly concerned with global existence, stability, and long-time dynamics, and many results may be found in the literature. It has been stabilized by means of different controls, for example, internal damping, boundary controls, dynamic boundary conditions, distributed damping and heat damping, and so on. See, for example, Brito [2], Cavalcanti *et al.* [3–5], Jorge Silva and Ma [6], Ma [7], Ma and Narciso [8], Oliveira and Lima [9], Park [10], Patcheu [11], Munõz Rivera [12, 13], Yang [14, 15], and the references therein. We would like here to mention the work of Andrade *et al.* [16]. In this paper the authors studied a viscoelastic plate equation with p-Laplacian and memory terms with strong damping,

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)\,ds - \Delta u_t + f(u) = 0, \tag{1.4}$$

and proved the existence of weak solutions by using Faedo-Galerkin approximations to the IBVP of (1.4). In addition, they obtained the uniqueness of strong solutions and the exponential stability of solutions to (1.4) under some suitable conditions on the memory kernel g and a forcing term f. For $\sigma(t) > 0$, Messaoudi [17] considered the following viscoelastic wave equation:

$$u_{tt} - \Delta u + \sigma(t)\int_0^t g(t-\tau)\Delta u(\tau)\,d\tau = 0.$$

Under some assumptions on the relaxation function g and the potential σ, the author established a general decay property which depends on the behavior of σ and g. Jorge Silva *et al.* [18] studied the following viscoelastic Kirchhoff plate equation:

$$u_{tt} - \sigma(t)\Delta u_{tt} + \Delta^2 u - \operatorname{div} F(\nabla u) - \int_0^t g(t-s)\Delta^2 u(s)\,ds = 0,$$

and they mainly proved the global well-posedness of the solution for $\sigma(t) = 1$ and similarly the result holds for $\sigma(t) = 0$. Moreover, the authors established the general rates of energy decay of the system for $\sigma \in [0,\infty)$. For more results on viscoelastic equations, we can refer to Berrimi and Messaoudi [19], Messaoudi [20], Messaoudi and Tartar [21, 22], Tatar [23], and the references therein.

In recent years, there has been published much work concerning the wave equation with time delay effects and the delay effects often appear in many practical problems. In Nicaise and Pignotti [24], the authors studied a wave equation with time delay,

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0,$$

and established stability results under the assumption $0 < \mu_2 < \mu_1$. In [25], Kirane and Said-Houari studied a viscoelastic wave equation with a delay term in internal feedbacks, and they proved the global well-posedness of the IBVP to the equation by using some suitable assumptions on the relaxation function and some restriction on the parameters μ_1 and μ_2. Furthermore, under the assumption $\mu_2 \leq \mu_1$, they obtained a general decay result of the total energy to the system. Dai and Yang [26] improved the results in [25] under weaker conditions. For the plate equation with time delay term, Park [27] considered

$$u_{tt} + \Delta^2 u - M\bigl(\|\nabla u\|^2\bigr)\Delta u + \sigma(t)\int_0^t g(t-s)\Delta u(s)\,ds + a_0 u_t + a_1 u_t(t-\tau) = 0,$$

which can be regarded as an extensive weak viscoelastic plate equation with a linear time delay term. The author obtained a general decay result of energy by using suitable energy and Lyapunov functionals. In [28], one of the present authors investigated an extensible plate equation with a weak viscoelastic term and a time delay term in the internal feedback,

$$u_{tt} + \Delta^2 u - M\bigl(\|\nabla u\|^2\bigr)\Delta u - \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0,$$

and established the global well-posedness of the initial and boundary value problem by using the Faedo-Galerkin approximations and some energy estimates. Moreover, the author proved a general rate result of energy decay when the weight of the delay is less than the weight of the damping. Recently, Yang [29] studied a viscoelastic plate equation with a linear time delay term

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0.$$

The author obtained the global well-posedness of the IBVP to the equation and established the decay property of energy for either $0 < |\mu_2| < \mu_1$ or $\mu_1 = 0$, $0 < |\mu_2| < a$, and $\zeta_2 > \zeta_0$, but one needs more assumptions on the kernel g. For more some results concerning the different boundary conditions under an appropriate assumption between μ_1 and μ_2, one can refer to Datko *et al.* [30], Kafini *et al.* [31], Nicaise and Pignotti [32], Nicaise *et al.* [33], Nicaise and Valein [34], and the references therein.

Equation (1.1) is a Kirchhoff plate equation with a memory term and a nonlinear time delay term in the internal feedback. To the best of our knowledge, the general rate of energy decay for system (1.1)-(1.3) were not previously considered. So the main objective of the present work is to establish the stability of initial boundary value problem (1.1)-(1.3).

The outline of this paper is as follows. In Section 2, we give some preparations for our consideration and our main results. In Section 3, we establish the general decay result of the energy by using energy perturbation method.

2 Assumptions and main results

We first introduce the following Hilbert spaces:

$$V_0 = L^2(\Omega), \qquad V_1 = H_0^1(\Omega), \qquad V_2 = H^2(\Omega) \cap H_0^1(\Omega),$$

with norms

$$\|u\|_{V_0} = \|u\|, \qquad \|u\|_{V_1} = \|\nabla u\|_2 \quad \text{and} \quad \|u\|_{V_2} = \|\Delta u\|,$$

respectively. The notation $\|\cdot\|_p$ denotes the L^p-norm, and $(\cdot,\cdot)$ is the L^2-inner product. In particular, we write $\|\cdot\|$ instead of $\|\cdot\|_2$ when $p = 2$. The constants $\lambda_0, \lambda_1, \lambda_2, \lambda > 0$

represent the embedding constants

$$\lambda_0\|u\|^2 \le \|\nabla u\|^2, \qquad \lambda_1\|u\|^2 \le \|\Delta u\|^2, \qquad \lambda_2\|\nabla u\|^2 \le \|\Delta u\|^2, \qquad \lambda = \frac{1}{\lambda_1} + \frac{1}{\lambda_2},$$

for $u \in V_2$.

For the relaxation function g and the potential σ, we assume

(A_1) $g, \sigma : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing differentiable functions satisfying

$$\begin{aligned} &g(0) > 0, \qquad l_0 = \int_0^\infty g(s)\,ds > \infty, \qquad \sigma(t) > 0, \\ &1 - 2\sigma(t)\int_0^t g(s)\,ds \ge l > 0, \quad \text{for } t \ge 0, \end{aligned} \tag{2.1}$$

with $l = 1 - l_0$, and there exists a nonincreasing differentiable function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\zeta(t) > 0, \quad g'(t) \le -\zeta(t)g(t) \quad \text{for } t \ge 0, \lim_{t\to\infty} \frac{-\sigma'(t)}{\zeta(t)\sigma(t)} = 0. \tag{2.2}$$

(A_2) The constant m satisfies

$$m \ge 1 \quad \text{if } n = 1, 2, \qquad 1 \le m \le \frac{n+2}{n-2} \quad \text{if } n \ge 3. \tag{2.3}$$

(A_3) The function $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-vector field given by $F = (F_1, \ldots, F_n)$ satisfying for every $j = 1, 2, \ldots, n$,

$$\left|\nabla F_j(u)\right| \le k_j\left(1 + |u|^{\frac{p_j-1}{2}}\right), \quad \forall u \in \mathbb{R}^n, \tag{2.4}$$

where k_j are positive constants and the constants p_j satisfy

$$p_j \ge 1 \quad \text{if } n = 1, 2, \qquad 1 \le p_j \le \frac{n+2}{n-2} \quad \text{if } n \ge 3. \tag{2.5}$$

Moreover, the function F is a conservative vector field with $F = \nabla f$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a real valued function satisfying

$$0 \le f(u) \le F(u)u + \alpha l|u|^2, \quad \forall u \in \mathbb{R}^n, \tag{2.6}$$

where $\alpha \in [0, \mu)$ with $\mu = \lambda_2 \frac{1-2\sigma(t)\int_0^t g(s)\,ds}{2l}$.

The vector field F satisfying a condition like (2.4) possesses an interesting property. One can find the detailed proof in [18].

Remark 2.1 Let $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-vector field given by $F = (F_1, \ldots, F_n)$. If there exist positive constants $k_1, \ldots, k_n$ and $q_1, \ldots, q_n$ such that, for every $j = 1, \ldots, n$,

$$\left|\nabla F_j(u)\right| \le k_j\left(1 + |u|^{q_j}\right), \quad \forall u \in \mathbb{R}^n.$$

Then there exists a positive constant $K = K(k_j, q_j, n), j = 1, \ldots, n$, such that, for all $x, y \in \mathbb{R}^n$,

$$\left|F(x) - F(y)\right| \leq K \sum_{j=1}^{n} \left(1 + |x|^{q_j} + |y|^{q_j}\right)|x - y|.$$

In particular, we have

$$\left|F(x)\right| \leq \left|F(0)\right| + K \sum_{j=1}^{n} \left(1 + |x|^{q_j}\right)|x|, \quad \forall x \in \mathbb{R}^n. \tag{2.7}$$

Now we give some estimates related to the convolution operator. By direct calculations, we shall see below that

$$\begin{aligned} &\sigma(t)(g * u, u_t) \\ &\quad = -\frac{\sigma(t)}{2} g(t)\left\|u(t)\right\|^2 - \frac{d}{dt}\left[\frac{\sigma(t)}{2}(g \circ u)(t) - \frac{\sigma(t)}{2}\left(\int_0^t g(s)\,ds\right)\left\|u(t)\right\|^2\right] \\ &\qquad + \frac{\sigma(t)}{2}\left(g' \circ u\right)(t) + \frac{\sigma'(t)}{2}(g \circ u)(t) - \frac{\sigma'(t)}{2}\int_0^t g(s)\,ds \left\|u(t)\right\|^2, \end{aligned} \tag{2.8}$$

$$(g * u, u) \leq 2\left(\int_0^t g(s)\,ds\right)\left\|u(t)\right\|^2 + \frac{1}{4}(g \circ u)(t), \tag{2.9}$$

where

$$(g * u)(t) = \int_0^t g(t - s)u(s)\,ds, \qquad (g \circ u)(t) = \int_0^t g(t - s)\left\|u(t) - u(s)\right\|^2\,ds.$$

Motivated by [32, 34], we introduce the following new dependent variable to deal with the delay feedback term:

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \quad x \in \Omega, \rho \in (0, 1), t > 0, \tag{2.10}$$

which gives us

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad \text{in } \Omega \times (0, 1) \times (0, \infty). \tag{2.11}$$

Thus, problem (1.1)-(1.3) is equivalent to

$$\begin{cases} u_{tt} + \Delta^2 u - \operatorname{div} F(\nabla u) - \sigma(t)\int_0^t g(t - s)\Delta^2 u(s)\,ds \\ \quad + \mu_1 |u_t|^{m-1} u_t + \mu_2 |z(x, 1, t)|^{m-1} z(x, 1, t) = 0, \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \end{cases} \tag{2.12}$$

where $x \in \Omega$, $\rho \in (0, 1)$ and $t > 0$, and the initial and boundary conditions are

$$\begin{cases} u(x, 0) = u_0, \qquad u_t(x, 0) = u_1, \quad x \in \Omega \\ z(x, \rho, 0) = f_0(x, -\rho\tau), \quad (x, t) \in \Omega \times (0, \tau), \\ u = \Delta u = 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(x, 0, t) = u_t(x, t), \quad x \in \Omega, t > 0. \end{cases} \tag{2.13}$$

Let ξ be a positive constant satisfying

$$\tau \frac{\mu_2 m}{m+1} < \xi < \tau \frac{(m+1)\mu_1 - \mu_2}{m+1}. \tag{2.14}$$

Now we define the weak solutions of (1.1)-(1.3): for given initial data $(u_0, u_1) \in V_2 \times V_0$, we say that a function $U = (u, u_t) \in C(\mathbb{R}^+, V_2 \times V_0)$ is a weak solution to the problem (1.1)-(1.3) if $U(0) = (u_0, u_1)$ and

$$\begin{aligned}&(u_{tt}, \omega) + (\Delta u, \Delta \omega) + \big(F(\nabla u), \nabla \omega\big) - \sigma(t) \int_0^t g(t-s)\big(\Delta u(s), \Delta \omega\big)\, ds \\ &\quad + \mu_1\big(|u_t|^{m-1} u_t, \omega\big) + \mu_2\big(\big|u_t(t-\tau)\big|^{m-1} u_t(t-\tau), \omega\big) = 0,\end{aligned}$$

for all $\omega \in V_2$.

The following theorem is concerned with the global well-posedness of problem (2.12)-(2.13). By using the classical Faedo-Galerkin method, see, *e.g.*, [16, 18, 28, 35], we can prove the theorem, and we omit the proof here.

Theorem 2.1 *Let $\mu_2 \leq m\mu_1$, and assume the assumptions* (2.1)-(2.6) *hold. If the initial data $(u_0, u_1) \in (V_2 \times V_0)$, $f_0 \in L^2(\Omega \times (0,1))$, then problem* (2.12)-(2.13) *has a unique weak solution $(u, u_t) \in C(0, T; V_2 \times V_0)$ such that, for any $T > 0$,*

$$u \in L^\infty(0, T; V_2), \qquad u_t \in L^\infty(0, T; V_0).$$

We introduce the modified energy functional to problem (2.12)-(2.13) by

$$\begin{aligned}E(t) &= \frac{1}{2}\big\|u_t(t)\big\|^2 + \frac{1}{2}\bigg[1 - \sigma(t)\bigg(\int_0^t g(s)\, ds\bigg)\bigg]\big\|\Delta u(t)\big\|^2 + \frac{1}{2}\sigma(t)(g \circ \Delta u) \\ &\quad + \frac{\xi}{2}\int_\Omega \int_0^1 |z|^{m+1}(x, \rho, t)\, d\rho\, dx + \int_\Omega f\big(\nabla u(t)\big)\, dx.\end{aligned} \tag{2.15}$$

Our main result is the general decay rate of the energy, which is given by the following theorem.

Theorem 2.2 *Let $\mu_2 < m\mu_1$, and assume the assumptions* (2.1)-(2.6) *hold. Let (u, u_t) be the weak solutions of problem* (2.12)-(2.13) *with the initial data $(u_0, u_1) \in (V_2 \times V_0)$, $f_0 \in L^2(\Omega \times (0,1))$. Then there exist two constants $\beta > 0$ and $\gamma > 0$ such that the energy $E(t)$ defined by* (2.15) *satisfies*

$$E(t) \leq \beta \exp\bigg(-\gamma \int_0^t \zeta(s)\sigma(s)\, ds\bigg), \quad \textit{for all } t \geq 0. \tag{2.16}$$

Remark 2.2 Generally speaking, the energy of problem (2.12)-(2.13) is usually defined by

$$F(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{\xi}{2}\int_\Omega \int_0^1 |z|^{m+1}\, d\rho\, dx + \int_\Omega f(\nabla u)\, dx.$$

From Theorem 2.2, we can also get the decay

$$F(t) \leq \beta' \exp\bigg(-\gamma \int_0^t \zeta(s)\sigma(s)\, ds\bigg). \tag{2.17}$$

Indeed, by (2.15) and (2.1), we have

$$\begin{aligned} E(t) &= F(t) - \frac{1}{2}\sigma(t)\int_0^t g(s)\,ds\|\Delta u\|^2 + \frac{1}{2}\sigma(t)(g \circ \Delta u) \\ &\geq \frac{3+l}{4}F(t), \end{aligned}$$

which, together with (2.16), implies (2.17) with $\beta' = \frac{4\beta}{l+3}$.

3 General decay rate

In this section, we shall establish the general decay property of the solution for problem (2.12)-(2.13) in the case $\mu_2 < m\mu_1$. For this purpose we define

$$\mathcal{L}(t) := E(t) + \varepsilon_1\sigma(t)\Phi(t) + \varepsilon_2\sigma(t)\Psi(t), \tag{3.1}$$

where ε_1 and ε_2 are positive constants and

$$\Phi(t) = \int_\Omega u_t u\,dx, \tag{3.2}$$

$$\Psi(t) = -\int_\Omega u_t \int_0^t g(t-s)\big(u(t) - u(s)\big)\,ds\,dx. \tag{3.3}$$

To prove Theorem 2.2, we need the following technical lemmas.

Lemma 3.1 *Under the assumptions in Theorem* 2.2, *the modified energy functional defined by* (2.15) *satisfies there exist two positive constants* c_1 *and* c_2 *such that, for any* $t \geq 0$,

$$\begin{aligned} E'(t) \leq{}& -c_1\|u_t\|_{m+1}^{m+1} - c_2\big\|z(x,1,t)\big\|_{m+1}^{m+1} + \frac{\sigma(t)}{2}\big(g' \circ \Delta u\big) \\ &- \frac{\sigma'(t)}{2}\int_0^t g(s)\,ds\|\Delta u\|^2. \end{aligned} \tag{3.4}$$

Proof First the direct calculation yields

$$\int_\Omega F(\nabla u)\cdot\nabla u\,dx = \int_\Omega \nabla f(\nabla u)\cdot\nabla u\,dx = \frac{d}{dt}\int_\Omega f(\nabla u)\,dx. \tag{3.5}$$

Multiplying the first equation in (2.12) by u_t, integrating the result over Ω, and using integration by parts, (2.8) and (3.5), we can obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{\sigma(t)}{2}(g\circ\Delta) - \frac{\sigma(t)}{2}\left(\int_0^t g(s)\,ds\right)\|\Delta u\|^2 + \int_\Omega f(\nabla u)\,dx\right] \\ &\quad + \mu_1\|u_t\|_{m+1}^{m+1} + \mu_2\int_\Omega \big|z(x,1,t)\big|^{m-1}z(x,1,t)u_t\,dx + \frac{\sigma(t)}{2}g(t)\|\Delta u\|^2 \\ &\quad - \frac{\sigma(t)}{2}\big(g'\circ\Delta u\big) - \frac{\sigma'(t)}{2}(g\circ\Delta u) + \frac{\sigma'(t)}{2}\int_0^t g(s)\,ds\|\Delta u\|^2 = 0. \end{aligned} \tag{3.6}$$

Multiplying the second equation in (2.12) by ξz and integrating the result over $\Omega \times (0,1)$, we have

$$
\begin{aligned}
&\xi \frac{d}{dt}\int_\Omega \int_0^1 \left|z(x,\rho,t)\right|^{m-1} z(x,\rho,t)\,d\rho\,dx \\
&\quad = -\frac{\xi}{\tau(m+1)}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\left|z(x,\rho,t)\right|^{m+1} d\rho\,dx \\
&\quad = -\frac{\xi}{\tau(m+1)}\int_\Omega \left(\left|z(x,1,t)\right|^{m+1} - \left|z(x,0,t)\right|^{m+1}\right) dx \\
&\quad = -\frac{\xi}{\tau}\int_\Omega \left|z(x,1,t)\right|^{m+1} dx + \frac{\xi}{\tau}\int_\Omega |u_t|^{m+1}\,dx.
\end{aligned}
\tag{3.7}
$$

By using Young's inequality, we get

$$
\begin{aligned}
&\mu_2 \left|\int_\Omega \left|z(x,1,t)\right|^{m-1} z(x,1,t) u_t\,dx\right| \\
&\quad \le \frac{\mu_2 m}{m+1}\int_\Omega \left|z(x,1,t)\right|^{m+1} dx + \frac{\mu_2}{m+1}\int_\Omega |u_t|^{m+1}\,dx,
\end{aligned}
$$

which, together with (3.6)-(3.7), gives us

$$
\begin{aligned}
E'(t) \le{}& -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{m+1}\right)\|u_t\|_{m+1}^{m+1} - \left(\frac{\xi}{2\tau} - \frac{\mu_2 m}{m+1}\right)\left\|z(x,1,t)\right\|_{m+1}^{m+1} \\
& + \frac{\sigma(t)}{2}\left(g' \circ \Delta u\right) - \frac{\sigma'(t)}{2}\int_0^t g(s)\,ds \|\Delta u\|^2.
\end{aligned}
$$

By using condition (2.14), we get

$$
c_1 := \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{m+1} > 0 \quad \text{and} \quad c_2 := \frac{\xi}{\tau} - \frac{\mu_2 m}{m+1} > 0,
$$

which implies the desired inequality (3.4). The proof is now complete. □

Lemma 3.2 *Under the assumptions in Theorem* 2.2, *for the functional* $\Phi(t)$ *defined in* (3.2) *there exists a positive constant* c_3 *such that, for any* $t \ge 0$,

$$
\Phi'(t) \le \|u_t\|^2 - c_3\|\Delta u\|^2 + C_\varepsilon \|u_t\|_{m+1}^{m+1} + C_\varepsilon \left\|z(x,1,t)\right\|_{m+1}^{m+1} + \frac{\sigma(t)}{4}(g \circ \Delta u),
\tag{3.8}
$$

where $C_\varepsilon > 0$ *is a constant depending for any* $\varepsilon > 0$.

Proof By using the first equation of (2.12), we obtain

$$
\begin{aligned}
\frac{d}{dt}\Phi(t) ={}& \int_\Omega u_{tt} u\,dx + \|u_t\|^2 \\
={}& \|u_t\|^2 + \int_\Omega \biggl(-\Delta^2 u + \operatorname{div} F(\nabla u) + \sigma(t)\int_0^t g(t-s)\Delta^2 u(s)\,ds \\
& - \mu_1 |u_t|^{m_1} u_t - \mu_2 \left|z(x,1,t)\right|^{m-1} z(x,1,t)\biggr) \cdot u\,dx
\end{aligned}
$$

$$= \|u_t\|^2 - \|\Delta u\|^2 - \int_\Omega F(\nabla u)\nabla u\,dx - \mu_2 \int_\Omega \big|z(x,1,t)\big|^{m-1} z(x,1,t)u\,dx$$
$$+ \sigma(t)\int_\Omega \int_0^t g(t-s)\Delta u(s)\,ds \cdot \Delta u\,dx - \mu_1 \int_\Omega |u_t|^{m-1}u_t u\,dx. \tag{3.9}$$

Using Young's inequality, the embedding theorem, and (2.15), we know that, for any $\varepsilon > 0$,

$$\begin{aligned}\left|\mu_1 \int_\Omega |u_t|^{m-1}u_t u\,dx\right| &\le \int_\Omega |u_t|^m |u|\,dx \\ &\le \varepsilon \|u\|_{m+1}^{m+1} + C_\varepsilon \|u_t\|_{m+1}^{m+1} \\ &\le C\varepsilon \|\nabla u\|^{m+1} + C_\varepsilon \|u_t\|_{m+1}^{m+1} \\ &\le \frac{C\varepsilon}{\lambda_2}\left(\frac{2E(0)}{l}\right)^{m-1} \|\Delta u\|^2 + C_\varepsilon \|u_t\|_{m+1}^{m+1},\end{aligned} \tag{3.10}$$

where the constant $C > 0$ is the embedding constant.

Similarly we get

$$\left|\mu_2 \int_\Omega \big|z(x,1,t)\big|^{m-1} z(x,1,t)u\,dx\right| \le \frac{C\varepsilon}{\lambda_2}\left(\frac{2E(0)}{l}\right)^{m-1} \|\Delta u\|^2 + C_\varepsilon \big\|z(x,1,t)\big\|_{m+1}^{m+1}. \tag{3.11}$$

We infer from (2.6) that

$$-\int_\Omega F(\nabla u)\nabla u\,dx \le \alpha l \int_\Omega |\nabla u|^2\,dx \le \frac{\alpha l}{\lambda_2}\|\Delta u\|^2. \tag{3.12}$$

Combining (2.11) and (3.10)-(3.12) with (3.9), we can get

$$\Phi'(t) \le \|u_t\|^2 - \left[\left(1 - 2\sigma(t)\int_0^t g(s)\,ds\right) - \frac{2C\varepsilon}{\lambda_2}\left(\frac{2E(0)}{l}\right)^{m-1} - \frac{\alpha l}{\lambda_2}\right]\|\Delta u\|^2$$
$$+ C_\varepsilon \|u_t\|_{m+1}^{m+1} + C_\varepsilon \big\|z(x,1,t)\big\|_{m+1}^{m+1} + \frac{\sigma(t)}{4}(g \circ \Delta u). \tag{3.13}$$

Due to (2.1) and (2.6) and choosing $\varepsilon > 0$ small enough, we know that

$$c_3 := \left(1 - 2\sigma(t)\int_0^t g(s)\,ds\right) - \frac{2C\varepsilon}{\lambda_2}\left(\frac{2E(0)}{l}\right)^{m-1} - \frac{\alpha l}{\lambda_2} > 0,$$

which, together with (3.13), give us (3.8). The proof is hence complete. □

Lemma 3.3 *Under the assumptions in Theorem* 2.2, *and for any* $\delta > 0$, *there exists a positive constant* C_δ *such that the functional* $\Psi(t)$ *defined in* (2.3) *satisfies*

$$\Psi'(t) \le -\left(\int_0^t g(s)\,ds - \delta\right)\|u_t\|^2 + \big[2\delta + 2\delta(1-l)^2\sigma(t)\big]\|\Delta u\|^2 + \delta\mu_2\|u_t\|_{m+1}^{m+1}$$
$$+ \delta\mu_2 \big\|z(x,1,t)\big\|_{m+1}^{m+1} + C_\delta\big[1 + (1-l)\sigma(t) + \big(E(0)\big)^{\frac{p-1}{2}}\big](g \circ \Delta u)$$
$$- \frac{Cg(0)}{4\delta\lambda_1}\big(g' \circ \Delta u\big), \tag{3.14}$$

where

$$p = \begin{cases} \max_{j=1,\ldots,n}\{p_j\}, & \text{if } E(0) \geq 1, \\ \min_{j=1,\ldots,n}\{p_j\}, & \text{if } E(0) < 1. \end{cases}$$

Proof The straightforward computation implies that

$$\begin{aligned}
\Psi'(t) = {} & -\int_\Omega u_{tt} \cdot \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
& - \int_\Omega u_t \left[u_t \int_0^t g(t-s)\,ds + \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds \right] dx \\
= {} & -\int_\Omega \bigg(-\Delta^2 u + \operatorname{div} F(\nabla u) + \sigma(t)\int_0^t g(t-s)\Delta^2 u(s)\,ds - \mu_1 |u_t|^{m-1}u_t \\
& - \mu_2 \big|z(x,1,t)\big|^{m-1} z(x,1,t) \bigg) \cdot \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
& - \int_0^t g(s)\,ds \|u_t\|^2 - \int_\Omega u_t \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
= {} & \int_\Omega \Delta u \int_0^t g(t-s)\big(\Delta u(t) - \Delta u(s)\big)\,ds\,dx \\
& + \int_\Omega F(\nabla u) \int_0^t g(t-s)\big(\nabla u(t) - \nabla u(s)\big)\,ds\,dx \\
& - \sigma(t) \int_\Omega \left(\int_0^t g(t-s)\Delta u(s)\,ds \right) \left(\int_0^t g(t-s)\big(\Delta u(t) - \Delta u(s)\big)\,ds \right) dx \\
& + \mu_1 \int_\Omega |u_t|^{m-1}u_t \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx + \mu_2 \int_\Omega \big|z(x,1,t)\big|^{m-1} z(x,1,t) \\
& \times \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx - \int_0^t g(s)\,ds \|u_t\|^2 \\
& - \int_\Omega u_t \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx.
\end{aligned} \tag{3.15}$$

By using Hölder's inequality, Young's inequality, and the embedding theorem, we can infer that, for any $\delta > 0$,

$$\int_\Omega \Delta u \int_0^t g(t-s)\big(\Delta u(t) - \Delta u(s)\big)\,ds\,dx \leq \delta \|\Delta u\|^2 + \frac{1-l}{4\delta}(g \circ \Delta u), \tag{3.16}$$

$$\begin{aligned}
& -\sigma(t) \int_\Omega \left(\int_0^t g(t-s)\Delta u(s)\,ds \right) \left(\int_0^t g(t-s)\big(\Delta u(t) - \Delta u(s)\big)\,ds \right) dx \\
& \quad \leq 2\delta(1-l)^2 \sigma(t)\|\Delta u\|^2 + \left(2\delta + \frac{1}{4\delta} \right)(1-l)\sigma(t)(g \circ \Delta u),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
& \mu_1 \int_\Omega |u_t|^{m-1}u_t \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
& \quad \leq \delta\mu_1 \|u_t\|_{m+1}^{m+1} + \frac{\mu_1}{4\delta} \int_0^t g(t-s)\big\|u(t)-u(s)\big\|_{m+1}^{m+1}\,ds \\
& \quad \leq \delta\mu_1 \|u_t\|_{m+1}^{m+1} + \frac{C\mu_1}{4\delta\lambda_1}(g \circ \Delta u),
\end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&\mu_2 \int_\Omega \bigl|z(x,1,t)\bigr|^{m-1} z(x,1,t) \int_0^t g(t-s)\bigl(u(t)-u(s)\bigr)\,ds\,dx \\
&\quad \le \delta\mu_2 \bigl\|z(x,1,t)\bigr\|_{m+1}^{m+1} + \frac{C\mu_2}{4\delta\lambda_1}(g\circ\Delta u),
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
&-\int_\Omega u_t \int_0^t g'(t-s)\bigl(u(t)-u(s)\bigr)\,ds\,dx \\
&\quad \le \delta\|u_t\|^2 + \frac{1}{4\delta}\left(\int_0^t \bigl(-g'(t-s)\bigr)\bigl\|u(t)-u(s)\bigr\|\,ds\right)^2 \\
&\quad \le \delta\|u_t\|^2 - \frac{Cg(0)}{4\delta\lambda_1}\bigl(g'\circ\Delta u\bigr).
\end{aligned}
\tag{3.20}
$$

Now we estimate the term $\int_\Omega F(\nabla u)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx$ by using the method in [18]. Noting (2.5) and the embedding $V_2 \hookrightarrow W_0^{1,p_j+1}(\Omega)$, $j=1,\ldots,n$, we know that there exist positive constants $\mu_{p_1},\ldots,\mu_{p_n}$ satisfying

$$
\|\nabla u\|_{p_j+1} \le \mu_{p_j}\|\Delta u\|, \quad \forall j=1,\ldots,n.
$$

Using (2.7) with $F(0)=0$, Hölder's inequality, Young's inequality, and the embedding theorem, we can conclude that, for any $\delta>0$,

$$
\begin{aligned}
&\int_\Omega \bigl|F(\nabla u)\bigr|\bigl|\nabla u(t)-\nabla u(s)\bigr|\,dx \\
&\quad \le K\int_\Omega\left(\sum_{j=1}^n\bigl(1+|\nabla u|^{\frac{p_j-1}{2}}\bigr)\right)|\nabla u|\bigl|\nabla u(t)-\nabla u(s)\bigr|\,dx \\
&\quad \le K\sum_{j=1}^n\bigl(|\Omega|^{\frac{p_j-1}{2(p_j+1)}}+\|\nabla u\|_{p_j+1}^{\frac{p_j-1}{2}}\bigr)\|\nabla u\|_{p_j+1}\bigl\|\nabla u(t)-\nabla u(s)\bigr\| \\
&\quad \le \frac{K}{\lambda_2}\sum_{j=1}^n\mu_{p_j}\bigl(|\Omega|^{\frac{p_j-1}{2(p_j+1)}}+\|\nabla u\|_{p_j+1}^{\frac{p_j-1}{2}}\bigr)\|\Delta u\|\bigl\|\Delta u(t)-\Delta u(s)\bigr\| \\
&\quad \le \delta\|\Delta u\|^2+\frac{1}{4\delta}\left[\frac{K}{\lambda_2}\sum_{j=1}^n\mu_{p_j}\bigl(|\Omega|^{\frac{p_j-1}{2(p_j+1)}}+\|\nabla u\|_{p_j+1}^{\frac{p_j-1}{2}}\bigr)\right]^2\bigl\|\Delta u(t)-\Delta u(s)\bigr\|^2 \\
&\quad \le \delta\|\Delta u\|^2+\left[\frac{2K^2}{\lambda_2^2}\left(\sum_{j=1}^n\mu_{p_j}|\Omega|^{\frac{p_j-1}{2(p_j+1)}}\right)^2+\frac{2K^2}{\lambda_2^2}\left(\sum_{j=1}^n\mu_{p_j}^{\frac{p_j+1}{2}}\left(\frac{2}{l}\right)^{\frac{p_j-1}{4}}\bigl(E(0)\bigr)^{\frac{p_j-1}{4}}\right)^2\right] \\
&\qquad \times\frac{1}{4\delta}\bigl\|\Delta u(t)-\Delta u(s)\bigr\|^2 \\
&:= \delta\|\Delta u\|^2+\frac{1}{4\delta}\bigl(\alpha_1+\alpha_2\bigl(E(0)\bigr)^{\frac{p-1}{2}}\bigr)\bigl\|\Delta u(t)-\Delta u(s)\bigr\|^2,
\end{aligned}
\tag{3.21}
$$

where

$$
p=\begin{cases}\max_{j=1,\ldots,n}\{p_j\}, & \text{if } E(0)\ge 1,\\ \min_{j=1,\ldots,n}\{p_j\}, & \text{if } E(0)<1,\end{cases}
$$

and

$$\alpha_1 := \frac{2K^2}{\lambda_2^2}\left(\sum_{j=1}^{n}\mu_{p_j}|\Omega|^{\frac{p_j-1}{2(p_j+1)}}\right)^2,$$

$$\alpha_2 := \frac{2K^2}{\lambda_2^2}\left(\sum_{j=1}^{n}\mu_{p_j}^{\frac{p_j+1}{2}}\left(\frac{2}{l}\right)^{\frac{p_j-1}{4}}\bigl(E(0)\bigr)^{\frac{p_j-1}{4}}\right)^2.$$

It follows from (3.21), Hölder's inequality, and Young's inequality that

$$\begin{aligned}
&\int_\Omega F(\nabla u)\int_0^t g(t-s)\bigl(\nabla u(t)-\nabla u(s)\bigr)\,ds\,dx \\
&\quad\le \int_0^t g(t-s)\left(\int_\Omega \bigl|F(\nabla u)\bigr|\bigl|\nabla u(t)-\nabla u(s)\bigr|\,dx\right)ds \\
&\quad\le \delta\|\Delta u\|^2+\frac{1}{4\delta}\bigl(\alpha_1+\alpha_2\bigl(E(0)\bigr)^{\frac{p-1}{2}}\bigr)(g\circ\Delta u). \qquad (3.22)
\end{aligned}$$

Inserting (3.16)-(3.20) and (3.22) into (3.15), we obtain (3.14). The proof is therefore complete. □

Lemma 3.4 *For $\varepsilon_1>0$ and $\varepsilon_2>0$ small enough, there exist two positive constants β_1 and β_2 such that*

$$\beta_1 E(t)\le \mathcal{L}(t)\le \beta_2 E(t). \qquad (3.23)$$

Proof Using Hölder's inequality, Young's inequality, and Poincaré's inequality, we can easily get

$$\begin{aligned}
\bigl|\mathcal{L}(t)-E(t)\bigr| &\le \frac{\varepsilon_1}{2}\sigma(0)\|u_t\|^2+\frac{\varepsilon_1}{2\lambda_1}\sigma(0)\|\Delta u\|^2+\frac{\varepsilon_2}{2}\sigma(0)\|u_t\|^2 \\
&\quad+\frac{\varepsilon_2}{2}\int_\Omega\left(\int_0^t g(t-s)\bigl(u(t)-u(s)\bigr)\,ds\right)^2dx \\
&\le \frac{\varepsilon_1+\varepsilon_2}{2}\sigma(0)\|u_t\|^2+\frac{\varepsilon_1}{2\lambda_1}\sigma(0)\|\Delta u\|^2+\frac{\varepsilon_2 l_0}{2\lambda_1}\sigma(0)(g\circ\Delta u),
\end{aligned}$$

which, choosing $\varepsilon_1>0$ and $\varepsilon_2>0$ small enough, implies (3.23). The proof is complete. □

Proof of Theorem 2.2 Combining (3.4), (3.8), and (3.14) with assumption (A_1), we can obtain

$$\begin{aligned}
\mathcal{L}'(t) &= E'(t)+\varepsilon_1\sigma(t)\Phi(t)+\varepsilon_1\sigma(t)\Phi'(t)+\varepsilon_2\sigma(t)\Psi(t)+\varepsilon_2\sigma(t)\Psi'(t) \\
&\le -\sigma(t)\left(\frac{c_1}{\sigma(0)}-\varepsilon_1-\varepsilon_2\delta\mu_2\right)\|u_t\|_{m+1}^{m+1}-\sigma(t)\left[\varepsilon_2\left(\int_0^t g(s)\,ds-\delta\right)-\varepsilon_1\right]\|u_t\|^2 \\
&\quad-\sigma(t)\left(\frac{c_2}{\sigma(0)}-C_\varepsilon\varepsilon_1-\varepsilon_2\delta\mu_2\right)\bigl\|z(x,1,t)\bigr\|_{m+1}^{m+1}+\sigma(t)\left(\frac{1}{2}-\varepsilon_2\frac{Cg(0)}{4\delta\lambda_1}\right)\bigl(g'\circ\Delta u\bigr) \\
&\quad-\sigma(t)\bigl[c_3\varepsilon_1-2\varepsilon_2\delta-2\varepsilon_2\sigma(t)(1-l)^2\bigr]\|\Delta u\|^2-\frac{1}{2}\sigma'(t)\left(\int_0^t g(s)\,ds\right)\|\Delta u\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\sigma(t)\biggl[\frac{\varepsilon_1}{4}\sigma(t)+\varepsilon_2 C_\delta\bigl[1+(1-l)\sigma(t)+\bigl(E(0)\bigr)^{\frac{p-1}{2}}\bigr]\biggr](g\circ\Delta u)\\
&+\varepsilon_1\sigma'(t)\int_\Omega uu_t\,dx+\varepsilon_2\sigma'(t)\int_\Omega u_t\int_0^t g(t-s)\bigl(u(s)-u(t)\bigr)\,ds\,dx. \qquad (3.24)
\end{aligned}
$$

By using the Young inequality and the Poincaré inequality, we shall see that

$$
\begin{aligned}
&\sigma'(t)\int_\Omega uu_t\,dx+\sigma'(t)\int_\Omega u_t\int_0^t g(t-s)\bigl(u(s)-u(t)\bigr)\,ds\,dx\\
&\quad\le-\frac{\sigma'(t)}{2\lambda_1}\|\Delta u\|^2-\sigma'(t)\|u_t\|^2-\frac{\sigma'(t)}{2\lambda_1}\biggl(\int_0^t g(s)\,ds\biggr)(g\circ\Delta u). \qquad (3.25)
\end{aligned}
$$

For any fixed $t_0>0$, we know that, for any $t\ge t_0$,

$$
\int_0^t g(s)\,ds\ge\int_0^{t_0}g(s)\,ds:=g_0,
$$

which, along with (3.24)-(3.25), implies for any $t\ge t_0$,

$$
\begin{aligned}
L'(t)\le{}&-\sigma(t)\biggl(\frac{c_1}{\sigma(0)}-\varepsilon_1-\varepsilon_2\delta\mu_2\biggr)\|u_t\|_{m+1}^{m+1}-\sigma(t)\biggl[\varepsilon_2(g_0-\delta)-\varepsilon_1+\frac{\sigma'(t)}{\sigma(t)}\biggr]\|u_t\|^2\\
&-\sigma(t)\biggl(\frac{c_2}{\sigma(0)}-C_\varepsilon\varepsilon_1-\varepsilon_2\delta\mu_2\biggr)\bigl\|z(x,1,t)\bigr\|_{m+1}^{m+1}+\sigma(t)\biggl(\frac12-\varepsilon_2\frac{Cg(0)}{4\delta\lambda_1}\biggr)\bigl(g'\circ\Delta u\bigr)\\
&-\sigma(t)\biggl[c_3\varepsilon_1-2\varepsilon_2\delta-2\varepsilon_2\sigma(t)(1-l)^2+\frac{1}{2\lambda_1}\frac{\sigma'(t)}{\sigma(t)}\biggr]\|\Delta u\|^2\\
&+\sigma(t)\biggl[\frac{\varepsilon_1}{4}\sigma(t)+\varepsilon_2 C_\delta\bigl[1+(1-l)\sigma(t)+\bigl(E(0)\bigr)^{\frac{p-1}{2}}\bigr]-\frac{g_0}{2\lambda_1}\frac{\sigma'(t)}{\sigma(t)}\biggr](g\circ\Delta u). \qquad (3.26)
\end{aligned}
$$

At this point we first choose $0<\delta<\min\{\frac{g_0}{2},\frac{c_3g_0}{4[1+(1-l)^2]}\}$, and we get

$$
g_0-\delta>\frac12 g_0\quad\text{and}\quad\frac{\delta}{2c_3}\bigl[2+2(1-l)^2\bigr]<\frac14 g_0.
$$

For any fixed $\delta>0$, we take $\varepsilon_1>0$ and $\varepsilon_2>0$ satisfying

$$
\frac{g_0}{4}\varepsilon_2<\varepsilon_1<\frac{g_0}{2}\varepsilon_2 \qquad (3.27)
$$

so small that

$$
\begin{aligned}
&\eta_1:=\varepsilon_2(g_0-\delta)-\varepsilon_1>0,\\
&\eta_2:=c_3\varepsilon_1-2\varepsilon_2\delta\sigma-2\varepsilon_2\delta(1-l)^2>0.
\end{aligned}
$$

We at last choose $\varepsilon_1>0$ and $\varepsilon_2>0$ small enough for (3.23) and (3.27) to remain valid, and further,

$$
\frac{c_1}{\sigma(0)}-\varepsilon_1-\varepsilon_2\delta\mu_2>0,\qquad\frac{c_2}{\sigma(0)}-C_\varepsilon\varepsilon_1-\varepsilon_2\delta\mu_2>0,\qquad\frac12-\varepsilon_2\frac{Cg(0)}{4\delta\lambda_1}>0.
$$

From this it follows that, for positive constants η_1, η_2, and η_3,

$$\begin{aligned}\mathcal{L}'(t) &\le -\sigma(t)\left(2\eta_1 + \frac{1}{2}\frac{\sigma'(t)}{\sigma(t)}\right)\|u_t\|^2 - \sigma(t)\left(2\eta_2 + \frac{1}{2\lambda_1}\frac{\sigma'(t)}{\sigma(t)}\right)\|\Delta u\|^2 \\ &\quad + \left(\eta_3 - \frac{g_0}{2\lambda_1}\frac{\sigma'(t)}{\sigma(t)}\right)\sigma(t)(g\circ\Delta u), \quad \forall t\ge t_0. \end{aligned} \tag{3.28}$$

Since $\lim_{t\to\infty}\frac{\sigma'(t)}{\sigma(t)} = 0$, we choose $t_1 \ge t_0$ and use (2.16) to get

$$\begin{aligned}\mathcal{L}'(t) &\le -\sigma(t)\bigl(\eta_1\|u_t\|^2 + \eta_2\|\Delta u\|^2\bigr) + \eta_3\sigma(t)(g\circ\Delta u) \\ &\le -\eta_4\sigma(t)E(t) + \eta_5\sigma(t)(g\circ\Delta u), \quad \forall t_1\ge t_0, \end{aligned} \tag{3.29}$$

where η_4 and η_5 are positive constants.

Multiplying (3.29) by $\zeta(t)$ and using (3.4), we obtain

$$\begin{aligned}\zeta(t)\mathcal{L}'(t) &\le -\eta_4\zeta(t)\sigma(t)E(t) + \eta_5\zeta(t)\sigma(t)(g\circ\Delta u) \\ &\le -\eta_4\zeta(t)\sigma(t)E(t) - \eta_5\left[2E'(t) + \sigma'(t)\left(\int_0^t g(s)\,ds\right)\|\Delta u\|^2\right], \end{aligned}$$

which, combining with (2.16), gives us for any $t \ge t_1$,

$$\zeta(t)\mathcal{L}'(t) + 2\eta_5E'(t) \le -\sigma(t)\zeta(t)\left[\eta_4 + \frac{2l\sigma'(t)}{\zeta(t)\sigma(t)}\left(\int_0^t g(s)\,ds\right)\right]E(t). \tag{3.30}$$

Since $\lim_{t\to\infty}\frac{\sigma'(t)}{\zeta(t)\sigma(t)} = 0$, we can choose $t_2 \ge t_1$ so that

$$\zeta(t)\mathcal{L}'(t) + 2\eta_5E'(t) \le -\frac{\eta_4}{2}\sigma(t)\zeta(t)E(t), \quad \forall t\ge t_2. \tag{3.31}$$

Let $\mathcal{E}(t) = \zeta(t)\mathcal{L}(t) + 2\eta_5E(t)$, then it is easy to see that $\mathcal{E}(t)$ is equivalent to the modified energy $E(t)$ by using (3.23), that is, there exist two positive constants β_3 and β_4 such that

$$\beta_3E(t) \le \mathcal{E}(t) \le \beta_4E(t), \tag{3.32}$$

which, together with (3.31) and using $\zeta'(t)\le 0$, shows that there exists a positive constant $\gamma_1 > 0$ such that, for any $t\ge t_2$,

$$\mathcal{E}'(t) \le -\frac{\eta_4}{\beta_4}\zeta(t)\sigma(t)\mathcal{E}(t). \tag{3.33}$$

Integrating (3.33) over (t_2, t) with respect to t, we get for any $t \ge t_2$,

$$\begin{aligned}\mathcal{E}(t) &\le \mathcal{E}(t_2)\exp\left(-\frac{\eta_4}{\beta_4}\int_{t_2}^t \zeta(s)\sigma(s)\,ds\right) \\ &\le \mathcal{E}(t_2)\exp\left(\frac{\eta_4}{\beta_4}\int_0^{t_2} \zeta(s)\sigma(s)\,ds\right)\exp\left(-\frac{\eta_4}{\beta_4}\int_0^t \zeta(s)\sigma(s)\,ds\right), \end{aligned}$$

which, using (3.32), implies for any $t\ge t_2$,

$$E(t) \le \frac{\beta_4}{\beta_3}\gamma_1E(t_2)\exp\left(-\frac{\eta_4}{\beta_4}\int_0^t \zeta(s)\sigma(s)\,ds\right). \tag{3.34}$$

Therefore (2.16) follows by renaming the constants, and by the continuity and boundedness of $E(t)$. The proof is hence complete. □

Remark 3.5 We illustrate several rates of energy decay through the following examples, some of which can be found in [17, 23].

Example 1 If g decays exponentially, *i.e.*, $\zeta(t) = a$, and $\sigma(t) = \frac{b}{1+t}$, then (2.16) gives us

$$E(t) \le \frac{\beta}{(1+t)^{\gamma ab}}.$$

Example 2 If g decays exponentially, *i.e.*, $\zeta(t) = a$, and $\sigma(t) = b$, then (2.16) gives us

$$E(t) \le \beta e^{-ab\gamma t}.$$

Example 3 When $g(t) = ae^{-b(1+t)^\alpha}$ and $\sigma(t) = \frac{1}{1+t}$ for $a, b > 0$ and $0 < \alpha < 1$, then $\zeta(t) = b\alpha(1+t)^{\alpha-1}$ satisfies (2.1)-(2.2). Estimate (2.16) takes the form

$$E(t) \le \beta \exp\left(-\frac{b\alpha\gamma}{\alpha-1}(1+t)^{\alpha-1}\right).$$

Example 4 If $g(t) = a\exp(-b\ln^\alpha(1+t))$ and $\sigma(t) = \frac{1}{\ln(1+t)}$ for $a, b > 0$ and $\alpha > 1$, we know that $\zeta(t) = \frac{b\alpha\ln^{\alpha-1}(1+t)}{1+t}$ satisfies (2.1)-(2.2). Estimate (2.16) takes the form

$$E(t) \le \beta \exp\left(-\frac{b\alpha\gamma}{\alpha-1}\ln^{\alpha-1}(1+t)\right).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Faculty of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu, 611130, P.R. China.
[2]College of Mathematics and Information Science, Beifang University of Nationalities, Yinchuan, 750021, P.R. China.

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