Luo et al. *Advances in Difference Equations* (2016) 2016:256
DOI 10.1186/s13662-016-0984-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Robust convergence analysis of iterative learning control for impulsive Riemann-Liouville fractional-order systems

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Abstract

In this paper, we explore *P*-type learning laws for impulsive Riemann-Liouville fractional-order controlled systems ($0 < \alpha < 1$) with initial state offset bounded to track the varying reference accurately by using a few iterations in a finite time interval. By using the Gronwall inequality and fundamental inequalities, we obtain open-loop and closed-loop *P*-type robust convergence results in the sense of $(PC_{1-\alpha}, \lambda)$-norm $\|\cdot\|_{PC_{1-\alpha},\lambda}$. Finally, numerical examples are given to illustrate our theoretical results.

MSC: 34A37; 93C15; 93C40

Keywords: impulsive fractional-order system; iterative learning control; robust convergence; weighted norm

1 Introduction

Since Uchiyama and Arimoto put forward the concept of iterative learning control (ILC for short), ILC has been extended to tracking tasks with iteratively varying reference trajectories [1–3] extensively. Up to now, a wide variety of iterative learning control problems and related issues have been proposed and studied in many fields. For example, ILC for fractional differential systems [4–6], ILC for impulsive differential systems [7, 8], research on the robustness of ILC [9–11], and so on.

Recently, the fractional-order differential system has played an important role in various fields such as electricity, signal and image processing, neural networks [12–14], and control problems [15]. Furthermore, the qualitative theory of fractional differential systems has been studied extensively. The existence theory of solutions to fractional-order differential equations involving Riemann-Liouville and Caputo derivatives has been investigated in [16–25]. Meanwhile, it is remarkable that some interesting existence and controllability results have been obtained for fractional controlled systems involving the Caputo derivative [26–29]. Moreover, the concept and existence of solutions for impulsive fractional differential equations involving Riemann-Liouville and Caputo derivatives have been studied in [30–34]. There are few papers on ILC for integer-order and Caputo type fractional-order impulsive differential systems [35–44]. Since Riemann-Liouville fractional-order systems play the same important role in theory analysis and application, it is necessary to deal with ILC problems for Riemann-Liouville type fractional impulsive differential systems.

In this paper, we discuss ILC for impulsive Riemann-Liouville fractional controlled systems with initial state offset bounded and present the robust convergence analysis results. More precisely, we study

$$\begin{cases} (D_{0,t}^{\alpha}x_k)(t) = \mu x_k(t) + f(t,x_k(t),u_k(t)) + \xi_k(t), & t \in [0,T] \setminus \{t_1,\ldots,t_m\}, \mu < 0, \\ \lim_{t\to 0+}(I_{0,t}^{1-\alpha}x_k)(t) = x_k(0), \\ \Theta(I_{0,t_j}^{1-\alpha}x_k)(t_j) = G_j(t_j^-, x_k(t_j^-)), & t_j \in \{t_1,\ldots,t_m\}, \\ y_k(t) = g(t,x_k(t)) + Bu_k(t) + \eta_k(t), \end{cases} \tag{1}$$

where $D_{0,t}^{\alpha}$ denotes Riemann-Liouville fractional derivatives of the order $\alpha \in (0,1)$ from lower limit zero and $I_{0,t}^{1-\alpha}$ denotes Riemann-Liouville fractional integral the order $1-\alpha$ from lower limit zero (see Definition 2.1), k denotes the kth learning iteration, T denotes pre-fixed iteration domain length, impulsive term

$$\Theta\left(I_{0,t_j}^{1-\alpha}x\right)(t_j) := I_{0,t_j^+}^{1-\alpha}x\left(t_j^+\right) - I_{0,t_j^-}^{1-\alpha}x\left(t_j^-\right) = \Gamma(\alpha)\Big[\lim_{t\to t_j^+}(t-t_j)^{1-\alpha}x(t) - \lim_{t\to t_j^-}(t-t_j)^{1-\alpha}x(t)\Big],$$

where $I_{0,t_j^+}^{1-\alpha}x(t_j^+)$ and $I_{0,t_j^-}^{1-\alpha}x(t_j^-)$ denote the right and the left limits of $I_{0,t}^{1-\alpha}x(t)$ at $t_j \in \{t_1,\ldots,t_m\}$. For more details on $\Theta(I_{0,t_j}^{1-\alpha}x)(t_j)$, one can see [16], Lemma 3.2, Chapter 3. Also, $t_j, j = 1,2,\ldots,m$, denotes the jth impulsive points satisfying $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. The nonlinear terms $f : J \times R^n \times R^n \to R^n$ and $G_j, g : J \times R^n \to R^n$ are given functions. The functions ξ_k, $\eta_k : J \to R^n$ represent the state interference and output disturbance, respectively. The variables $x_k, u_k, y_k \in R^n$ denote state, input, and output, respectively. Moreover, B is a $n \times n$ real matrix.

According to [32], (2.5), the continuous solution of the system (1) can be formulated by the solution of the fractional integral equations

$$x_k(t) = \begin{cases} t^{\alpha-1}E_{\alpha,\alpha}(\mu t^{\alpha})x_k(0) \\ \quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\mu(t-s)^{\alpha})[f(s,x_k(s),u_k(s)) + \xi_k(s)]\,ds, & t \in [0,t_1], \\ t^{\alpha-1}E_{\alpha,\alpha}(\mu t^{\alpha})x_k(0) \\ \quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\mu(t-s)^{\alpha})[f(s,x_k(s),u_k(s)) + \xi_k(s)]\,ds \\ \quad + \sum_{i=1}^{j} E_{\alpha,\alpha}(\mu(t-t_i)^{\alpha})(t-t_i)^{\alpha-1}G_i(t_i^-, x_k(t_i^-)), \\ \qquad t \in (t_j, t_{j+1}], j = 1,2,\ldots,m, \end{cases} \tag{2}$$

where $E_{\alpha,\alpha}$ denotes Mittag-Leffler type function (see Definition 2.2).

The ILC problems for Riemann-Liouville type fractional impulsive differential systems have not been studied extensively. The main difficulties are the following two facts:

(i) The initial value involving singular term in Riemann-Liouville fractional differential equations of order $\alpha \in (0,1)$ is much different from Caputo fractional differential equations with the same order.

(ii) Impulsive conditions make the formula of solutions to fractional differential equations more complex due to the memory property of the fractional derivative.

After carefully observing, we have to introduce the piecewise continuous space with weighted norm to deal with the singular term appearing in the initial condition via a new singular impulsive Gronwall inequality, which is the main difficult to be solved by us.

For the system (1), we consider an open-loop P-type ILC updating law with the initial state offset bounded,

$$\Delta u_k = P_o e_k(t), \qquad \|\Delta x_k(0)\| \le d_0, \tag{3}$$

and a closed-loop P-type ILC updating law learning with initial state offset bounded,

$$\Delta u_k = P_d e_{k+1}(t), \qquad \|\Delta x_k(0)\| \le d_0, \tag{4}$$

where $\Delta u_k = u_{k+1} - u_k$, $e_k = y_d - y_k$, $\Delta x_k = x_{k+1} - x_k$ denote the tracking error and y_d the iteratively varying reference trajectory, and d_0 is positive constant, P_o and P_d are unknown $n \times n$ matrix parameters to be determined.

The main objective of this paper is to generate the control input u_k such that the impulsive fractional system output y_k tracking the iteratively varying reference trajectories y_d (may be continuous or discontinuous) as accurately as possible when $k \to \infty$ uniformly on $[0, T]$ in the sense of $(PC_{1-\alpha}, \lambda)$-norm by adopting a P-type ILC updating law with initial state offset bounded.

The main contribution of this paper are as follows.

(i) We establish a standard study framework of the ILC problem for an impulsive Riemann-Liouville fractional system associated with an impulsive Gronwall inequality with singular kernel given by us (see [31], Lemma 2.8).

(ii) Sufficient conditions ensuring the robust convergence of ILC problem for impulsive Riemann-Liouville fractional system with order lying in $(0,1)$ are derived.

The rest of this paper is organized as follows. In Section 2, we give some necessary notations, concepts, and lemmas. In Section 3, two sufficient conditions ensuring convergence results of the system (1) are presented. An interesting example is given in the final section to demonstrate the application of our main results.

2 Preliminaries

Set $J = [0, T]$. Let $C(J, R^n)$ be the Banach space of vector-value continuous functions from $J \to R^n$ endowed with the standard norm $\|\cdot\|$. In order to define the solutions of system (1), we consider a Banach space $PC(J, R^n) = \{x : (t - t_j)^{1-\alpha} x(t) \in C((t_j, t_{j+1}], R^n)$, and $\lim_{t \to t_j^+} (t - t_j)^{1-\alpha} x(t)$ exists, $j = 1, 2, \ldots, m\}$ endowed with the $(PC_{1-\alpha}, \lambda)$-norm

$$\|x\|_{PC_{1-\alpha},\lambda} = \max\{(t - t_j)^{1-\alpha} e^{-\lambda(t-t_j)} \|x(t)\| : j = 0, 1, \ldots, m\}.$$

Next, we recall some basic definitions on fractional calculus.

Definition 2.1 (see [16], Formula (2.1.1)) For a given function f, the Riemann-Liouville fractional integral $I_{a,x}^{\alpha} f$ is defined by

$$\left(I_{a,x}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > a; 0 < \alpha < 1,$$

and the Riemann-Liouville fractional derivative $D_{a,x}^{\alpha} f$ is defined by

$$\left(D_{a,x}^{\alpha} f\right)(x) := \frac{d}{dx}\left(I_{a,x}^{1-\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}}\, dt,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (see [45], (4.1.1)) The two-parameter Mittag-Leffler type function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta \in R, z \in R.$$

The following lemmas will be used in the sequel.

Lemma 2.3 (see [34], Lemma 2) *Let $\alpha \in (0,1]$ and $\lambda > 0$ be arbitrary. The functions $E_\alpha(\cdot)$, $E_{\alpha,\alpha}(\cdot)$ are nonnegative and*

$$E_\alpha\bigl(-t^\alpha \lambda\bigr) \le 1, \qquad E_{\alpha,\alpha}\bigl(-t^\alpha \lambda\bigr) \le \frac{1}{\Gamma(\alpha)}.$$

Lemma 2.4 (see [31], Lemma 2.8) *Let $v \in PC(J, R^+)$ satisfy the following inequality*:

$$v(t) \le c_1(t) + c_2 \int_0^t (t-s)^{\beta-1} v(s)\, ds + \sum_{j=1}^{k} \theta_j v\bigl(t_j^-\bigr),$$

where $c_1(t)$ is nonnegative continuous and nondecreasing on J, and $c_2, \theta_j > 0$ are constants. Then

$$v(t) \le c_1(t)\bigl(1 + \theta E_\beta\bigl(c_2 \Gamma(\beta) t^\beta\bigr)\bigr)^j E_\beta\bigl(c_2 \Gamma(\beta) t^\beta\bigr), \quad \textit{for } t \in (t_j, t_{j+1}],$$

where $\theta = \max\{\theta_j : j = 1, 2, \ldots, m\}$.

Lemma 2.5 (see [9], Lemma 1) *Let d_k be a sequence of real number which converges to the limit d_∞ as $k \to \infty$. Suppose that a_k is a sequence of real number such that*

$$p a_k + q a_{k-1} \le d_k, \quad p > -q \ge 0.$$

Then we have

$$\limsup_{k \to \infty} a_k \le \frac{d_\infty}{p+q}.$$

3 Robust convergence analysis of *P*-type

In this section, we discuss robust convergence results for (1) via an ILC of an open-loop *P*-type ILC (3) and closed-loop (4), respectively.

For a start, we impose the following assumptions:

(A_1) The function $f : J \times R^n \times R^n \to R^n$ is continuous and there exist two nonnegative functions $L_f(\cdot)$ and $I_f(\cdot)$ such that

$$\bigl\| f(t,x,u) - f(t,\hat{x},\hat{u}) \bigr\| \le L_f(t)\|x - \hat{x}\| + I_f(t)\|u - \hat{u}\|,$$

for any $x, \hat{x}, u, \hat{u} \in R^n$ and all $t \in J$.

The function $g : J \times R^n \to R^n$ is continuous and there exists a constant $L_g > 0$ such that

$$\bigl\| g(t,x) - g(t,\hat{x}) \bigr\| \le L_g \|x - \hat{x}\|,$$

for any $x, \hat{x} \in R^n$ and all $t \in J$.

We have the function $G_j : J \times R^n \to R^n$, $j = 1, 2, \ldots, m$, and there exists a nonnegative function $L_{G_j}(\cdot)$ such that

$$\left\| G_j(t,x) - G_j(t,\hat{x}) \right\| \le L_{G_j}(t)\|x - \hat{x}\|,$$

for any $x, \hat{x} \in R^n$ and all $t \in J$.

(A_2) For nonnegative functions $L_f(\cdot)$, $I_f(\cdot)$, and $L_{G_j}(\cdot)$, we set

$$\begin{aligned} M_1 &= \max\left\{ \sup_{t\in(t_j,t_{j+1}]} \frac{L_f(t)}{(t-t_j)^{1-\alpha}}, \sup_{t\in(t_j,t_{j+1}]} \frac{I_f(t)}{(t-t_j)^{1-\alpha}}, j = 0,1,\ldots,m \right\}, \\ M_2 &= \max\left\{ L_{G_j}(t_j), j = 1,\ldots,m \right\}, \\ M_L &= \max\{M_1, M_2\}. \end{aligned}$$

(A_3) For uncertainty and disturbance terms $\xi_k(t) \in R^n$, $\eta_k(t) \in R^n$ and the initial value $x_k(0) \in R^n$ are bounded as follows, for all $t \in (t_j, t_{j+1}]$, $j = 1, 2, \ldots, m$, and for any k, $\|\xi_{k+1}(t) - \xi_k(t)\| \le d_\xi$, $\|\eta_{k+1}(t) - \eta_k(t)\| \le d_\eta$, where d_ξ and d_η are positive constants.

Now we are ready to present the robust convergence analysis result for an open-loop P-type ILC.

Theorem 3.1 *For the system* (1), *the assumptions* (A_1)-(A_3) *hold. If* $\|I - BP_o\| < 1$, *then, for arbitrary initial input* u_0, (3) *guarantees that* $y_k(t)$ *is uniformly bounded for* $t \in J$ *as* $k \to \infty$ *in the sense of* $(PC_{1-\alpha}, \lambda)$*-norm. Further,* $y_k(t)$ *uniform convergent to* $y_d(t)$ *for* $t \in J$ *if disturbance is converge asymptotically to zero.*

Proof Without loss of generality, we only consider $t \in (t_j, t_{j+1}]$, $j = 0, 1, 2, \ldots, m$. Linking (1) and (3), we have

$$e_{k+1}(t) = (I - BP_o)e_k(t) + g\bigl(t, x_k(t)\bigr) - g\bigl(t, x_{k+1}(t)\bigr) + \eta_k(t) - \eta_{k+1}(t). \tag{5}$$

Taking the norm $\|\cdot\|$ on both sides of (5), one can derive that

$$\left\| e_{k+1}(t) \right\| \le \|I - BP_o\| \left\| e_k(t) \right\| + L_g \left\| \Delta x_k(t) \right\| + d_\eta. \tag{6}$$

In the following, we prove $\|e_{k+1}\|_{PC_{1-\alpha},\lambda}$ is uniformly bounded as $k \to \infty$.

Taking the norm $\|\cdot\|$ on both sides of (2), one can apply (A_1) and (A_3) to derive that

$$\begin{aligned} \left\| \Delta x_k(t) \right\| \le{}& \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left\| \Delta x_k(0) \right\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl[L_f(s) \left\| \Delta x_k(s) \right\| + I_f(s) \left\| \Delta u_k(s) \right\| \bigr]\,ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d_\xi \,ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{j} (t-t_i)^{\alpha-1} L_{G_i}(t_i) \left\| \Delta x_k\bigl(t_i^-\bigr) \right\|. \end{aligned} \tag{7}$$

Multiplying $(t - t_j)^{1-\alpha}$ on both sides of (7), using (A_2) we have

$$\begin{aligned} &(t-t_j)^{1-\alpha} \left\| \Delta x_k(t) \right\| \\ &\quad\le \frac{(1-\frac{t_j}{t})^{1-\alpha}}{\Gamma(\alpha)} d_0 + \frac{(t-t_j)^{1-\alpha} e^{\lambda(t-t_j)} M_L}{\lambda^\alpha} \|P_o\| \|\Delta e_k\|_{PC_{1-\alpha},\lambda} \end{aligned}$$

$$+\frac{(t_{j+1}-t_j)^{1-\alpha}M_L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(s-t_j)^{1-\alpha}\left\|\Delta x_k(s)\right\|ds+\frac{(t-t_j)^{1-\alpha}t^\alpha}{\Gamma(\alpha+1)}d_\xi$$
$$+\frac{(t-t_j)^{1-\alpha}}{\Gamma(\alpha)}\sum_{i=1}^{j}(t-t_i)^{\alpha-1}(t_i-t_{i-1})^{\alpha-1}L_{G_i}(t_i)(t_i-t_{i-1})^{1-\alpha}\left\|\Delta x_k\left(t_i^-\right)\right\|. \tag{8}$$

Note that the fact $\frac{t-t_j}{t-t_i}\le 1$ since $t_j\ge t_i$ and

$$(t-t_j)^{1-\alpha}(t-t_i)^{\alpha-1}=\left(\frac{t-t_j}{t-t_i}\right)^{1-\alpha}\le 1.$$

Then (8) reduces to

$$\begin{aligned}
&(t-t_j)^{1-\alpha}\left\|\Delta x_k(t)\right\|\\
&\quad\le\frac{(1-\frac{t_j}{t})^{1-\alpha}}{\Gamma(\alpha)}d_0+\frac{(t-t_j)^{1-\alpha}e^{\lambda(t-t_j)}M_L}{\lambda^\alpha}\|P_o\|\|\Delta e_k\|_{PC_{1-\alpha},\lambda}\\
&\qquad+\frac{(t_{j+1}-t_j)^{1-\alpha}M_L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(s-t_j)^{1-\alpha}\left\|\Delta x_k(s)\right\|ds+\frac{(t-t_j)^{1-\alpha}t^\alpha}{\Gamma(\alpha+1)}d_\xi\\
&\qquad+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{j}(t_i-t_{i-1})^{\alpha-1}M_L(t_i-t_{i-1})^{1-\alpha}\left\|\Delta x_k\left(t_i^-\right)\right\|.
\end{aligned}\tag{9}$$

For the inequality (9), we set $v(t)=(t-t_i)^{1-\alpha}\|\Delta x_k(t)\|$. Then one can apply Lemma 2.4 to derive that

$$\begin{aligned}
&(t-t_j)^{1-\alpha}\left\|\Delta x_k(t)\right\|\\
&\quad\le\left(\frac{(1-\frac{t_j}{t})^{1-\alpha}}{\Gamma(\alpha)}d_0+\frac{(t-t_j)^{1-\alpha}M_Le^{\lambda(t-t_j)}}{\lambda^\alpha}\|P_o\|\|\Delta e_k\|_{PC_{1-\alpha},\lambda}\right.\\
&\qquad\left.+\frac{(t-t_j)^{1-\alpha}t^\alpha}{\Gamma(\alpha+1)}d_\xi\right)\left(1+\theta E_\alpha\left((t_{j+1}-t_j)^{1-\alpha}t^\alpha M_L\right)\right)^jE_\alpha\left((t_{j+1}-t_j)^{1-\alpha}t^\alpha M_L\right),
\end{aligned}\tag{10}$$

where

$$\theta=\max\left\{\frac{(t_{j+1}-t_j)^{\alpha-1}M_L}{\Gamma(\alpha)}:j=0,1,2,\ldots,m\right\}.$$

Multiplying $e^{-\lambda(t-t_j)}$ on both sides of (10) and noting the fact that $E_\alpha(z),z>0$ is an increasing function, we have

$$\begin{aligned}
&(t-t_j)^{1-\alpha}e^{-\lambda(t-t_j)}\left\|\Delta x_k(t)\right\|\\
&\quad\le N_j\left(\frac{d_0}{t_{j+1}\Gamma(\alpha)}+\frac{M_L}{t_j^\alpha\lambda^\alpha}\|P_o\|\|\Delta e_k\|_{PC_{1-\alpha},\lambda}+\frac{d_\xi}{\Gamma(\alpha+1)}\right)\\
&\qquad\times\left(1+\theta E_\alpha(N_jM_L)\right)^jE_\alpha(N_jM_L),
\end{aligned}\tag{11}$$

where

$$N_j=\left[1-\frac{t_j}{t_{j+1}}\right]^{1-\alpha}\times t_{j+1}.$$

For (11), one can take the $(PC_{1-\alpha}, \lambda)$-norm to derive that

$$\begin{aligned}\|\Delta x_k\|_{PC_{1-\alpha},\lambda} \leq N_{\max}\left(\frac{d_0}{t_1\Gamma(\alpha)} + \frac{M_L}{t_1^\alpha \lambda^\alpha}\|P_o\|\|\Delta e_k\|_{PC_{1-\alpha},\lambda} + \frac{d_\xi}{\Gamma(\alpha+1)}\right) \\ \times \left(1+\theta E_\alpha(N_{\max}M_L)\right)^m E_\alpha(N_{\max}M_L),\end{aligned} \tag{12}$$

where

$$N_{\max} = \max\{N_j : j = 1,2,\ldots,m\}.$$

Linking (6) and (12), we have

$$\|e_{k+1}\|_{PC_{1-\alpha},\lambda} \leq \|I - BP_o\|\|e_k\|_{PC_{1-\alpha},\lambda} + L_g\|\Delta x_k\|_{PC_{1-\alpha},\lambda} + \Delta t_{j\max}^{1-\alpha} d_\eta, \tag{13}$$

where

$$\Delta t_{j\max}^{1-\alpha} = \max\left\{(t_{j+1}-t_j)^{1-\alpha} : j = 0,1,\ldots,m\right\}.$$

Submitting (12) into (13), we obtain

$$\|e_{k+1}\|_{PC_{1-\alpha},\lambda} \leq \tilde{q}\|\Delta e_k\|_{PC_{1-\alpha},\lambda} + \tilde{M}, \tag{14}$$

where

$$\begin{aligned}\tilde{M} &= \left[\frac{L_g N_{\max} d_0}{t_1\Gamma(\alpha)} + \frac{L_g N_{\max} d_\xi}{\Gamma(\alpha+1)}\right]\left(1+\theta E_\alpha(N_{\max}M_L)\right)^m E_\alpha(N_{\max}M_L) + \Delta t_{j\max}^{1-\alpha} d_\eta, \\ \tilde{q} &= \|I - BP_o\| + \frac{L_g N_{\max} M_L}{t_1^\alpha \lambda^\alpha}\|P_o\|\left(1+\theta E_\alpha(N_{\max}M_L)\right)^m E_\alpha(N_{\max}M_L).\end{aligned}$$

Note that there exists a large enough λ such that $\tilde{q} < 1$ due to $\|I - BP_o\| < 1$. Concerning (14), one can use Lemma 2.5 to derive that

$$\lim_{k\to\infty} \sup \|e_{k+1}\|_{PC_{1-\alpha},\lambda} \leq \frac{\tilde{M}}{1-\tilde{q}},$$

which shows that $y_k(t)$ is uniformly bounded in the sense of $(PC_{1-\alpha}, \lambda)$-norm. Further, if the disturbance has asymptotic convergence, which means that $d_\xi \to 0$, $d_\eta \to 0$, and $d_0 \to 0$, as $k \to \infty$, then $y_k(t)$ uniform convergent to $y_d(t)$ for $t \in J$ if the disturbance converges asymptotically to zero. □

Remark 3.2 In Theorem 3.1, If we set $\alpha = 0$, $x_k(0) = x_{k+1}(0)$, $\xi_k(t) = \eta_k(t) = 0$, $0 < \beta_3 < \frac{\partial g}{\partial x} < \beta_4$, $G_j(t,x) = G_j(x)$, $L_f(t) = L_f$, $I_f(t) = I_f$, and $L_{G_j}(t) = L_G$, then $y_k(\cdot)$ is uniform convergent to $y_d(\cdot)$ in the sense of (PC, λ)-norm, which is a parallel result to [40], Theorem 3.1, in the sense of the L^2-norm.

Next, we present the robust convergence analysis result for a closed-loop P-type ILC.

Theorem 3.3 *For the system* (1), *the assumptions* (A_1)-(A_3) *hold. If* $(I+BP_d)^{-1}$ *exists and* $\|(I+BP_d)^{-1}\|<1$, (4) *guarantees that* $y_k(t)$ *is uniformly bounded for* $t\in J$ *as* $k\to\infty$ *in the sense of the* $(PC_{1-\alpha},\lambda)$*-norm. Moreover, if the disturbance converges asymptotically to zero, then* $y_k(t)$ *is uniform convergent to* $y_d(t)$ *for* $t\in J$.

Proof Similar to the proof of Theorem 3.1, we consider $t\in(t_j,t_{j+1}]$, $j=1,2,\ldots,m$. Linking (1) and (4), we have

$$\begin{aligned} e_{k+1}(t) &= e_k(t)+g\bigl(t,x_k(t)\bigr)-g\bigl(t,x_{k+1}(t)\bigr)-BP_d e_{k+1}(t)+\eta_k(t)-\eta_{k+1}(t) \\ &= (I+BP_d)^{-1}e_k(t)+(I+BP_d)^{-1}\bigl(g\bigl(t,x_k(t)\bigr)-g\bigl(t,x_{k+1}(t)\bigr)\bigr) \\ &\quad +(I+BP_d)^{-1}\bigl(\eta_k(t)-\eta_{k+1}(t)\bigr). \end{aligned} \tag{15}$$

Taking the norm $\|\cdot\|$ on both sides of (15), we have

$$\bigl\|e_{k+1}(t)\bigr\| \le \bigl\|(I+BP_d)^{-1}\bigr\|\bigl\|e_k(t)\bigr\|+L_g\bigl\|(I+BP_d)^{-1}\bigr\|\bigl\|\Delta x_k(t)\bigr\|+\bigl\|(I+BP_d)^{-1}\bigr\|d_\eta. \tag{16}$$

Next, we apply the analogy method in Theorem 3.1 to prove that $\|e_{k+1}\|_{PC_{1-\alpha},\lambda}$ is uniformly bounded as $k\to\infty$.

By repeating the procedure to derive (12), one has

$$\begin{aligned} \|\Delta x_k\|_{PC_{1-\alpha},\lambda} &\le N_{\max}\left(\frac{d_0}{t_1\Gamma(\alpha)}+\frac{M_L}{t_1^\alpha\lambda^\alpha}\|P_d\|\|\Delta e_{k+1}\|_{PC_{1-\alpha},\lambda}+\frac{d_\xi}{\Gamma(\alpha+1)}\right) \\ &\quad\times\bigl(1+\theta E_\alpha(N_{\max}M_L)\bigr)^m E_\alpha(N_{\max}M_L), \end{aligned} \tag{17}$$

where N_j, θ, and $N_{\max}$ are defined in Theorem 3.1.

Substituting (16) into (17), we have

$$\begin{aligned} \|e_{k+1}\|_{PC_{1-\alpha},\lambda} &\le \bigl\|(I+BP_d)^{-1}\bigr\|\|e_k\|_{PC_{1-\alpha},\lambda}+L_g\bigl\|(I+BP_d)^{-1}\bigr\|\|\Delta x_k\|_{PC_{1-\alpha},\lambda} \\ &\quad+\bigl\|(I+BP_d)^{-1}\bigr\|\Delta t_{j\max}^{1-\alpha}d_\eta. \end{aligned} \tag{18}$$

Taking (17) into (18), we have

$$\begin{aligned} \|e_{k+1}\|_{PC_{1-\alpha},\lambda} &\le \bigl\|(I+BP_d)^{-1}\bigr\|\|e_k\|_{PC_{1-\alpha},\lambda} \\ &\quad+L_g\bigl\|(I+BP_d)^{-1}\bigr\|N_{\max}\left(\frac{d_0}{t_1\Gamma(\alpha)}+\frac{M_L}{t_1^\alpha\lambda^\alpha}\|P_d\|\|e_{k+1}\|_{PC_{1-\alpha},\lambda}+\frac{d_\xi}{\Gamma(\alpha+1)}\right) \\ &\quad\times\bigl(1+\theta E_\alpha(N_{\max}M_L)\bigr)^m E_\alpha(N_{\max}M_L)+\bigl\|(I+BP_d)^{-1}\bigr\|\Delta t_{j\max}^{1-\alpha}d_\eta, \end{aligned}$$

which implies that

$$\begin{aligned} &\|e_{k+1}\|_{PC_{1-\alpha},\lambda} \\ &\quad\le \frac{\|(I+BP_d)^{-1}\|}{H}\|e_k\|_{PC_{1-\alpha},\lambda}+\frac{L_g\|(I+BP_d)^{-1}\|N_{\max}}{H}\left(\frac{d_0}{t_1\Gamma(\alpha)}+\frac{d_\xi}{\Gamma(\alpha+1)}\right) \\ &\quad\quad\times\bigl(1+\theta E_\alpha(N_{\max}M_L)\bigr)^m E_\alpha(N_{\max}M_L)+\frac{\|(I+BP_d)^{-1}\|}{H}\Delta t_{j\max}^{1-\alpha}d_\eta, \end{aligned} \tag{19}$$

where

$$H = 1 - \frac{L_g \|(I+BP_d)^{-1}\| M_L}{t_1^{\alpha}\lambda^{\alpha}} \|P_d\| \bigl(1+\theta E_\alpha(N_{\max}M_L)\bigr)^m E_\alpha(N_{\max}M_L).$$

Let

$$\begin{aligned}\bar{M} &= \frac{L_g \|(I+BP_d)^{-1}\| N_{\max}}{H}\left(\frac{d_0}{t_1\Gamma(\alpha)} + \frac{d_\xi}{\Gamma(\alpha+1)}\right)\bigl(1+\theta E_\alpha(N_{\max}M_L)\bigr)^m E_\alpha(N_{\max}M_L) \\ &\quad + \frac{\|(I+BP_d)^{-1}\|}{H}\Delta t_{j\max}^{1-\alpha} d_\eta, \\ \bar{q} &= \frac{\|(I+BP_d)^{-1}\|}{H}.\end{aligned}$$

Then (19) reduces to

$$\|e_{k+1}\|_{PC_{1-\alpha},\lambda} \leq \bar{q}\|e_k\|_{PC_{1-\alpha},\lambda} + \bar{M}.$$

Note that $\|(I+BP_d)^{-1}\| < 1$. It is not difficult to see that $\tilde{q} < 1$ for some large enough λ. By Lemma 2.5, we have

$$\lim_{k\to\infty} \sup \|e_{k+1}\|_{PC_{1-\alpha},\lambda} \leq \frac{\tilde{M}}{1-\tilde{q}}. \tag{20}$$

Thus, the demised results are obtained immediately. The proof is finished. □

Remark 3.4 In Theorem 3.3, if we set $\alpha = 0$, $x_k(0) = x_{k+1}(0)$, $\xi_k(t) = \eta_k(t) = 0$, $0 < \beta_3 < \frac{\partial g}{\partial x} < \beta_4$, $G_j(t,x) = G_j(x)$, $L_f(t) = L_f$, $I_f(t) = I_f$, and $L_{G_j}(t) = L_G$, then $y_k(\cdot)$ is uniform convergent to $y_d(\cdot)$ in the sense of (PC,λ)-norm, which is another parallel result to [40], Theorem 3.2, in the sense of L^2-norm.

4 Simulation examples

In this section, one numerical example is presented to demonstrate the validity of the designed method. In order to describe the stability of the system which is associated with the increase of the iterations, we denote the total energy in the kth iteration as $\mathscr{E}_k = \|u_k\|_\infty = \max_{t\in[0,T]} \|u_k(t)\|$.

Example 4.1 Consider the following impulsive fractional controlled systems:

$$\begin{cases} (D_{0,t}^{0.5}x_k)(t) = -x_k(t) + t(t-0.5)^2 u_k(t) + \frac{t}{k+1}, & t \in [0,1] \setminus \{0.5\}, \\ \lim_{t\to 0+}(I_{0,t}^{0.5}x_k)(t) = 0, \\ \Theta(I_{0,t_1}^{0.5}x)(t_1^-) = t_1^- x_k(t_1^-), \quad t_1 = 0.5, \\ y_k(t) = x_k(t) + 1.2u_k(t) + e^{-kt}, \end{cases} \tag{21}$$

and the P-type ILC

$$u_{k+1}(t) = u_k(t) + P_o e_k(t).$$

Figure 1 The system output and the tracking error.

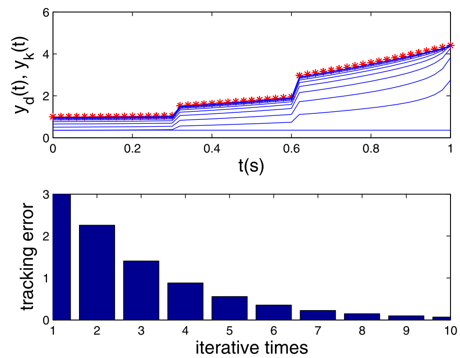
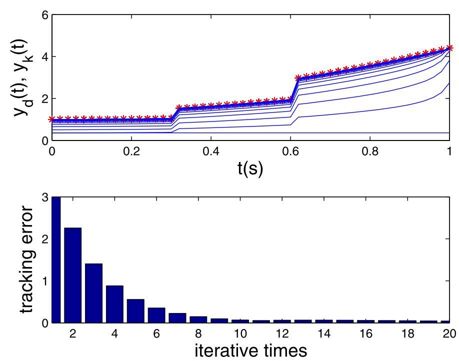


Figure 2 The system output and the tracking error.



Set $\alpha = 0.5$, $\mu = -1$, $f(t,x_k,u_k) = t(t-0.5)^2u_k$, $G_j(t_j^-, x_k(t_j^-)) = t_1^- x_k(t_1^-)$, $j = 1$, $\xi_k(t) = \frac{t}{k+1}$, $\eta_k(t) = e^{-kt}$, $t \in [0,1]$. Obviously, $L_f(t) = 0$, $I_f(t) = t(t-0.5)^2$, $L_{G_j}(t) = 0.5$, $t \in [0,1]$. It is not difficult to verify that $M_1 = 0.25$, $M_2 = 0.5$. Set $M_L = 0.5$. Then (A_1)-(A_3) are satisfied.

Case 1: The original reference trajectory is a piecewise continuous function,

$$y_d(t) = \begin{cases} \frac{1}{2}t^3 + 1, & t \in [0,0.3], \\ \frac{1}{2}t^3 + (t^4 + 0.3) + 1, & t \in (0.3,0.6], \\ \frac{1}{2}t^3 + (t^4 + 0.3) + (t^5 + 0.6) + 1, & t \in (0.6,1]. \end{cases}$$

(i) We set $u_k(0) = 0$ and $B = 1.2$, $P_o = 0.3$. Obviously, $|1 - BP_o| = 0.64 < 1$. $\xi_k(t) = \frac{t}{k+1}$, $\eta_k(t) = e^{-kt}$, $t \in [0,1]$. All the conditions of Theorem 3.1 are satisfied. Meanwhile, the disturbances have asymptotic convergence, then $y_k(t)$ is uniform convergent to $y_d(t)$, for $t \in [0,1]$.

⋆ Figure 1 shows the output y_k of equation (21) of the 10th iteration and the reference trajectory y_d. The lower figure of Figure 1 shows the ∞-norm of the tracking error in each iteration and the error is 0.0647.

⋆ Figure 2 shows the output y_k of equation (21) of the 20th iteration and the reference trajectory y_d. The lower figure of Figure 2 shows the ∞-norm of the tracking error in each iteration and the error is 0.0405.

(ii) We set $u_k(0) = 0$ and $B = 1.2$, $P_o = 0.7$. Obviously, $|1 - BP_o| = 0.16 < 1$. Then all the conditions of Theorem 3.1 are satisfied. The $y_k(t)$ is uniform convergent to $y_d(t)$, for $t \in [0,1]$.

Figure 3 The system output and the tracking error.

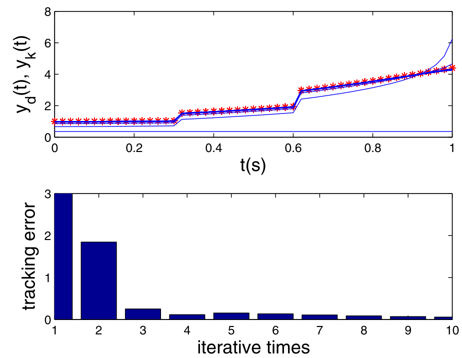
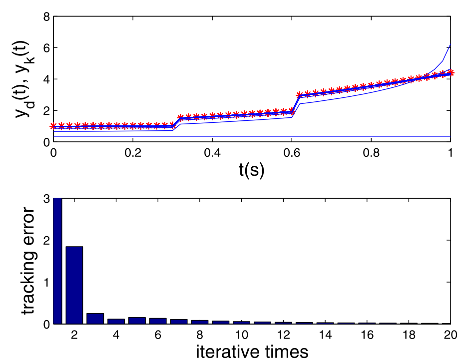


Figure 4 The system output and the tracking error.



⋆ Figure 3 shows the output y_k of equation (21) of the 10th iteration and the reference trajectory y_d. The lower figure of Figure 3 shows the ∞-norm of the tracking error in each iteration and the error is 0.0575.

⋆ Figure 4 shows the output y_k of equation (21) of the 20th iteration and the reference trajectory y_d. The lower figure of Figure 4 shows the ∞-norm of the tracking error in each iteration and the error is 0.0153.

Conclusions:

From Figures 1 and 2 or 3 and 4, we find that the tracking error decreases with k increasing.

From Figures 1 and 3 or 2 and 4, we find that the tracking error decreases with P_o increasing.

Case 2: The second original reference trajectory is continuous,

$$y_d(t) = 6t^2(1-t) + t - 1.$$

(iii) We set $u_k(0) = 0$ and $B = 1.2$, $P_o = 0.3$. Obviously, $|1 - BP_o| = 0.64 < 1$. All the conditions of Theorem 3.1 are satisfied. Meanwhile, the disturbances are asymptotic convergence, then $y_k(t)$ uniform convergent to $y_d(t)$, for $t \in [0,1]$.

⋆ Figure 5 shows the output y_k of equation (21) of the 10th iteration and the reference trajectory y_d. The lower figure of Figure 5 shows the ∞-norm of the tracking error in each iteration and the error is 0.01754.

Figure 5 The system output and the tracking error.

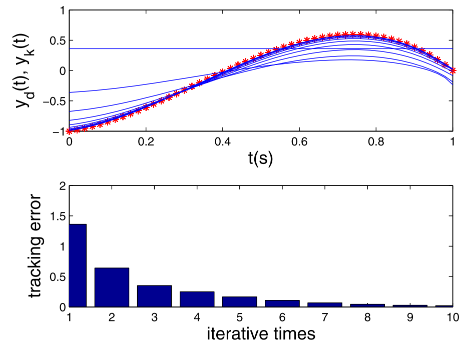


Figure 6 The system output and the tracking error.

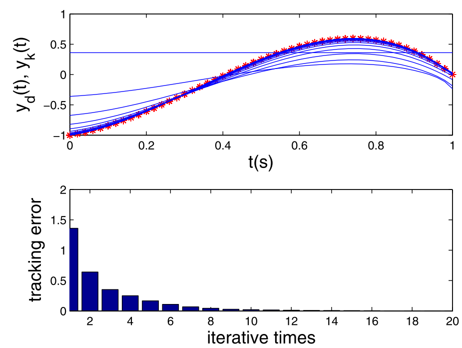
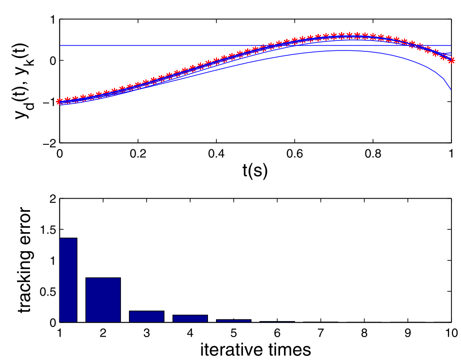


Figure 7 The system output and the tracking error.



⋆ Figure 6 shows the output y_k of equation (21) of the 20th iteration and the reference trajectory y_d. The lower figure of Figure 6 shows the ∞-norm of the tracking error in each iteration and the error is 0.00033936.

(iv) We set $u_k(0) = 0$ and $B = 1.2$, $P_o = 0.7$. Obviously, $|1 - BP_o| = 0.16 < 1$. Then all the conditions of Theorem 3.1 are satisfied. The $y_k(t)$ uniform convergent to $y_d(t)$, for $t \in [0, 1]$.

⋆ Figure 7 shows the output y_k of equation (21) of the 10th iteration and the reference trajectory y_d. The lower figure of Figure 7 shows the ∞-norm of the tracking error in each iteration and the error is 0.00058637.

⋆ Figure 8 shows the output y_k of equation (21) of the 20th iteration and the reference trajectory y_d. The lower figure of Figure 8 shows the ∞-norm of the tracking error in each iteration and the error is 0.00017541.

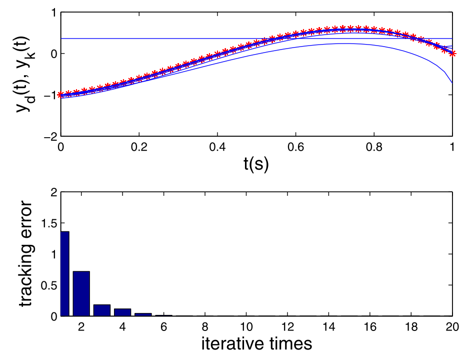


Figure 8 The system output and the tracking error.

Conclusions:

From Figures 5 and 6 or 7 and 8, we can see that the tracking error decreases with k increasing.

From Figures 5 and 7 or 6 and 8, we can see that the tracking error decreases with P_o increasing.

5 Conclusions

Due to the fact that impulse phenomenon and fractional-order systems widely exist in engineering, we investigated the P-type learning laws for impulsive Riemann-Liouville fractional-order controlled systems ($0 < \alpha < 1$) with initial state offset bounded. We obtain open-loop and closed-loop P-type robust convergence results in the sense of $(PC_{1-\alpha}, \lambda)$-norm $\|\cdot\|_{PC_{1-\alpha},\lambda}$ via an impulsive Gronwall inequality. Furthermore, one example is given to verify the effectiveness and feasibility of the obtained results. The proposed scheme can deal with the robust convergence of impulsive Riemann-Liouville fractional systems. We would like to point out that it is possible to extend our results to other impulsive fractional-order models such as non-instantaneous impulsive fractional-order systems and so on.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

This work was carried out in collaboration between all authors. ZJL, WW, and JRW proved the theorems, interpreted the results, and wrote the article. All authors defined the research theme, and they read and approved the manuscript.

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Acknowledgements

This work is partially supported by NNSFC (No. 11661016;11261011) and Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006), 2011 Collaborative Innovation Project of Guizhou Province ([2014]02), Unite Foundation of Guizhou Province ([2015]7640), and Graduate ZDKC ([2015]003).

Received: 14 July 2016 Accepted: 27 September 2016 Published online: 11 October 2016

References

1. Bristow, D, Tharayil, M, Alleyne, AG: A survey of iterative learning control. IEEE Control Syst. **26**, 96-114 (2006)
2. Xu, JX, Xu, J: On iterative learning for different tracking tasks in the presence of time-varying uncertainties. IEEE Trans. Syst. Man Cybern., Part B, Cybern. **34**, 589-597 (2004)
3. Ahn, HS, Moore, KL, Chen, YQ: Iterative Learning Control: Robustness and Monotonic Convergence in the Iteration Domain. Springer, Berlin (2007)
4. Luo, Y, Chen, YQ: Fractional order [proportional derivative] controller for a class of fractional order systems. Automatica **45**, 2446-2450 (2009)

5. Li, Y, Chen, YQ, Ahn, HS: On the PD^{α}-type iterative learning control for the fractional-order nonlinear systems. In: Proceedings of the American Control Conference, pp. 4320-4325 (2011)
6. Li, Y, Chen, YQ, Ahn, HS, Tian, G: A survey on fractional-order iterative learning control. J. Optim. Theory Appl. **156**, 127-140 (2013)
7. Bien, ZZ, Xu, JX (eds.): Iterative Learning Control: Analysis, Design, Integration and Applications. Springer, Berlin (2012)
8. Li, ZG, Chang, YW, Soh, YC: Analysis and design of impulsive control systems. IEEE Trans. Autom. Control **46**, 894-897 (2001)
9. Sun, MX: Robust convergence analysis of iterative learning control systems. Control Theory Appl. **3**, 320-326 (1998)
10. Lee, HS, Bien, Z: Design issues on robustness and convergence of iterative learning controller. Intell. Autom. Soft Comput. **8**, 95-106 (2002)
11. Lazarević, MP, Tzekis, P: Robust second-order PD^{α} type iterative learning control for a class of uncertain fractional order singular systems. J. Vib. Control **22**, 2004-2018 (2016)
12. Bao, H, Park, JH, Cao, J: Adaptive synchronization of fractional-order memristor-based neural networks with time delay. Nonlinear Dyn. **82**, 1343-1354 (2015)
13. Bao, H, Cao, J: Projective synchronization of fractional-order memristor-based neural networks. Neural Netw. **63**, 1-9 (2015)
14. Huang, X, Zhao, Z, Wang, Z, Li, Y: Chaos and hyperchaos in fractional-order cellular neural networks. Neurocomputing **94**, 13-21 (2012)
15. Shen, J, Lam, J: State feedback H_{∞} control of commensurate fractional-order systems. Int. J. Syst. Sci. **45**, 363-372 (2014)
16. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
17. Bai, C: Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative. J. Math. Anal. Appl. **384**, 211-231 (2011)
18. Li, K, Peng, J, Jia, J: Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives. J. Funct. Anal. **263**, 476-510 (2012)
19. Jankowshi, T: Fractional equations of Volterra type involving a Riemann-Liouville derivative. Appl. Math. Lett. **26**, 344-350 (2013)
20. Ren, Y, Qin, Y, Sakthivel, R: Existence results for fractional order semilinear integro-differential evolution equations with infinite delay. Integral Equ. Oper. Theory **67**, 33-49 (2010)
21. Wang, J, Fečkan, M, Zhou, Y: Ulam's type stability of impulsive ordinary differential equations. J. Math. Anal. Appl. **395**, 258-264 (2012)
22. Wang, J, Zhou, Y, Fečkan, M: On recent developments in the theory of boundary value problems for impulsive fractional differential equations. Comput. Math. Appl. **64**, 3008-3020 (2012)
23. Wang, J, Zhang, Y: On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives. Appl. Math. Lett. **39**, 85-90 (2015)
24. Wang, J, Ibrahim, AG, Fečkan, M: Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces. Appl. Math. Comput. **257**, 103-118 (2015)
25. Wang, J, Fečkan, M, Zhou, Y: A survey on impulsive fractional differential equations. Fract. Calc. Appl. Anal. **19**, 806-831 (2016)
26. Ganesh, R, Sakthivel, R, Mahmudov, NI: Approximate controllability of fractional functional equations with infinite delay. Topol. Methods Nonlinear Anal. **43**, 345-364 (2014)
27. Sakthivel, R, Ganesh, R, Anthoni, SM: Approximate controllability of fractional nonlinear differential inclusions. Appl. Math. Comput. **225**, 708-717 (2013)
28. Li, Y, Chen, YQ, Ahn, HS: Fractional-order iterative learning control for fractional-order linear systems. Asian J. Control **13**, 54-63 (2011)
29. Lan, YH, Zhou, Y: D^{α} type iterative learning control for fractional order linear time-delay systems. Asian J. Control **15**, 669-677 (2013)
30. Wang, J, Fěckan, M, Zhou, Y: On the new concept of solutions and existence results for impulsive fractional evolution equations. Dyn. Partial Differ. Equ. **8**, 345-361 (2011)
31. Wang, J, Zhou, Y, Fěckan, M: Nonlinear impulsive problems for fractional differential equations and Ulam stability. Comput. Math. Appl. **64**, 3389-3405 (2012)
32. Liu, ZH, Bin, MJ: Approximate controllability of impulsive Riemann-Liouville fractional equations in Banach spaces. J. Integral Equ. Appl. **26**, 527-551 (2014)
33. Wang, Q, Lu, D, Fang, Y: Stability analysis of impulsive fractional differential systems with delay. Appl. Math. Lett. **40**, 1-6 (2015)
34. Wang, J, Fečkan, M, Zhou, Y: Presentation of solutions of impulsive fractional Langevin equations and existence results. Eur. Phys. J. Spec. Top. **222**, 1857-1874 (2013)
35. Wang, J, Zhou, Y: Analysis of nonlinear fractional control systems in Banach spaces. Nonlinear Anal. TMA **74**, 5929-5942 (2011)
36. Wang, J, Zhou, Y: A class of fractional evolution equations and optimal controls. Nonlinear Anal., Real World Appl. **12**, 262-272 (2011)
37. Wang, J, Zhou, Y, Wei, W: Fractional Schrödinger equations with potential and optimal controls. Nonlinear Anal., Real World Appl. **13**, 2755-2766 (2012)
38. Wang, J, Zhou, Y: Existence and controllability results for fractional semilinear differential inclusions. Nonlinear Anal., Real World Appl. **12**, 3642-3653 (2011)
39. Wang, J, Fěckan, M, Zhou, Y: Controllability of Sobolev type fractional evolution equations. Dyn. Partial Differ. Equ. **11**, 71-87 (2014)
40. Liu, S, Wang, J, Wei, W: A study on iterative learning control for impulsive differential equations. Commun. Nonlinear Sci. Numer. Simul. **24**, 4-10 (2015)
41. Yu, X, Wang, J: Uniform design and analysis of iterative learning control for a class of impulsive first-order distributed parameter systems. Adv. Differ. Equ. **2015**, 261 (2015)

42. Liu, S, Wang, J, Wei, W: Iterative learning control based on a noninstantaneous impulsive fractional-order system. J. Vib. Control **22**, 1972-1979 (2016)
43. Yu, X, Debbouche, A, Wang, J: On the iterative learning control of fractional impulsive evolution equations in Banach spaces. Math. Methods Appl. Sci. (2015). doi:10.1002/mma.3726
44. Liu, S, Debbouche, A, Wang, J: On the iterative learning control for stochastic impulsive differential equations with randomly varying trial lengths. J. Comput. Appl. Math. (2015). doi:10.1016/j.cam.2015.10.028
45. Gorenflo, R, Kilbas, AA, Mainardi, F, Rogosin, SV: Mittag-Leffler Functions, Related Topics and Applications. Springer, Berlin (2014)