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Research article

A regularity criterion of smooth solution for the 3D viscous Hall-MHD equations

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Abstract: In this work, we investigate the regularity criterion for the solution of the Hall-MHD system in three-dimensions. It is proved that if the pressure π and the gradient of magnetic field ∇B satisfies some kind of space-time integrable condition on $[0, T]$, then the corresponding solution keeps smoothness up to time T. This result improves some previous works to the Morrey space $\dot{\mathcal{M}}_{2,\frac{3}{r}}$ for $0 \leq r < 1$ which is larger than $L^{\frac{3}{r}}$.

Keywords: Hall-MHD; regularity criterion; Morrey space; Besov space $\dot{B}^{-1}_{\infty,\infty}$
Mathematics Subject Classification: 35Q35, 76D03

1. Introduction

This work is devoted to the study of the regularity criterion of smooth solutions for the 3D Hall-magnetohydrodynamics (Hall-MHD) equations [1, 29] :

$$\begin{cases} \partial_t u + u \cdot \nabla u - B \cdot \nabla B - \Delta u + \nabla \left(\pi + \frac{|B|^2}{2} \right) = 0, \\ \partial_t B + u \cdot \nabla B - B \cdot \nabla u + \text{curl}\,[\text{curl}\, B \times B] - \Delta B = 0, \\ \nabla \cdot u = \nabla \cdot B = 0, \\ (u, B)(x, 0) = (u_0(x), B_0(x)), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^3$ and $t \geq 0$. Here $u = u(x, t) \in \mathbb{R}^3$, $B = B(x, t) \in \mathbb{R}^3$ and $\pi = \pi(x, t)$ are non-dimensional quantities corresponding to the fluid velocity field, the magnetic field and the pressure at the point

(x,t), while $u_0(x)$ and $B_0(x)$ are the given initial velocity and initial magnetic field with $\nabla\cdot u_0 = 0$ and $\nabla\cdot B_0 = 0$, respectively. The Hall-MHD equations are of relevance to study a number of models coming from magnetic reconnection in space plasmas [19], star formation [2] and also neutron stars [32].

The Hall term curl [curl $B\times B$] included in $(1.1)_2$ due to the Ohm's law plays an important role in magnetic reconnection which is happening in the case of large magnetic shear. For the physical background of the Hall-MHD, we refer the readers to [19, 29] and references therein. When the Hall term is absent, it is obvious to see that the system (1.1) reduces to the classical magnetohydrodynamic (MHD) equations.

The global existence of weak solutions and the local well-posedness of classical solution in the whole space $\mathbb{R}^3$ were established by Chae-Degond-Liu in [4]. But due to the presence of Navier-Stokes equations in the system (1.1) whether this unique local solution can exist globally is an outstanding challenge problem. For this reason, there have been a lot of literatures devoted to finding sufficient conditions to ensure the smoothness of the solutions (see [4, 5, 7–13, 16–18, 25–27, 33–35, 38, 39] and reference therein. Meanwhile, in [4], the authors obtained a blow-up criterion and the global existence of smooth solution for small initial data. Later, both blow-up criterion and the small data results were refined by Chae-Lee [5]. In particular, Chae and Lee proved the following regularity criteria

$$u\in L^{\frac{2p}{p-3}}(0,T;L^p(\mathbb{R}^3)) \quad\text{and}\quad \nabla B\in L^{\frac{2\beta}{\beta-3}}(0,T;L^\beta(\mathbb{R}^3)) \ \text{ with } \ 3<p,\beta\le\infty \tag{1.2}$$

or

$$u\in L^2(0,T;BMO(\mathbb{R}^3)) \ \text{ and } \ \nabla B\in L^2(0,T;BMO(\mathbb{R}^3)), \tag{1.3}$$

which is an improvement of the Prodi-Serrin condition (1.2). Here BMO is the space of functions of bounded mean oscillation of John and Nirenberg. The regularity criterion (1.3) was improved by [15] as

$$u\in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)) \ \text{ and } \ \nabla B\in L^2(0,T;BMO(\mathbb{R}^3)). \tag{1.4}$$

On the other hand, based on the well-known pressure-velocity-magnetic relation of the Hall-MHD equations (1.1) in $\mathbb{R}^3$, certain growth conditions in terms of pressure were proposed to ensure the regularity criterion of smooth solutions. Fan et al. [15] showed that if the pressure satisfies one of the following two conditions :

$$\pi\in L^{\frac{2p}{p-3}}(0,T;L^p(\mathbb{R}^3)) \quad\text{and}\quad \nabla B\in L^{\frac{2p}{p-3}}(0,T;L^p(\mathbb{R}^3)) \ \text{ with } \ 3<p\le\infty, \tag{1.5}$$

or

$$\nabla\pi\in L^{\frac{2p}{3p-3}}(0,T;L^p(\mathbb{R}^3)) \quad\text{and}\quad \nabla B\in L^{\frac{2p}{p-3}}(0,T;L^p(\mathbb{R}^3)) \ \text{ with } \ 3<p\le\infty, \tag{1.6}$$

with $0<T<\infty$, then the solution (u,B) can be smoothly extended beyond time T. Recently, Fan et al. [10] proved the following regularity criterion, which can be regarded as the end-point cases of (1.5) and (1.6) :

$$\pi\in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)) \quad\text{and}\quad \nabla B\in L^{\frac{2\beta}{\beta-3}}(0,T;L^\beta(\mathbb{R}^3)) \ \text{ with } \ 3<\beta\le\infty \tag{1.7}$$

or

$$\nabla\pi\in L^{\frac{2}{3}}(0,T;BMO(\mathbb{R}^3)) \quad\text{and}\quad \nabla B\in L^{\frac{2\beta}{\beta-3}}(0,T;L^\beta(\mathbb{R}^3)) \ \text{ with } \ 3<\beta\le\infty. \tag{1.8}$$

Motivated by the above cited works, our aim is to establish a regularity criterion of the smooth solution in terms of $\pi \in L^2(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$ and $\nabla B \in L^{\frac{2}{1-r}}(0,T;\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)) \cap L^2(0,T;\dot{\mathcal{M}}_{2,3}(\mathbb{R}^3))$ with $0 < r \leq 1$. Due to the facts that

$$L^3(\mathbb{R}^3) \subset \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3) \quad \text{and} \quad L^3(\mathbb{R}^3) \neq \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3),$$

and from a mathematical viewpoint, Besov space $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ is the largest scaling invariant space of system (1.1).

2. Preliminaries and main result

First, we recall the definition and some properties of the space we are going to use (see e.g. [3]).

Definition 2.1. *Let* $\left\{\varphi_j\right\}_{j\in\mathbb{Z}}$ *be the Littlewood-Paley dyadic decomposition of unity that satisfies* $\widehat{\varphi} \in C_0^\infty\left(B_2\backslash B_{\frac{1}{2}}\right)$, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}\left(2^{-j}\xi\right)$ *and*

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}_j(\xi) = 1 \ \text{ for any } \ \xi \neq 0,$$

where B_R *is the ball in* $\mathbb{R}^3$ *centered at the origin with radius* $R > 0$. *The homogeneous Besov space is defined by*

$$\dot{B}^s_{p,q} = \left\{f \in \mathcal{S}'/\mathcal{P} : \|f\|_{\dot{B}^s_{p,q}} < \infty\right\}$$

with norm

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} \left\|2^{js}\varphi_j * f\right\|^q_{L^p}\right)^{\frac{1}{q}}$$

for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, *where* $\mathcal{S}'$ *is the space of tempered distributions and* $\mathcal{P}$ *is the space of polynomials.*

The following is a key lemma to prove Theorem 2.7 due to Meyer–Gerard–Oru [31], which a simple proof can be found in [24].

Lemma 2.2. *For any function* f *belonging to* $\dot{H}^1(\mathbb{R}^3) \cap \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$, *one has*

$$\|f\|^2_{L^4} \leq C \|\nabla f\|_{L^2} \|f\|_{\dot{B}^{-1}_{\infty,\infty}}.$$

It is well known that $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ is the biggest critical homogeneous space of degree -1 and as shown by Frazier, Jawerth and Weiss [40] any critical homogeneous space continuously embedded in $\mathcal{S}'(\mathbb{R}^3)$ is also continuously embedded into $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$.

Next, we give the definition of the Morrey spaces. For more details see [28].

Definition 2.3. *For* $0 < r < \frac{3}{2}$, *the homogeneous Morrey space* $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ *is defined as*

$$\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3) = \left\{f \in L^2_{loc}(\mathbb{R}^3) : \|f\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} < +\infty\right\},$$

where

$$\|f\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} = \sup_{x\in\mathbb{R}^3}\sup_{0<R<\infty} R^{r-\frac{3}{2}}\left(\int_{B(x,R)} |f(y)|^2\right)^{\frac{1}{2}}.$$

Here $B(x,R)$ denotes the closed ball in $\mathbb{R}^3$ with center x and radius R.

In order to prove our result, we need the following lemma which plays a very important role in the proof. This lemma can be found in [28] (see also [20, 21, 36, 37]) which gives an equivalence between $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ and a multiplier space $\mathcal{Z}_r(\mathbb{R}^3)$.

Lemma 2.4. *For $0 < r < \frac{3}{2}$, let the space $\mathcal{Z}_r(\mathbb{R}^3)$ as the space of functions which are locally square integrable on $\mathbb{R}^3$ and such that pointwise multiplication with these functions maps boundedly the Besov space $\dot{B}^r_{2,1}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The norm in $\mathcal{Z}_r(\mathbb{R}^3)$ is given by the operator norm of pointwise multiplication:*

$$\|f\|_{\mathcal{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}}\leq 1} \|fg\|_{L^2} < \infty.$$

Then f belongs to $\dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ if and only if f belongs to $\mathcal{Z}_r(\mathbb{R}^3)$ with equivalence of norms.

The following simple lemma is fundamental and shows that any function in $L^{\frac{3}{r}}$ is also in $\dot{\mathcal{M}}_{2,\frac{3}{r}}$.

Lemma 2.5. *Let $0 < r < \frac{3}{2}$. If $f \in L^{\frac{3}{r}}(\mathbb{R}^3)$, then $f \in \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)$ and $\|f\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} \leq C\|f\|_{L^{\frac{3}{r}}}$.*

Proof: Let $f \in L^{\frac{3}{r}}(\mathbb{R}^3)$. By using the following well-known Sobolev embedding $\dot{B}^r_{2,1}(\mathbb{R}^3) \subset \dot{H}^r(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$ with $\frac{1}{q} = \frac{1}{2} - \frac{r}{3}$, we have by Hölder's inequality,

$$\|fg\|_{L^2} \leq \|f\|_{L^{\frac{3}{r}}}\|g\|_{L^q} \leq \|f\|_{L^{\frac{3}{r}}}\|g\|_{\dot{H}^r} \leq \|f\|_{L^{\frac{3}{r}}}\|g\|_{\dot{B}^r_{2,1}},$$

Then, it follows that

$$\|f\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}} \approx \|f\|_{\mathcal{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}}\leq 1}\|fg\|_{L^2} \leq C\|f\|_{L^{\frac{3}{r}}}.$$

□

While $L^{\frac{3}{r}} \subset \dot{\mathcal{M}}_{2,\frac{3}{r}}$, clearly $\dot{\mathcal{M}}_{2,\frac{3}{r}}$ is a larger space than $L^{\frac{3}{r}}$: for example ,

$$|x|^{-r} \in \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3),$$

but this function is not an element of $L^{\frac{3}{r}}(\mathbb{R}^3)$.

Remark 2.1. *By the embedding $L^{\frac{3}{r}} \subsetneq \dot{\mathcal{M}}_{2,\frac{3}{r}}$, we see that our results generalize that in [10] and [15].*

We will use the following inequality (see [23, 36, 37]).

Lemma 2.6. *If $f \in H^1(\mathbb{R}^3)$ and $\nabla f \in \dot{\mathcal{M}}_{2,3}(\mathbb{R}^3)$, then $f \in BMO(\mathbb{R}^3)$. Furthermore, one has*

$$\|f\|^2_{L^{2q}} \leq C\|f\|_{L^2}\|f\|_{BMO} \leq C\|f\|_{L^2}\|\nabla f\|_{\dot{\mathcal{M}}_{2,3}}, \quad 1 < q < \infty.$$

Our result on the regularity criterion now reads as follows.

Theorem 2.7 (Regularity criterion). *Assume that* $(u_0, B_0) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ *with* $s > \frac{5}{2}$ *and* $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ *in* $\mathbb{R}^3$*, in the sense of distributions. Let* (u, B) *be the corresponding local smooth solution to the Hall-MHD equations (1.1) on* $[0, T)$ *for some* $T > 0$*. If the pressure* π *and the magnetic field* B *satisfy*

$$\pi \in L^2(0, T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)) \tag{2.1}$$

and

$$\nabla B \in L^{\frac{2}{1-r}}(0, T; \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)) \cap L^2(0, T; \dot{\mathcal{M}}_{2,3}(\mathbb{R}^3)) \tag{2.2}$$

with $0 < r < 1$*, then* (u, B) *can be extended beyond* T.

Our result (2.2) is just refer to Morrey space with the combined assumption (2.1). As an application of Theorem 2.7, we also obtain the following regularity criterion

Corollary 2.8. *Let* (u, B)*,* (u_0, B_0) *be as in Theorem 2.7. Suppose that the pressure* π *and the magnetic field* B *satisfy*

$$\pi \in L^1(0, T; BMO(\mathbb{R}^3))$$

and

$$\nabla B \in L^{\frac{2}{1-r}}(0, T; \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)) \cap L^2(0, T; \dot{\mathcal{M}}_{2,3}(\mathbb{R}^3))$$

with $0 < r < 1$*, then* (u, B) *can be extended beyond* T.

Remark 2.2. *Therefore, Corollary 2.8 is a further improvement of the result of work [10].*

Since $\dot{\mathcal{M}}_{2,3}(\mathbb{R}^3) \subset \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ (for the proof, see e.g. [24]), we have the following result.

Corollary 2.9. *Let* (u, B)*,* (u_0, B_0) *be as in Theorem 2.7. Suppose that the pressure* π *and the magnetic field* B *satisfy*

$$\pi \in L^2(0, T; \dot{\mathcal{M}}_{2,3}(\mathbb{R}^3))$$

and

$$\nabla B \in L^{\frac{2}{1-r}}(0, T; \dot{\mathcal{M}}_{2,\frac{3}{r}}(\mathbb{R}^3)) \cap L^2(0, T; \dot{\mathcal{M}}_{2,3}(\mathbb{R}^3))$$

with $0 < r < 1$*, then* (u, B) *can be extended beyond* T.

3. Proof of Theorem 2.7

Now we are in a position to prove Theorem 2.7.

Proof: Firstly, we derive the energy inequality. For this purpose, we take the $L^2(\mathbb{R}^3)$ inner product of u and B with equations (1.1), respectively, sum the resulting equations and then integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(|u|^2 + |B|^2)dx + \int_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla B|^2)dx = 0,$$

where we used $\nabla \cdot u = \nabla \cdot B = 0$. This proves

$$\|(u, B)\|_{L^\infty(0,T;L^2)} + \|(u, B)\|_{L^2(0,T;H^1)} \leq C.$$

In the following, from Serrin type criteria (1.2) with $p = \beta = 4$ on the $3D$ Hall-MHD equations (1.1), it is sufficient to prove that

$$u \in L^\infty(0, T; L^4(\mathbb{R}^3)) \subset L^8(0, T; L^4(\mathbb{R}^3)).$$

To this end, let $T > 0$ be a given fixed time. Multiplying $(1.1)_1$ by $|u|^2 u$ and integrating by parts over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
& \frac{1}{4}\frac{d}{dt}\|u\|_{L^4}^4 + \||u|\,|\nabla u|\|_{L^2}^2 + \frac{1}{2}\left\|\nabla |u|^2\right\|_{L^2}^2 \\
= & \int_{\mathbb{R}^3}\left(B\cdot\nabla B - \frac{1}{2}\nabla |B|^2\right)\cdot |u|^2 u dx - \int_{\mathbb{R}^3} u\cdot\nabla\pi\,|u|^2\,dx \\
= & -\sum_{i=1}^{3}\int_{\mathbb{R}^3} B_i B\partial_i(|u|^2 u)dx + \frac{1}{2}\int_{\mathbb{R}^3}|B|^2\,(\nabla\cdot(|u|^2 u))dx - \int_{\mathbb{R}^3} u\cdot\nabla\pi\,|u|^2\,dx \\
\leq & C\left\||B|^2\right\|_{L^4}\|u\|_{L^4}\||u|\,|\nabla u|\|_{L^2} + \left|\int_{\mathbb{R}^3} u\cdot\nabla\pi\,|u|^2\,dx\right| \\
\leq & C\|B\|_{L^8}^2\|u\|_{L^4}\||u|\,|\nabla u|\|_{L^2} + 2\left|\int_{\mathbb{R}^3}\pi u\,|u|\cdot\nabla u dx\right| \\
\leq & C\|B\|_{L^4}\|\nabla B\|_{\dot{\mathcal{M}}_{2,3}}\|u\|_{L^4}\||u|\,|\nabla u|\|_{L^2} + 2\|\pi u\|_{L^2}\||u|\,|\nabla u|\|_{L^2} \\
\leq & C\|\nabla B\|_{\dot{\mathcal{M}}_{2,3}}^2\left(\|B\|_{L^4}^4 + \|u\|_{L^4}^4\right) + \frac{1}{2}\||u|\,|\nabla u|\|_{L^2}^2 + C\|\pi\|_{L^4}^2\|u\|_{L^4}^2 \\
\leq & C\|\nabla B\|_{\dot{\mathcal{M}}_{2,3}}^2\left(\|B\|_{L^4}^4 + \|u\|_{L^4}^4\right) + \frac{1}{4}\||u|\,|\nabla u|\|_{L^2}^2 + C\|\pi\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\pi\|_{L^2}\|u\|_{L^4}^2 \\
\leq & C\|\nabla B\|_{\dot{\mathcal{M}}_{2,3}}^2\left(\|B\|_{L^4}^4 + \|u\|_{L^4}^4\right) + \frac{1}{4}\||u|\,|\nabla u|\|_{L^2}^2 \\
& + C\|\pi\|_{\dot{B}^{-1}_{\infty,\infty}}\left(\||u|\,|\nabla u|\|_{L^2} + \||B|\,|\nabla B|\|_{L^2}\right)\|u\|_{L^4}^2 \\
\leq & C\|\nabla B\|_{\dot{\mathcal{M}}_{2,3}}^2\left(\|B\|_{L^4}^4 + \|u\|_{L^4}^4\right) + \frac{1}{2}\||u|\,|\nabla u|\|_{L^2}^2 + \frac{1}{2}\||B|\,|\nabla B|\|_{L^2}^2 \\
& + C\|\pi\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|u\|_{L^4}^4,
\end{aligned} \tag{3.1}$$

where we have used the Lemma 2.6 and the fact :

$$\begin{aligned}
\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot |u|^2 u dx &= 0, \\
(\nabla u)\cdot(\nabla(|u|^2 u)) &= |\nabla u|^2\,|u|^2 + \frac{1}{2}\left|\nabla |u|^2\right|^2.
\end{aligned}$$

In a similar way, multipying $(1.1)_2$ by $|B|^2 B$, and integrating by parts yield

$$\frac{1}{4}\frac{d}{dt}\|B\|_{L^4}^4 + \frac{1}{2}\||B|\,|\nabla B|\|_{L^2}^2 + \frac{1}{2}\left\|\nabla |B|^2\right\|_{L^2}^2$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} (B\cdot\nabla)u\cdot|B|^2 B dx + \int_{\mathbb{R}^3} (B\times \text{curl } B)\text{curl } (|B|^2 B)dx \\
&= -\int_{\mathbb{R}^3} (B\cdot\nabla)\,|B|^2\, B\cdot u dx + \int_{\mathbb{R}^3} (B\times \text{curl } B)(\nabla\,|B|^2\times B)dx \\
&\le C\int_{\mathbb{R}^3} |u|\,|B|^3\,|\nabla B|\,dx + C\int_{\mathbb{R}^3} |B|^3\,|\nabla B|^2\,dx \\
&\le C\left\||B|^2\,|\nabla B|\right\|_{L^2}\|B\|_{L^4}\|u\|_{L^4} + C\left\||\nabla B|\,|B|^2\right\|_{L^2}\||B|\,|\nabla B|\|_{L^2} \\
&\le C\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}\left\||B|^2\right\|_{\dot{B}^r_{2,1}}\|B\|_{L^4}\|u\|_{L^4} + C\||B|\,|\nabla B|\|_{L^2}\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}\left\||B|^2\right\|_{\dot{B}^r_{2,1}} \\
&\le C\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}\left\||B|^2\right\|_{L^2}^{1-r}\left\|\nabla\,|B|^2\right\|_{L^2}^{r}\|B\|_{L^4}\|u\|_{L^4} \\
&\quad +C\||B|\,|\nabla B|\|_{L^2}\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}\left\||B|^2\right\|_{L^2}^{1-r}\left\|\nabla\,|B|^2\right\|_{L^2}^{r} \\
&\le C\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}\|B\|_{L^4}^{3-2r}\left\|\nabla\,|B|^2\right\|_{L^2}^{r}\|u\|_{L^4} \\
&\quad +C\||B|\,|\nabla B|\|_{L^2}\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}\|B\|_{L^4}^{2(1-r)}\left\|\nabla\,|B|^2\right\|_{L^2}^{r} \\
&\le C\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{2-r}}\|B\|_{L^4}^{2\frac{(3-2r)}{2-r}}\|u\|_{L^4}^{\frac{2}{2-r}} + \frac{1}{8}\left\|\nabla\,|B|^2\right\|_{L^2}^2 \\
&\quad +\frac{1}{8}\||B|\,|\nabla B|\|_{L^2}^2 + C\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{1-r}}\|B\|_{L^4}^4 + \frac{1}{8}\left\|\nabla\,|B|^2\right\|_{L^2}^2 \\
&\le C\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{2-r}}\left(\|B\|_{L^4}^4+\|u\|_{L^4}^4\right) + \frac{1}{4}\left\|\nabla\,|B|^2\right\|_{L^2}^2 + \frac{1}{2}\||B|\,|\nabla B|\|_{L^2}^2 + C\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{1-r}}\|B\|_{L^4}^4 \\
&\le C\left(1+\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{1-r}}\right)\left(\|B\|_{L^4}^4+\|u\|_{L^4}^4\right) + \frac{1}{4}\left\|\nabla\,|B|^2\right\|_{L^2}^2 + \frac{1}{2}\||B|\,|\nabla B|\|_{L^2}^2 + C\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{1-r}}\|B\|_{L^4}^4 \\
&\le C\left(1+\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{1-r}}\right)\left(\|B\|_{L^4}^4+\|u\|_{L^4}^4\right) + \frac{1}{4}\left\|\nabla\,|B|^2\right\|_{L^2}^2 + \frac{1}{2}\||B|\,|\nabla B|\|_{L^2}^2, \qquad (3.2)
\end{aligned}
$$

where we have used the following interpolation inequality due to [30]:

$$
\|f\|_{\dot{B}^r_{2,1}} \le C\|f\|_{L^2}^{1-r}\|\nabla f\|_{L^2}^{r} \quad \text{with } 0<r<1.
$$

Summing (3.1) and (3.2), we get

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left(\|u\|_{L^4}^4+\|B\|_{L^4}^4\right)+\frac{1}{2}\||u|\,|\nabla u|\|_{L^2}^2+\frac{1}{2}\||B|\,\nabla\,|B|\|_{L^2}^2+\frac{1}{2}\left(\left\|\nabla\,|u|^2\right\|_{L^2}^2+\left\|\nabla\,|B|^2\right\|_{L^2}^2\right) \\
\le\ & C\left(1+\|\nabla B\|_{\dot{\mathcal{M}}_{2,3}}^2+\|\nabla B\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{1-r}}\right)\left(\|B\|_{L^4}^4+\|u\|_{L^4}^4\right)+\|\pi\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|u\|_{L^4}^4.
\end{aligned}
$$

Using Gronwall's inequality, we obtain

$$
\sup_{0<t<T}\left(\|u(\cdot,t)\|_{L^4}^4+\|B(\cdot,t)\|_{L^4}^4\right)
$$

$$\leq \left(\|u_0\|_{L^4}^4 + \|B_0\|_{L^4}^4\right) \exp\left(\int_0^t \left\{\|\nabla B(\cdot,\tau)\|_{\dot{\mathcal{M}}_{2,3}}^2 + \|\nabla B(\cdot,\tau)\|_{\dot{\mathcal{M}}_{2,\frac{3}{r}}}^{\frac{2}{1-r}} + \|\pi(\cdot,\tau)\|_{\dot{B}_{\infty,\infty}^{-1}}^2 + 1\right\} d\tau\right)$$
$$< \infty.$$

This implies that

$$u \in L^\infty(0,T;L^4(\mathbb{R}^3)) \subset L^8(0,T;L^4(\mathbb{R}^3)).$$

Now due to regularity criterion (1.2), the proof of Theorem 2.7 is complete. □

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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