Xie and Viouonu *Boundary Value Problems* (2016) 2016:136
DOI 10.1186/s13661-016-0642-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Some new results on the boundary behaviors of harmonic functions with integral boundary conditions

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Abstract
In this paper, using a generalized Carleman formula, we prove two new results on the boundary behaviors of harmonic functions with integral boundary conditions in a smooth cone, which generalize some recent results.

Keywords: boundary behavior; harmonic function; boundary condition

1 Introduction

Let $\mathbf{R}^n$ ($n \geq 2$) be the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $V = (X, y)$, where $X = (x_1, x_2, \dots, x_{n-1})$. The boundary and the closure of a set E in $\mathbf{R}^n$ are denoted by ∂E and $\overline{E}$, respectively.

We introduce a system of spherical coordinates (l, Λ), $\Lambda = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in $\mathbf{R}^n$ that are related to Cartesian coordinates $(x_1, x_2, \dots, x_{n-1}, y)$ by $y = l \cos\theta_1$.

The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. For simplicity, a point $(1, \Lambda)$ on $\mathbf{S}^{n-1}$ and the set $\{\Lambda; (1, \Lambda) \in \Gamma\}$ for a set $\Gamma \subset \mathbf{S}^{n-1}$ are often identified with Λ and Γ, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Gamma \subset \mathbf{S}^{n-1}$, the set $\{(l, \Lambda) \in \mathbf{R}^n; l \in \Xi, (1, \Lambda) \in \Gamma\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Gamma$.

We denote the set $\mathbf{R}_+ \times \Gamma$ in $\mathbf{R}^n$ with the domain Γ on $\mathbf{S}^{n-1}$ by $T_n(\Gamma)$. We call it a cone. In particular, the half-space $\mathbf{R}_+ \times \mathbf{S}_+^{n-1}$ is denoted by $T_n(\mathbf{S}_+^{n-1})$. The sets $I \times \Gamma$ and $I \times \partial\Gamma$ with an interval on $\mathbf{R}$ are denoted by $T_n(\Gamma; I)$ and $\mathcal{S}_n(\Gamma; I)$, respectively. We denote $T_n(\Gamma) \cap S_l$ by $\mathcal{S}_n(\Gamma; l)$, and we denote $\mathcal{S}_n(\Gamma; (0, +\infty))$ by $\mathcal{S}_n(\Gamma)$.

The ordinary Poisson in $T_n(\Gamma)$ is defined by

$$c_n \mathbb{PI}_\Gamma(V, W) = \frac{\partial \mathbb{G}_\Gamma(V, W)}{\partial n_W},$$

where $\partial/\partial n_W$ denotes the differentiation at W along the inward normal into $T_n(\Gamma)$, and $\mathbb{G}_\Gamma(V, W)$ ($P, Q \in T_n(\Gamma)$) is the Green function in $T_n(\Gamma)$. Here, $c_2 = 2$ and $c_n = (n-2)w_n$ for $n \geq 3$, where w_n is the surface area of $\mathbf{S}^{n-1}$.

Let Δ_n^* be the spherical part of the Laplace operator, and Γ be a domain on $\mathbf{S}^{n-1}$ with smooth boundary $\partial\Gamma$. Consider the Dirichlet problem (see [1])

$$(\Delta_n^* + \tau)\psi = 0 \quad \text{on } \Gamma,$$

$$\psi = 0 \quad \text{on } \partial\Gamma.$$

We denote the least positive eigenvalue of this boundary problem by τ and the normalized positive eigenfunction corresponding to τ by $\psi(\Lambda)$. In the sequel, for brevity, we shall write χ instead of $\aleph^+ - \aleph^-$, where

$$2\aleph^{\pm} = -n + 2 \pm \sqrt{(n-2)^2 + 4\tau}.$$

The estimate we deal with has a long history tracing back to known Matsaev's estimate of harmonic functions from below in the half-plane (see, e.g., Levin [2], p.209).

Theorem A *Let A_1 be a constant, and let $h(z)$ ($|z| = R$) be harmonic on $T_2(\mathbf{S}_+^1)$ and continuous on $\overline{T_2(\mathbf{S}_+^1)}$. Suppose that*

$$h(z) \le A_1 R^{\rho}, \quad z \in T_2(\mathbf{S}_+^1), R > 1, \rho > 1,$$

and

$$|h(z)| \le A_1, \quad R \le 1, z \in \overline{T_2(\mathbf{S}_+^1)}.$$

Then

$$h(z) \ge -A_1 A_2 (1 + R^{\rho}) \sin^{-1}\alpha,$$

where $z = Re^{i\alpha} \in T_2(\mathbf{S}_+^1)$, and A_2 is a constant independent of A_1, R, α, and the function $h(z)$.

In 2014, Xu and Zhou [3] considered Theorem A in the half-space. Pan *et al.* [4], Theorems 1.2 and 1.4, obtained similar results, slightly different from the following Theorem B.

Theorem B *Let A_3 be a constant, and $h(V)$ ($|V| = R$) be harmonic on $T_n(\mathbf{S}_+^{n-1})$ and continuous on $\overline{T_n(\mathbf{S}_+^{n-1})}$. If*

$$h(V) \le A_3 R^{\rho}, \quad P \in T_n(\mathbf{S}_+^{n-1}), R > 1, \rho > n-1, \tag{1.1}$$

and

$$|h(V)| \le A_3, \quad R \le 1, P \in \overline{T_n(\mathbf{S}_+^{n-1})}, \tag{1.2}$$

then

$$h(V) \ge -A_3 A_4 (1 + R^{\rho}) \cos^{1-n}\theta_1,$$

where $V \in T_n(\mathbf{S}_+^{n-1})$, and A_4 is a constant independent of A_3, R, θ_1, and the function $h(V)$.

Recently, Pang and Ychussie [5], Theorem 1, further extended Theorems A and B and proved Matsaev's estimates for harmonic functions in a smooth cone.

Theorem C *Let K be a constant, and $h(V)$ ($V = (R,\Lambda)$) be harmonic on $T_n(\Gamma)$ and continuous on $\overline{T_n(\Gamma)}$. If*

$$h(V) \leq KR^{\rho(R)}, \quad V = (R,\Lambda) \in T_n\bigl(\Gamma;(1,\infty)\bigr), \quad \rho(R) > \aleph^+, \tag{1.3}$$

and

$$h(V) \geq -K, \quad R \leq 1, \quad V = (R,\Lambda) \in \overline{T_n(\Gamma)}, \tag{1.4}$$

then

$$h(V) \geq -KM\left(1 + \left(\frac{N+1}{N}R\right)^{\rho(\frac{N+1}{N}R)}\right)\psi^{1-n}(\Lambda),$$

where $V \in T_n(\Gamma)$, N (≥ 1) is a sufficiently large number, and M is a constant independent of K, R, $\psi(\Lambda)$, and the function $h(V)$.

In this paper, we obtain two new results on the lower bounds of harmonic functions with integral boundary conditions in a smooth cone (Theorems 1 and 2), which further extend Theorems A, B, and C. Our proofs are essentially based on the Riesz decomposition theorem (see [6]) and a modified Carleman formula for harmonic functions in a smooth cone (see [5], Lemma 1).

In order to avoid complexity of our proofs, we assume that $n \geq 3$. However, our results in this paper are also true for $n = 2$. We use the standard notations $h^+ = \max\{h, 0\}$ and $h^- = -\min\{h, 0\}$. All constants appearing further in expressions will be always denoted M because we do not need to specify them. We will always assume that $\eta(t)$ and $\rho(t)$ are nondecreasing real-valued functions on an interval $[1, +\infty)$ and $\rho(t) > \aleph^+$ for any $t \in [1, +\infty)$.

2 Main results

First of all, we shall state the following result, which further extends Theorem C under weak boundary integral conditions.

Theorem 1 *Let $h(V)$ ($V = (R,\Lambda)$) be harmonic on $T_n(\Gamma)$ and continuous on $\overline{T_n(\Gamma)}$.*

Suppose that the following conditions (I) *and* (II) *are satisfied*:

(I) *For any $V = (R,\Lambda) \in T_n(\Gamma;(1,\infty))$, we have*

$$\int_{S_n(\Gamma;(1,R))} h^- t^{\aleph^-} \partial\psi/\partial n \, d\sigma_W \leq M\eta(R)(cR)^{\rho(cR)-\aleph^+} \tag{2.1}$$

and

$$\chi \int_{S_n(\Gamma;R)} h^- R^{\aleph^- -1} \psi \, dS_R \leq M\eta(R)(cR)^{\rho(cR)-\aleph^+}. \tag{2.2}$$

(II) *For any $V = (R,\Lambda) \in T_n(\Gamma;(0,1])$, we have*

$$h(V) \geq -\eta(R). \tag{2.3}$$

Then

$$h(V) \geq -M\eta(R)\bigl(1+(cR)^{\rho(cR)}\bigr)\psi^{1-n}(\Lambda),$$

where $V \in T_n(\Gamma)$, N (≥ 1) *is a sufficiently large number, and* M *is a constant independent of* R, $\psi(\Lambda)$, *and the functions* $\eta(R)$ *and* $h(V)$.

Remark 1 From the proof of Theorem 1 it is easy to see that condition (I) in Theorem 1 is weaker than that in Theorem C in the case $c \equiv (N+1)/N$ and $\eta(R) \equiv K$, where N (≥ 1) is a sufficiently large number, and K is a constant.

Theorem 2 *The conclusion of Theorem* 1 *remains valid if* (I) *in Theorem* 1 *is replaced by*

$$h(V) \leq \eta(R)R^{\rho(R)}, \quad V = (R,\Lambda) \in T_n\bigl(\Gamma;(1,\infty)\bigr). \tag{2.4}$$

Remark 2 In the case $c \equiv (N+1)/N$ and $\eta(R) \equiv K$, where N (≥ 1) is a sufficiently large number and K is a constant, Theorem 2 reduces to Theorem C.

3 Proof of Theorem 1

By the Riesz decomposition theorem (see [6]) we have

$$-h(V) = \int_{\mathcal{S}_n(\Gamma;(0,R))} \mathcal{PI}_\Gamma(V,W) - h(W)\,d\sigma_W + \int_{\mathcal{S}_n(\Gamma;R)} \frac{\partial \mathbb{G}_{\Gamma,R}(V,W)}{\partial R} - h(W)\,dS_R, \tag{3.1}$$

where $V = (l,\Lambda) \in T_n(\Gamma;(0,R))$.

We next distinguish three cases.

Case 1. $V = (l,\Lambda) \in T_n(\Gamma;(5/4,\infty))$ and $R = 5l/4$.

Since $-h(V) \leq h^-(V)$, we have

$$-h(V) = \sum_{i=1}^{4} U_i(V) \tag{3.2}$$

from (3.1), where

$$U_1(V) = \int_{\mathcal{S}_n(\Gamma;(0,1])} \mathcal{PI}_\Gamma(V,W) - h(W)\,d\sigma_W,$$

$$U_2(V) = \int_{\mathcal{S}_n(\Gamma;(1,4l/5])} \mathcal{PI}_\Gamma(V,W) - h(W)\,d\sigma_W,$$

$$U_3(V) = \int_{\mathcal{S}_n(\Gamma;(4l/5,R))} \mathcal{PI}_\Gamma(V,W) - h(W)\,d\sigma_W,$$

and

$$U_4(V) = \int_{\mathcal{S}_n(\Gamma;R)} \mathcal{PI}_\Gamma(V,W) - h(W)\,d\sigma_W.$$

We have the following estimates:

$$U_1(V) \leq M\eta(R)\psi(\Lambda) \tag{3.3}$$

and

$$U_2(V) \le M\eta(R)(cR)^{\rho(cR)}\psi(\Lambda) \tag{3.4}$$

from [7, 8] and (2.1).

We consider the inequality

$$U_3(V) \le U_{31}(V) + U_{32}(V), \tag{3.5}$$

where

$$U_{31}(V) = M\int_{\mathcal{S}_n(\Gamma;(4l/5,R))} \frac{-h(W)\psi(\Lambda)}{t^{n-1}} \frac{\partial\phi(\Phi)}{\partial n_\Phi}\, d\sigma_W$$

and

$$U_{32}(V) = Mr\psi(\Lambda)\int_{\mathcal{S}_n(\Gamma;(4l/5,R))} \frac{-h(W)l\psi(\Lambda)}{|V-W|^n} \frac{\partial\phi(\Phi)}{\partial n_\Phi}\, d\sigma_W.$$

We first have

$$U_{31}(V) \le M\eta(R)(cR)^{\rho(cR)}\psi(\Lambda) \tag{3.6}$$

from (2.1).

We shall estimate $U_{32}(V)$. Take a sufficiently small positive number d such that

$$\mathcal{S}_n\big(\Gamma;(4l/5,R)\big) \subset B(P,l/2)$$

for any $V = (l,\Lambda) \in \Pi(d)$, where

$$\Pi(d) = \Big\{V = (l,\Lambda) \in T_n(\Gamma); \inf_{(1,z)\in\partial\Gamma}\big|(1,\Lambda) - (1,z)\big| < d, 0 < r < \infty\Big\},$$

and divide $T_n(\Gamma)$ into two sets $\Pi(d)$ and $T_n(\Gamma) - \Pi(d)$.

If $V = (l,\Lambda) \in T_n(\Gamma) - \Pi(d)$, then there exists a positive d' such that $|V - W| \ge d'l$ for any $Q \in \mathcal{S}_n(\Gamma)$, and hence

$$U_{32}(V) \le M\eta(R)(cR)^{\rho(cR)}\psi(\Lambda), \tag{3.7}$$

which is similar to the estimate of $U_{31}(V)$.

We shall consider the case $V = (l,\Lambda) \in \Pi(d)$. Now put

$$H_i(V) = \big\{W \in \mathcal{S}_n\big(\Gamma;(4l/5,R)\big); 2^{i-1}\delta(V) \le |V-W| < 2^i\delta(V)\big\},$$

where

$$\delta(V) = \inf_{Q\in\partial T_n(\Gamma)} |V - W|.$$

Since $\mathcal{S}_n(\Gamma) \cap \{W \in \mathbf{R}^n : |V - W| < \delta(V)\} = \emptyset$, we have

$$U_{32}(V) = M \sum_{i=1}^{i(V)} \int_{H_i(V)} \frac{-h(W) r\psi(\Lambda)}{|V - W|^n} \frac{\partial \psi(\Phi)}{\partial n_\Phi} \, d\sigma_W,$$

where $i(V)$ is a positive integer satisfying

$$2^{i(V)-1}\delta(V) \le \frac{r}{2} < 2^{i(V)}\delta(V).$$

Since $r\psi(\Lambda) \le M\delta(V)$ $(V = (l, \Lambda) \in T_n(\Gamma))$, similarly to the estimate of $U_{31}(V)$, we obtain

$$\int_{H_i(V)} \frac{-h(W) r\psi(\Lambda)}{|V - W|^n} \frac{\partial \psi(\Phi)}{\partial n_\Phi} \, d\sigma_W \le M\eta(R)(cR)^{\rho(cR)}\psi^{1-n}(\Lambda)$$

for $i = 0, 1, 2, \ldots, i(V)$.
So

$$U_{32}(V) \le M\eta(R)(cR)^{\rho(cR)}\psi^{1-n}(\Lambda). \tag{3.8}$$

From (3.5), (3.6), (3.7), and (3.8) we see that

$$U_3(V) \le M\eta(R)(cR)^{\rho(cR)}\psi^{1-n}(\Lambda). \tag{3.9}$$

On the other hand, we have from (2.2) that

$$U_4(V) \le M\eta(R)R^{\rho(cR)}\psi(\Lambda). \tag{3.10}$$

We thus obtain from (3.3), (3.4), (3.9), and (3.10) that

$$-h(V) \le M\eta(R)\bigl(1 + (cR)^{\rho(cR)}\bigr)\psi^{1-n}(\Lambda). \tag{3.11}$$

Case 2. $V = (l, \Lambda) \in T_n(\Gamma; (4/5, 5/4])$ and $R = 5l/4$.
It follows from (3.1) that

$$-h(V) = U_1(V) + U_5(V) + U_4(V),$$

where $U_1(V)$ and $U_4(V)$ are defined as in Case 1, and

$$U_5(V) = \int_{\mathcal{S}_n(\Gamma;(1,R))} \mathcal{PI}_\Gamma(V, W) - h(W) \, d\sigma_W.$$

Similarly to the estimate of $U_3(V)$ in Case 1, we have

$$U_5(V) \le M\eta(R)(cR)^{\rho(cR)}\psi^{1-n}(\Lambda),$$

which, together with (3.3) and (3.10), gives (3.11).

Case 3. $V = (l, \Lambda) \in T_n(\Gamma; (0, 4/5])$.

It is evident from (2.3) that

$$-h \leq \eta(R),$$

which also gives (3.11).

Finally, from (3.11) we have

$$h(V) \geq -\eta(R)M\bigl(1 + (cR)^{\rho(cR)}\bigr)\psi^{1-n}(\Lambda),$$

which is the conclusion of Theorem 1.

4 Proof of Theorem 2

We first apply a new type of Carleman's formula for harmonic functions (see [5], Lemma 1) to $h = h^+ - h^-$ and obtain

$$\begin{aligned} &\chi \int_{\mathcal{S}_n(\Gamma;R)} h^+ R^{\aleph^- -1}\psi\, dS_R \\ &\quad + \int_{\mathcal{S}_n(\Gamma;(1,R))} h^+\bigl(t^{\aleph^-} - t^{\aleph^+}R^{-\chi}\bigr)\partial\psi/\partial n\, d\sigma_W + d_1 + d_2 R^{-\chi} \\ &\quad = \chi \int_{\mathcal{S}_n(\Gamma;R)} h^- R^{\aleph^- -1}\psi\, dS_R + \int_{\mathcal{S}_n(\Gamma;(1,R))} h^-\bigl(t^{\aleph^-} - t^{\aleph^+}R^{-\chi}\bigr)\partial\psi/\partial n\, d\sigma_W, \end{aligned} \tag{4.1}$$

where dS_R denotes the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_R, and $\partial/\partial n$ denotes differentiation along the interior normal.

It is easy to see that

$$\chi \int_{\mathcal{S}_n(\Gamma;R)} h^+ R^{\aleph^- -1}\psi\, dS_R \leq M\eta(R)(cR)^{\rho(cR)-\aleph^+} \tag{4.2}$$

and

$$\int_{\mathcal{S}_n(\Gamma;(1,R))} h^+\bigl(t^{\aleph^-} - t^{\aleph^+}R^{-\chi}\bigr)\partial\psi/\partial n\, d\sigma_W \leq M\eta(R)(cR)^{\rho(cR)-\aleph^+} \tag{4.3}$$

from (2.4).

We remark that

$$d_1 + d_2 R^{-\chi} \leq M\eta(R)(cR)^{\rho(cR)-\aleph^+}. \tag{4.4}$$

We have (2.2) and

$$\int_{\mathcal{S}_n(\Gamma;(1,R))} h^-\bigl(t^{\aleph^-} - t^{\aleph^+}R^{-\chi}\bigr)\partial\psi/\partial n\, d\sigma_W \leq M\eta(R)(cR)^{\rho(cR)-\aleph^+}. \tag{4.5}$$

from (4.1), (4.2), (4.3), and (4.4).

Hence, (4.5) gives (2.1), which, together with Theorem 1, gives Theorem 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CV completed the main study. XX responded point by point to each reviewer comments and corrected the final proof. Both authors read and approved the final manuscript.

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Acknowledgements
This work was supported by the National Natural Science Foundation of China under Grant no. 61401368. We are grateful to the editor and anonymous reviewers for their valuable comments and corrections that helped improve the original version of this paper.

Received: 21 March 2016 Accepted: 11 July 2016 Published online: 27 July 2016

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