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Global exponential stability and existence of periodic solutions for delayed reaction-diffusion BAM neural networks with Dirichlet boundary conditions

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Abstract

In this paper, both global exponential stability and periodic solutions are investigated for a class of delayed reaction-diffusion BAM neural networks with Dirichlet boundary conditions. By employing suitable Lyapunov functionals, sufficient conditions of the global exponential stability and the existence of periodic solutions are established for reaction-diffusion BAM neural networks with mixed time delays and Dirichlet boundary conditions. The derived criteria extend and improve previous results in the literature. A numerical example is given to show the effectiveness of the obtained results.

Keywords: neural networks; reaction-diffusion; mixed time delays; global exponential stability; Poincaré mapping; Lyapunov functional

1 Introduction

Neural networks (NNs) have been extensively studied in the past few years and have found many applications in different areas such as pattern recognition, associative memory, combinatorial optimization, *etc*. Delayed versions of NNs were also proved to be important for solving certain classes of motion-related optimization problems. Various results concerning the dynamical behavior of NNs with delays have been reported during the last decade (see, *e.g.*, [1–7]). Recently, the authors in [1] and [2] considered the problem of exponential passivity analysis for uncertain NNs with time-varying delays and passivity-based controller design for Hopfield NNs, respectively.

Since NNs related to bidirectional associative memory (BAM) were proposed by Kosko [8], the BAM NNs have been one of the most interesting research topics and have attracted the attention of researchers. In the design and applications of networks, the stability of the designed BAM NNs is one of the most important issues (see, *e.g.*, [9–12]). Many important results concerning mainly the existence and stability of equilibrium of BAM NNs have been obtained (see, *e.g.*, [9–15]).

However, strictly speaking, diffusion effects cannot be avoided in the NNs when electrons are moving in asymmetric electromagnetic fields. So, we must consider that the activations vary in space as well as in time. In [16–34], the authors considered the stability of NNs with diffusion terms which were expressed by partial differential equations. In par-

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ticular, the existence and attractivity of periodic solutions for non-autonomous reaction-diffusion Cohen-Grossberg NNs with discrete time delays were investigated in [20]. The authors derived sufficient conditions on the stability and periodic solutions of delayed reaction-diffusion NNs (RDNNs) with Neumann boundary conditions in [21–25]. In these works, due to the divergence theorem employed, a negative integral term with gradient was removed in their deduction. Therefore, the stability criteria acquired by them do not contain diffusion terms; that is to say, the diffusion terms do not have any effect on their deduction and results. Meanwhile, some conditions dependent on the diffusion coefficients were given in [30, 32–34] to ensure the global exponential stability and periodicity of RDNNs with Dirichlet boundary conditions based on 2-norm.

To the best of our knowledge, there are few reports about global exponential stability and periodicity of RDNNs with mixed time delays and Dirichlet boundary conditions, which are very important in theories and applications and also are a very challenging problem. In this paper, by employing suitable Lyapunov functionals, we shall apply inequality techniques to establish global exponential stability criteria of the equilibrium and periodic solutions for RDNNs with mixed time delays and Dirichlet boundary conditions. The derived criteria extend and improve previous results in the literature [22, 29].

Throughout this paper, we need the following notations. R^n denotes the n-dimensional Euclidean space. We denote

$$\begin{aligned}
&\|u(t,x)-u^*\| = \int_\Omega \sum_{i=1}^m |u_i-u_i^*|^r\,dx,\\
&\|\varphi_u(s,x)-u^*\| = \sup_{-\infty\le s\le 0}\left[\int_\Omega \sum_{i=1}^m |\varphi_{ui}(s,x)-u_i^*|^r\,dx\right]
\end{aligned}$$

and

$$\begin{aligned}
\|v(t,x)-v^*\| &= \int_\Omega \sum_{j=1}^n |v_j-v_j^*|^r\,dx,\ \|\varphi_v(s,x)-v^*\| = \sup_{-\infty\le s\le 0}\left[\int_\Omega \sum_{j=1}^n |\varphi_{vj}(s,x)-v_j^*|^r\,dx\right],\\
&r\ge 2.
\end{aligned}$$

Let $u_i = u_i(t,x)$, $v_j = v_j(t,x)$.

The remainder of this paper is organized as follows. In Section 2, the basic notations, model description and assumptions are introduced. In Sections 3 and 4, criteria are proposed to determine global exponential stability, and periodic solutions are considered for reaction-diffusion recurrent neural networks with mixed time delays, respectively. An illustrative example is given to illustrate the effectiveness of the obtained results in Section 5. We also conclude this paper in Section 6.

2 Model description and preliminaries

In this paper, the RDNNs with mixed time delays are described as follows:

$$\begin{aligned}
\frac{\partial u_i}{\partial t} &= \sum_{k=1}^l \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right) - p_i\bigl(u_i(t,x)\bigr)\\
&\quad + \sum_{j=1}^n \bigl(b_{ji}f_j\bigl(v_j(t,x)\bigr)\bigr) + \sum_{j=1}^n \bigl(\tilde{b}_{ji}\tilde{f}_j\bigl(v_j\bigl(t-\theta_{ji}(t),x\bigr)\bigr)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{j=1}^{n} \bar{b}_{ji} \int_{-\infty}^{t} k_{ji}(t-s)\bar{f}_j\big(v_j(s,x)\big)\,ds + I_i(t), \\
\frac{\partial v_j}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_{jk}^{*}\frac{\partial v_j}{\partial x_k}\right) - q_j\big(v_j(t,x)\big) \\
&\quad + \sum_{i=1}^{m}\big(d_{ij}g_i\big(u_i(t,x)\big)\big) + \sum_{i=1}^{m}\big(\tilde{d}_{ij}\tilde{g}_i\big(u_i\big(t-\tau_{ij}(t),x\big)\big)\big) \\
&\quad + \sum_{i=1}^{m} \bar{d}_{ij} \int_{-\infty}^{t} \bar{k}_{ij}(t-s)\bar{g}_i\big(u_i(s,x)\big)\,ds + J_j(t).
\end{aligned}
\tag{1}
$$

The RDNNs model given in (1) can be regarded as RDNNs with two layers, where m is the number of neurons in the first layer and n is the number of neurons in the second layer. $x = (x_1, x_2, \dots, x_l)^T \in \Omega \subset R^l$, Ω is a compact set with smooth boundary $\partial\Omega$ and $\operatorname{mes}\Omega > 0$ in the space R^l; $u = (u_1, u_2, \dots, u_m)^T \in R^m$, $v = (v_1, v_2, \dots, v_n)^T \in R^n$. $u_i(t,x)$ and $v_j(t,x)$ represent the state of the ith neuron in the first layer and the jth neuron in the second layer at time t and in the space x, respectively. b_{ji}, $\tilde{b}_{ji}$, $\bar{b}_{ji}$, d_{ij}, $\bar{d}_{ij}$ and $\tilde{d}_{ij}$ are known constants denoting the synaptic connection strengths between the neurons in the two layers, respectively; f_j, $\tilde{f}_j$, $\bar{f}_j$, g_i, $\tilde{g}_i$ and $\bar{g}_i$ denote the activation functions of the neurons and the signal propagation functions, respectively. I_i and J_j denote the external inputs on the ith neuron and jth neuron, respectively; p_i and q_j are differentiable real functions with positive derivatives defining the neuron charging time; $\tau_{ij}(t)$ and $\theta_{ji}(t)$ represent continuous time-varying delay and discrete delay, respectively; $D_{ik} \ge 0$ and $D_{jk}^{*} \ge 0$, $i = 1, 2, \dots, m$, $k = 1, 2, \dots, l$ and $j = 1, 2, \dots, n$, stand for the transmission diffusion coefficient along the ith neuron and jth neuron, respectively.

System (1) is supplemented with the following boundary conditions and initial values:

$$
u_i(t,x) = 0, \qquad v_j(t,x) = 0, \quad t \ge 0, x \in \partial\Omega, \tag{2}
$$

$$
u_i(s,x) = \varphi_{ui}(s,x), \qquad v_j(s,x) = \varphi_{vj}(s,x), \quad (s,x) \in (-\infty, 0] \times \Omega \tag{3}
$$

for any $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, where $\bar{n}$ is the outer normal vector of $\partial\Omega$, $\varphi = \binom{\varphi_u}{\varphi_v} = (\varphi_{u1}, \dots, \varphi_{um}, \varphi_{v1}, \dots, \varphi_{vn})^T \in C$ are bounded and continuous, where $C = \{\varphi \mid \varphi = \binom{\varphi_u}{\varphi_v}, \varphi : \binom{(-\infty,0]\times R^m}{(-\infty,0]\times R^n} \to R^{m+n}\}$. It is the Banach space of continuous functions which maps $\binom{(-\infty,0]}{(-\infty,0]}$ into R^{m+n} with the topology of uniform convergence for the norm

$$
\|\varphi\| = \left\| \begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix} \right\| = \sup_{-\infty \le s \le 0} \left[\int_\Omega \sum_{i=1}^{m} |\varphi_{ui}|^r \,dx \right] + \sup_{-\infty \le s \le 0} \left[\int_\Omega \sum_{j=1}^{n} |\varphi_{vj}|^r \,dx \right].
$$

Remark 1 Some famous NN models became a special case of system (1). For example, when $D_{ik} = 0$ and $D_{jk}^{*} = 0$ ($i = 1, 2, \dots, m$, $k = 1, 2, \dots, l$), the special case of model (1) is the model which has been studied in [13–15]. When $\tilde{b}_{ji} = 0$ and $\tilde{d}_{ij} = 0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, system (1) became NNs with distributed delays and reaction-diffusion terms [18, 22, 29].

Throughout this paper, we assume that the following conditions are made.

(A1) The functions $\tau_{ij}(t)$, $\theta_{ji}(t)$ are piecewise-continuous of class C^1 on the closure of each continuity subinterval and satisfy

$$0 \le \tau_{ij}(t) \le \tau_{ij}, \qquad 0 \le \theta_{ji}(t) \le \theta_{ji}, \qquad \dot{\tau}_{ij}(t) \le \mu_\tau < 1, \qquad \dot{\theta}_{ji}(t) \le \mu_\theta < 1,$$
$$\tau = \max_{1\le i\le m, 1\le j\le n}\{\tau_{ij}\}, \qquad \theta = \max_{1\le i\le m, 1\le j\le n}\{\theta_{ji}\}$$

with some constants $\tau_{ij} \ge 0$, $\theta_{ji} \ge 0$, $\tau > 0$, $\theta > 0$ for all $t \ge 0$.

(A2) The functions $p_i(\cdot)$ and $q_j(\cdot)$ are piecewise-continuous of class C^1 on the closure of each continuity subinterval and satisfy

$$a_i = \inf_{\zeta\in R} p_i'(\zeta) > 0, \quad p_i(0) = 0,$$
$$c_j = \inf_{\zeta\in R} q_j'(\zeta) > 0, \quad q_j(0) = 0.$$

(A3) The activation functions and the signal propagation functions are bounded and Lipschitz continuous, *i.e.*, there exist positive constants L_j^f, $L_j^{\tilde{f}}$, $L_j^{\bar{f}}$, L_i^g, $L_i^{\tilde{g}}$ and $L_i^{\bar{g}}$ such that for all $\eta_1, \eta_2 \in R$,

$$\bigl|f_j(\eta_1) - f_j(\eta_2)\bigr| \le L_j^f|\eta_1 - \eta_2|, \qquad \bigl|\tilde{f}_j(\eta_1) - \tilde{f}_j(\eta_2)\bigr| \le L_j^{\tilde{f}}|\eta_1 - \eta_2|,$$
$$\bigl|\bar{f}_j(\eta_1) - \bar{f}_j(\eta_2)\bigr| \le L_j^{\bar{f}}|\eta_1 - \eta_2|, \qquad \bigl|g_i(\eta_1) - g_i(\eta_2)\bigr| \le L_i^g|\eta_1 - \eta_2|,$$
$$\bigl|\tilde{g}_i(\eta_1) - \tilde{g}_i(\eta_2)\bigr| \le L_i^{\tilde{g}}|\eta_1 - \eta_2|, \qquad \bigl|\bar{g}_i(\eta_1) - \bar{g}_i(\eta_2)\bigr| \le L_i^{\bar{g}}|\eta_1 - \eta_2|.$$

(A4) The delay kernels $K_{ji}(s), \bar{K}_{ij}(s) : [0,\infty) \to [0,\infty)$ ($i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$) are real-valued non-negative continuous functions that satisfy the following conditions:

(i) $\int_0^{+\infty} K_{ji}(s)\,ds = 1$, $\int_0^{+\infty} \bar{K}_{ji}(s)\,ds = 1$;
(ii) $\int_0^{+\infty} sK_{ji}(s)\,ds < \infty$, $\int_0^{+\infty} s\bar{K}_{ij}(s)\,ds < \infty$;
(iii) There exist a positive μ such that

$$\int_0^{+\infty} se^{\mu s}K_{ji}(s)\,ds < \infty, \qquad \int_0^{+\infty} se^{\mu s}\bar{K}_{ij}(s)\,ds < \infty.$$

Let $(u^*, v^*) = (u_1^*, u_2^*, \ldots, u_n^*, v_1^*, v_2^*, \ldots, v_n^*)$ be the equilibrium point of system (1).

Definition 1 The equilibrium point of system (1) is said to be globally exponentially stable if we can find $r \ge 2$ such that there exist constants $\alpha > 0$ and $\beta \ge 1$ such that

$$\begin{aligned}&\bigl\|u(t,x) - u^*\bigr\| + \bigl\|v(t,x) - v^*\bigr\| \\ &\quad \le \beta e^{-2\alpha t}\bigl(\bigl\|\varphi_u(s,x) - u^*\bigr\| + \bigl\|\varphi_v(s,x) - v^*\bigr\|\bigr)\end{aligned} \tag{4}$$

for all $t \ge 0$.

Remark 2 It is well known that bounded activation functions always guarantee the existence of an equilibrium point for system (1).

Lemma 1 [33] *Let Ω be a cube $|x_l| < d_l$ $(l = 1, \ldots, m)$, and let $h(x)$ be a real-valued function belonging to $C^1(\Omega)$ which vanishes on the boundary $\partial\Omega$ of Ω, i.e., $h(x)|_{\partial\Omega=0}$. Then*

$$\int_\Omega h^2(x)\,dx \le d_l^2 \int_\Omega \left|\frac{\partial h}{\partial x_l}\right|^2 dx. \tag{5}$$

3 Global exponential stability

Now we are in a position to investigate the global exponential stability of system (1). By constructing a suitable Lyapunov functional, we arrive at the following conclusion.

Theorem 1 *Let* (A1)-(A4) *be in force. If there exist $w_i > 0$ $(i = 1, 2, \ldots, n + m)$, $r \ge 2$, $\gamma_{ij} > 0$, $\beta_{ji} > 0$ such that*

$$\begin{aligned} &w_i\left(-rna_i^{r-1}D_i l - rna_i^r + 2(r-1)\sum_{j=1}^n a_i^r + (r-1)\sum_{j=1}^n a_i^r \beta_{ji}^{-\frac{r}{r-1}}\right) \\ &\quad + \sum_{j=1}^n w_{m+j} m^r \left(|d_{ij}|^r \left(L_i^g\right)^r + |\tilde{d}_{ij}|^r \frac{e^\tau}{1-\mu_\tau}\left(L_i^{\tilde{g}}\right)^r + |\bar{d}_{ij}|^r \gamma_{ij}^r \left(L_i^{\bar{g}}\right)^r\right) < 0 \end{aligned}$$

and

$$\begin{aligned} &w_{m+j}\left(-rmc_j^{r-1}D_j^* l - rmc_j^r + 2(r-1)\sum_{i=1}^m c_j^r + (r-1)\sum_{i=1}^m c_j^r \gamma_{ij}^{-\frac{r}{r-1}}\right) \\ &\quad + \sum_{i=1}^m w_i n^r \left(|b_{ji}|^r \left(L_j^f\right)^r + |\tilde{b}_{ji}|^r \frac{e^\theta}{1-\mu_\theta}\left(L_j^{\tilde{f}}\right)^r + |\bar{b}_{ji}|^r \beta_{ji}^r \left(L_j^{\bar{f}}\right)^r\right) < 0, \end{aligned} \tag{6}$$

in which $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, L_j^f, $L_j^{\tilde{f}}$, $L_j^{\bar{f}}$, L_i^g, $L_i^{\tilde{g}}$ and $L_i^{\bar{g}}$ are Lipschitz constants, $D_i = \min_{1\le k\le l}\{D_{ik}/d_k^2\}$, $D_j^* = \min_{1\le k\le l}\{D_{jk}^*/d_k^2\}$, then the equilibrium point (u^*, v^*) of system (1) *is unique and globally exponentially stable.*

Proof If (6) holds, we can always choose a positive number $\delta > 0$ (may be very small) such that

$$\begin{aligned} &w_i\left(-rna_i^{r-1}D_i l - rna_i^r + 2(r-1)\sum_{j=1}^n a_i^r + (r-1)\sum_{j=1}^n a_i^r \beta_{ji}^{-\frac{r}{r-1}}\right) \\ &\quad + \sum_{j=1}^n w_{m+j} m^r \left(|d_{ij}|^r \left(L_i^g\right)^r + |\tilde{d}_{ij}|^r \frac{e^\tau}{1-\mu_\tau}\left(L_i^{\tilde{g}}\right)^r + |\bar{d}_{ij}|^r \gamma_{ij}^r \left(L_i^{\bar{g}_i}\right)^r\right) + \delta < 0 \end{aligned}$$

and

$$\begin{aligned} &w_{m+j}\left(-rmc_j^{r-1}D_j^* l - rmc_j^r + 2(r-1)\sum_{i=1}^m c_j^r + (r-1)\sum_{i=1}^m c_j^r \gamma_{ij}^{-\frac{r}{r-1}}\right) \\ &\quad + \sum_{i=1}^m w_i n^r \left(|b_{ji}|^r \left(L_j^f\right)^r + |\tilde{b}_{ji}|^r \frac{e^\theta}{1-\mu_\theta}\left(L_j^{\tilde{f}}\right)^r + |\bar{b}_{ji}|^r \beta_{ji}^r \left(L_j^{\bar{f}}\right)^r\right) + \delta < 0, \end{aligned} \tag{7}$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Let us consider the functions

$$\begin{aligned} F_i(x_i^*) &= w_i\Biggl(-rna_i^{r-1}D_il - rna_i^r + 2(r-1)\sum_{j=1}^n a_i^r \\ &\quad + (r-1)\sum_{j=1}^n a_i^r\beta_{ji}^{-\frac{r}{r-1}}\int_0^{+\infty} k_{ji}(s)\,ds + 2x_i^* na_i^{r-1}\Biggr) \\ &\quad + \sum_{j=1}^n w_{m+j}m^r\biggl(|d_{ij}|^r\bigl(L_i^g\bigr)^r + |\tilde{d}_{ij}|^r\frac{e^\tau}{1-\mu_\tau}\bigl(L_i^{\tilde{g}}\bigr)^r \\ &\quad + |\bar{d}_{ij}|^r\gamma_{ij}^r\bigl(L_i^{\bar{g}_i}\bigr)^r\int_0^{+\infty} e^{2x_i^* s}\bar{k}_{ij}(s)\,ds\biggr) \end{aligned}$$

and

$$\begin{aligned} G_j(y_j^*) &= w_{m+j}\Biggl[-rmc_j^{r-1}D_j^*l - rmc_j^r + 2(r-1)\sum_{i=1}^m c_j^r + (r-1)\sum_{i=1}^m c_j^r\gamma_{ij}^{-\frac{r}{r-1}}\int_0^{+\infty}\bar{k}_{ij}(s)\,ds \\ &\quad + 2y_j^* mc_j^{r-1}\Biggr] + \sum_{i=1}^m w_in^r\biggl(|b_{ji}|^r\bigl(L_j^f\bigr)^r \\ &\quad + |\tilde{b}_{ji}|^r\frac{e^\theta}{1-\mu_\theta}\bigl(L_j^{\tilde{f}}\bigr)^r + |\bar{b}_{ji}|^r\beta_{ji}^r\bigl(L_j^{\bar{f}}\bigr)^r\int_0^{+\infty} e^{2y_j^* s}k_{ji}(s)\,ds\biggr), \end{aligned} \tag{8}$$

where $x_i^*, y_j^* \in [0,+\infty)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

From (8) and (A4), we derive $F_i(0) < -\delta < 0$, $G_j(0) < -\delta < 0$; $F_i(x_i^*)$ and $G_j(y_j^*)$ are continuous for $x_i^*, y_j^* \in [0,+\infty)$. Moreover, $F_i(x_i^*) \to +\infty$ as $x_i^* \to +\infty$ and $G_j(y_j^*) \to +\infty$ as $y_j^* \to +\infty$. Thus there exist constants $\varepsilon_i, \sigma_j \in [0,+\infty)$ such that

$$\begin{aligned} F_i(\varepsilon_i) &= w_i\Biggl(-rna_i^{r-1}D_il - rna_i^r + 2(r-1)\sum_{j=1}^n a_i^r \\ &\quad + (r-1)\sum_{j=1}^n a_i^r\beta_{ji}^{-\frac{r}{r-1}}\int_0^{+\infty} k_{ji}(s)\,ds + 2\varepsilon_i na_i^{r-1}\Biggr) + \sum_{j=1}^n w_{m+j}m^r\biggl(|d_{ij}|^r\bigl(L_i^g\bigr)^r \\ &\quad + |\tilde{d}_{ij}|^r\frac{e^\tau}{1-\mu_\tau}\bigl(L_i^{\tilde{g}}\bigr)^r + |\bar{d}_{ij}|^r\gamma_{ij}^r\bigl(L_i^{\bar{g}_i}\bigr)^r\int_0^{+\infty} e^{2\varepsilon_i s}\bar{k}_{ij}(s)\,ds\biggr) = 0 \end{aligned}$$

and

$$\begin{aligned} G_j(\sigma_j) &= w_{m+j}\Biggl(-rmc_j^{r-1}D_j^*l - rmc_j^r + 2(r-1)\sum_{i=1}^m c_j^r \\ &\quad + (r-1)\sum_{i=1}^m c_j^r\gamma_{ij}^{-\frac{r}{r-1}}\int_0^{+\infty}\bar{k}_{ij}(s)\,ds + 2\sigma_j mc_j^{r-1}\Biggr) + \sum_{i=1}^m w_in^r\biggl(|b_{ji}|^r\bigl(L_j^f\bigr)^r \\ &\quad + |\tilde{b}_{ji}|^r\frac{e^\theta}{1-\mu_\theta}\bigl(L_j^{\tilde{f}}\bigr)^r + |\bar{b}_{ji}|^r\beta_{ji}^r\bigl(L_j^{\bar{f}}\bigr)^r\int_0^{+\infty} e^{2\sigma_j s}k_{ji}(s)\,ds\biggr) = 0, \end{aligned} \tag{9}$$

where $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

By using $\alpha = \min_{1\le i\le m, 1\le j\le n}\{\varepsilon_i, \sigma_j\}$, obviously, we get

$$\begin{aligned}F_i(\alpha) &= w_i\Biggl(-rna_i^{r-1}D_i l - rna_i^r + 2(r-1)\sum_{j=1}^{n} a_i^r \\ &\quad + (r-1)\sum_{j=1}^{n} a_i^r \beta_{ji}^{-\frac{r}{r-1}} \int_0^{+\infty} k_{ji}(s)\,ds + 2\alpha n a_i^{r-1}\Biggr) + \sum_{j=1}^{n} w_{m+j} m^r \Biggl(|d_{ij}|^r \bigl(L_i^g\bigr)^r \\ &\quad + |\tilde{d}_{ij}|^r \frac{e^{\tau}}{1-\mu_\tau}\bigl(L_i^{\tilde{g}}\bigr)^r + |\bar{d}_{ij}|^r \gamma_{ij}^r \bigl(L_i^{\bar{g}_i}\bigr)^r \int_0^{+\infty} e^{2\alpha s}\bar{k}_{ij}(s)\,ds\Biggr) \le 0\end{aligned}$$

and

$$\begin{aligned}G_j(\alpha) &= w_{m+j}\Biggl(-rmc_j^{r-1}D_j^* l - rmc_j^r + 2(r-1)\sum_{i=1}^{m} c_j^r \\ &\quad + (r-1)\sum_{i=1}^{m} c_j^r \gamma_{ij}^{-\frac{r}{r-1}} \int_0^{+\infty} \bar{k}_{ij}(s)\,ds + 2\alpha m c_j^{r-1}\Biggr) + \sum_{i=1}^{m} w_i n^r \Biggl(|b_{ji}|^r \bigl(L_j^f\bigr)^r \\ &\quad + |\tilde{b}_{ji}|^r \frac{e^{\theta}}{1-\mu_\theta}\bigl(L_j^{\tilde{f}}\bigr)^r + |\bar{b}_{ji}|^r \beta_{ji}^r \bigl(L_j^{\bar{f}}\bigr)^r \int_0^{+\infty} e^{2\alpha s} k_{ji}(s)\,ds\Biggr) \le 0,\end{aligned} \tag{10}$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Suppose $(u, v) = (u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n)^T$ is any solution of model (1). Rewrite model (1) as

$$\begin{aligned}\frac{\partial(u_i - u_i^*)}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial(u_i - u_i^*)}{\partial x_k}\right) - \bigl(p_i\bigl(u_i(t,x)\bigr) - p_i\bigl(u_i^*\bigr)\bigr) \\ &\quad + \sum_{j=1}^{n}\bigl(b_{ji}\bigl(f_j\bigl(v_j(t,x)\bigr) - f_j\bigl(v_j^*\bigr)\bigr)\bigr) + \sum_{j=1}^{n}\bigl(\tilde{b}_{ji}\bigl(\tilde{f}_j\bigl(v_j\bigl(t-\theta_{ji}(t),x\bigr)\bigr) - \tilde{f}_j\bigl(v_j^*\bigr)\bigr)\bigr) \\ &\quad + \sum_{j=1}^{n}\bar{b}_{ji}\int_{-\infty}^{t} k_{ji}(t-s)\bigl(\bar{f}_j\bigl(v_j(s,x)\bigr) - \bar{f}_j\bigl(v_j^*\bigr)\bigr)\,ds,\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial(v_j - v_j^*)}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_{jk}^*\frac{\partial(v_j - v_j^*)}{\partial x_k}\right) - \bigl(q_j\bigl(v_j(t,x)\bigr) - q_j\bigl(v_j^*\bigr)\bigr) \\ &\quad + \sum_{i=1}^{m}\bigl(d_{ij}\bigl(g_i\bigl(u_i(t,x)\bigr) - g_i\bigl(u_i^*\bigr)\bigr)\bigr) + \sum_{i=1}^{m}\bigl(\tilde{d}_{ij}\bigl(\tilde{g}_i\bigl(u_i\bigl(t-\tau_{ij}(t),x\bigr)\bigr) - \tilde{g}_i\bigl(u_i^*\bigr)\bigr)\bigr) \\ &\quad + \sum_{i=1}^{m}\bar{d}_{ij}\int_{-\infty}^{t} \bar{k}_{ij}(t-s)\bigl(\bar{g}_i\bigl(u_i(s,x)\bigr) - \bar{g}_i\bigl(u_i^*\bigr)\bigr)\,ds.\end{aligned} \tag{12}$$

Multiplying (11) by $u_i - u_i^*$ and integrating over Ω yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \bigl(u_i - u_i^*\bigr)^2\,dx \\ &\quad = \int_\Omega \sum_{k=1}^{l}\bigl(u_i - u_i^*\bigr)\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial(u_i - u_i^*)}{\partial x_k}\right)dx\end{aligned}$$

$$
\begin{aligned}
&-p_i'(\xi_i)\int_\Omega \big(u_i-u_i^*\big)^2\,dx+\int_\Omega \sum_{j=1}^n \big(b_{ji}\big(u_i-u_i^*\big)\big(f_j(v_j)-f_j\big(v_j^*\big)\big)\big)\,dx\\
&+\sum_{j=1}^n \int_\Omega \big(\tilde{b}_{ji}\big(u_i-u_i^*\big)\big(\tilde{f}_j\big(v_j\big(t-\theta_{ji}(t),x\big)\big)-\tilde{f}_j\big(v_j^*\big)\big)\big)\,dx\\
&+\sum_{j=1}^n \int_\Omega \left[\left(\bar{b}_{ji}\big(u_i-u_i^*\big)\int_{-\infty}^t k_{ji}(t-s)\big(\bar{f}_j\big(v_j(s,x)\big)-\bar{f}_j\big(v_j^*\big)\big)\right)ds\right]dx.
\end{aligned}
\tag{13}
$$

According to Green's formula and the Dirichlet boundary condition, we get

$$
\int_\Omega \sum_{k=1}^l \big(u_i-u_i^*\big)\frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial(u_i-u_i^*)}{\partial x_k}\right)dx=-\sum_{k=1}^l \int_\Omega D_{ik}\left(\frac{\partial(u_i-u_i^*)}{\partial x_k}\right)^2 dx.
\tag{14}
$$

Moreover from Lemma 1, we have

$$
-\sum_{k=1}^l \int_\Omega D_{ik}\left(\frac{\partial(u_i-u_i^*)}{\partial x_k}\right)^2 dx\le-\int_\Omega \sum_{k=1}^l \frac{D_{ik}}{d_k^2}\big(u_i-u_i^*\big)^2\,dx\le -D_i l\big\|u_i-u_i^*\big\|_2^2.
\tag{15}
$$

From (11)-(15), (A2), (A3) and the Holder integral inequality, we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \big|u_i-u_i^*\big|^2\,dx\\
&\quad\le -2D_i l\int_\Omega \big|u_i-u_i^*\big|^2\,dx-2a_i\int_\Omega \big|u_i-u_i^*\big|^2\,dx\\
&\qquad+2\int_\Omega \sum_{j=1}^n \big(|b_{ji}|\big|u_i-u_i^*\big|L_j^f\big|v_j-v_j^*\big|\big)\,dx\\
&\qquad+2\sum_{j=1}^n \int_\Omega \big(|\tilde{b}_{ji}|\big|u_i-u_i^*\big|\big|\tilde{f}\big(v_j\big(t-\theta_{ji}(t),x\big)\big)-\tilde{f}\big(v_j^*\big)\big|\big)\,dx\\
&\qquad+2\sum_{j=1}^n \int_\Omega \left[\left(|\bar{b}_{ji}|\int_{-\infty}^t k_{ji}(t-s)\big|u_i-u_i^*\big|\big|\bar{f}_j\big(v_j(s,x)\big)-\bar{f}_j\big(v_j^*\big)\big|\right)ds\right]dx.
\end{aligned}
\tag{16}
$$

Multiplying both sides of (12) by $v_j-v_j^*$, similarly, we also have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \big|v_j-v_j^*\big|^2\,dx\\
&\quad\le -2D_j^* l\int_\Omega \big|v_j-v_j^*\big|^2\,dx-2c_j\int_\Omega \big|v_j-v_j^*\big|^2\,dx\\
&\qquad+2\int_\Omega \sum_{i=1}^m \big(|d_{ij}|L_i^g\big|u_i-u_i^*\big|\big|v_j-v_j^*\big|\big)\,dx\\
&\qquad+2\sum_{i=1}^m \int_\Omega \big(|\tilde{d}_{ij}|\big|\tilde{g}\big(u_i\big(t-\tau_{ij}(t),x\big)\big)-\tilde{g}\big(u_i^*\big)\big|\big|v_j-v_j^*\big|\big)\,dx\\
&\qquad+2\sum_{i=1}^m \int_\Omega \left[|\bar{d}_{ij}|\int_{-\infty}^t \bar{k}_{ij}(t-s)\big|\bar{g}_i\big(u_i(s,x)\big)-\bar{g}_i\big(u_i^*\big)\big|\big|v_j-v_j^*\big|\,ds\right]dx.
\end{aligned}
\tag{17}
$$

Choose a Lyapunov functional as follows:

$$\begin{aligned}
V(t) ={}& \int_\Omega \sum_{i=1}^m w_i \Bigg[n a_i^{r-1} \left|u_i - u_i^*\right|^r e^{2\alpha t} \\
&+ \sum_{j=1}^n |\tilde{b}_{ji}|^r n^r \frac{e^\theta}{1-\mu_\theta} \int_{t-\theta_{ji}(t)}^t e^{2\alpha\xi} \left|\tilde{f}_j\big(v_j(\xi,x)\big) - \tilde{f}_j\big(v_j^*\big)\right|^r d\xi \\
&+ \sum_{j=1}^n |\bar{b}_{ji}|^r n^r \beta_{ji}^r \int_0^{+\infty} k_{ji}(s) \int_{t-s}^t e^{2\alpha(s+\xi)} \left|\bar{f}_j\big(v_j(\xi,x)\big) - \bar{f}_j\big(v_j^*\big)\right|^r d\xi\, ds \Bigg] dx \\
&+ \int_\Omega \sum_{j=1}^n w_{m+j} \Bigg[m c_j^{r-1} \left|v_j - v_j^*\right|^r e^{2\alpha t} \\
&+ \sum_{i=1}^m |\tilde{d}_{ij}|^r m^r \frac{e^\tau}{1-\mu_\tau} \int_{t-\tau_{ij}(t)}^t e^{2\alpha\xi} \left|\tilde{g}_i\big(u_i(\xi,x)\big) - \tilde{g}_i\big(u_i^*\big)\right|^r d\xi \\
&+ \sum_{i=1}^m |\bar{d}_{ij}|^r m^r \gamma_{ij}^r \int_0^{+\infty} \bar{k}_{ij}(s) \int_{t-s}^t e^{2\alpha(s+\xi)} \left|\bar{g}_i\big(u_i(\xi,x)\big) - \bar{g}_i\big(u_i^*\big)\right|^r d\xi\, ds \Bigg] dx.
\end{aligned}$$

Its upper Dini-derivative along the solution to system (1) can be calculated as follows:

$$\begin{aligned}
D^+V(t) \le{}& \int_\Omega \sum_{i=1}^m w_i \Bigg[r n a_i^{r-1} \left|u_i - u_i^*\right|^{r-1} \frac{\partial |u_i - u_i^*|}{\partial t} e^{2\alpha t} + 2\alpha e^{2\alpha t} n a_i^{r-1} \left|u_i - u_i^*\right|^r \\
&+ e^{2\alpha t} \sum_{j=1}^n |\tilde{b}_{ji}|^r n^r \frac{e^\theta}{1-\mu_\theta} \left|\tilde{f}_j\big(v_j(t,x)\big) - \tilde{f}_j\big(v_j^*\big)\right|^r \\
&- \sum_{j=1}^n |\tilde{b}_{ji}|^r n^r \frac{e^\theta}{1-\mu_\theta} \big(1 - \dot{\theta}_{ji}(t)\big) e^{2\alpha(t-\theta_{ji}(t))} \left|\tilde{f}_j\big(v_j\big(t-\theta_{ji}(t),x\big)\big) - \tilde{f}_j\big(v_j^*\big)\right|^r \\
&+ e^{2\alpha t} \sum_{j=1}^n |\bar{b}_{ji}|^r n^r \beta_{ji}^r \int_0^{+\infty} e^{2\alpha s} k_{ji}(s) \left|\bar{f}_j\big(v_j(t,x)\big) - \bar{f}_j\big(v_j^*\big)\right|^r ds \\
&- e^{2\alpha t} \sum_{j=1}^n |\bar{b}_{ji}|^r n^r \beta_{ji}^r \int_0^{+\infty} k_{ji}(s) \left|\bar{f}_j\big(v_j(t-s,x)\big) - \bar{f}_j\big(v_j^*\big)\right|^r ds \Bigg] dx \\
&+ \int_\Omega \sum_{j=1}^n w_{m+j} \Bigg[r m c_j^{r-1} \left|v_j - v_j^*\right|^{r-1} \frac{\partial |v_j - v_j^*|}{\partial t} e^{2\alpha t} + 2\alpha e^{2\alpha t} m c_j^{r-1} \left|v_j - v_j^*\right|^r \\
&+ e^{2\alpha t} \sum_{i=1}^m |\tilde{d}_{ij}|^r m^r \frac{e^\tau}{1-\mu_\tau} \left|\tilde{g}_i\big(u_i(t,x)\big) - \tilde{g}_i\big(u_i^*\big)\right|^r \\
&- \sum_{i=1}^m |\tilde{d}_{ij}|^r m^r \frac{e^\tau}{1-\mu_\tau} e^{2\alpha(t-\tau_{ij}(t))} (1 - \dot{\tau}_{ij}) \left|\tilde{g}_i\big(u_i\big(t-\tau_{ij}(t),x\big)\big) - \tilde{g}_i\big(u_i^*\big)\right|^r \\
&+ e^{2\alpha t} \sum_{i=1}^m |\bar{d}_{ij}|^r m^r \gamma_{ij}^r \int_0^{+\infty} e^{2\alpha s} \bar{k}_{ij}(s) \left|\bar{g}_i\big(u_i(t,x)\big) - \bar{g}_i\big(u_i^*\big)\right|^r ds \\
&- e^{2\alpha t} \sum_{i=1}^m |\bar{d}_{ij}|^r m^r \gamma_{ij}^r \int_0^{+\infty} \bar{k}_{ij}(s) \left|\bar{g}_i\big(u_i(t-s,x)\big) - \bar{g}_i\big(u_i^*\big)\right|^r ds \Bigg] dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_\Omega \sum_{i=1}^{m} w_i \Biggl[rna_i^{r-1} \bigl|u_i - u_i^*\bigr|^{r-2} e^{2\alpha t} \Biggl(-D_i l \bigl|u_i - u_i^*\bigr|^2 - a_i \bigl|u_i - u_i^*\bigr|^2 \\
&\quad + \sum_{j=1}^{n} \bigl(|b_{ji}| \bigl|u_i - u_i^*\bigr| L_j^f \bigl|v_j - v_j^*\bigr|\bigr) \\
&\quad + \sum_{j=1}^{n} \bigl(|\tilde{b}_{ji}| \bigl|u_i - u_i^*\bigr| \bigl|\tilde{f}\bigl(v_j\bigl(t - \theta_{ji}(t), x\bigr)\bigr) - \tilde{f}\bigl(v_j^*\bigr)\bigr|\bigr) \\
&\quad + \sum_{j=1}^{n} \biggl(|\bar{b}_{ji}| \int_{-\infty}^{t} k_{ji}(t-s) \bigl|u_i - u_i^*\bigr| \bigl|\bar{f}_j\bigl(v_j(s,x)\bigr) - \bar{f}_j\bigl(v_j^*\bigr)\bigr| \biggr) \, ds \Biggr) \\
&\quad + 2\alpha e^{2\alpha t} na_i^{r-1} \bigl|u_i - u_i^*\bigr|^r + e^{2\alpha t} \sum_{j=1}^{n} |\tilde{b}_{ji}|^r n^r \frac{e^\theta}{1-\mu_\theta} \bigl|\tilde{f}_j\bigl(v_j(t,x)\bigr) - \tilde{f}_j\bigl(v_j^*\bigr)\bigr|^r \\
&\quad - e^{2\alpha t} \sum_{j=1}^{n} |\tilde{b}_{ji}|^r n^r \bigl|\tilde{f}_j\bigl(v_j(t - \theta_{ji}, x)\bigr) - \tilde{f}_j\bigl(v_j^*\bigr)\bigr|^r \\
&\quad + e^{2\alpha t} \sum_{j=1}^{n} |\bar{b}_{ji}|^r n^r \beta_{ji}^r \int_0^{+\infty} e^{2\alpha s} k_{ji}(s) \bigl|\bar{f}_j\bigl(v_j(t,x)\bigr) - \bar{f}_j\bigl(v_j^*\bigr)\bigr|^r \, ds \\
&\quad - e^{2\alpha t} \sum_{j=1}^{n} |\bar{b}_{ji}|^r n^r \beta_{ji}^r \int_0^{+\infty} k_{ji}(s) \bigl|\bar{f}_j\bigl(v_j(t-s,x)\bigr) - \bar{f}_j\bigl(v_j^*\bigr)\bigr|^r \, ds \Biggr] \, dx \\
&\quad + \int_\Omega \sum_{j=1}^{n} w_{m+j} \Biggl[rmc_j^{r-1} \bigl|v_j - v_j^*\bigr|^{r-2} e^{2\alpha t} \Biggl(-D_j^* l \bigl|v_j - v_j^*\bigr|^2 - c_j \bigl|v_j - v_j^*\bigr|^2 \\
&\quad + \sum_{i=1}^{m} \bigl(|d_{ij}| L_i^g \bigl|u_i - u_i^*\bigr| \bigl|v_j - v_j^*\bigr|\bigr) + \sum_{i=1}^{m} \bigl(|\tilde{d}_{ij}| \bigl|\tilde{g}\bigl(u_i(t - \tau_{ij}, x)\bigr) - \tilde{g}\bigl(u_i^*\bigr)\bigr| \bigl|v_j - v_j^*\bigr|\bigr) \\
&\quad + \sum_{i=1}^{m} \biggl(|\bar{d}_{ij}| \int_{-\infty}^{t} \bar{k}_{ij}(t-s) \bigl|\bar{g}_i\bigl(u_i(s,x)\bigr) - \bar{g}_i\bigl(u_i^*\bigr)\bigr| \bigl|v_j - v_j^*\bigr| \biggr) \, ds \Biggr) \\
&\quad + 2\alpha e^{2\alpha t} mc_j^{r-1} \bigl|v_j - v_j^*\bigr|^r + e^{2\alpha t} \sum_{i=1}^{m} |\tilde{d}_{ij}|^r m^r \frac{e^\tau}{1-\mu_\tau} \bigl|\tilde{g}_i\bigl(u_i(t,x)\bigr) - \tilde{g}_i\bigl(u_i^*\bigr)\bigr|^r \\
&\quad - e^{2\alpha t} \sum_{i=1}^{m} |\tilde{d}_{ij}|^r m^r \bigl|\tilde{g}_i\bigl(u_i(t - \tau_{ij}, x)\bigr) - \tilde{g}_i\bigl(u_i^*\bigr)\bigr|^r \\
&\quad + e^{2\alpha t} \sum_{i=1}^{m} |\bar{d}_{ij}|^r m^r \gamma_{ij}^r \int_0^{+\infty} e^{2\alpha s} \bar{k}_{ji}(s) \bigl|\bar{g}_i\bigl(u_i(t,x)\bigr) - \bar{g}_i\bigl(u_i^*\bigr)\bigr|^r \, ds \\
&\quad - e^{2\alpha t} \sum_{i=1}^{m} |\bar{d}_{ij}|^r m^r \gamma_{ij}^r \int_0^{+\infty} \bar{k}_{ij}(s) \bigl|\bar{g}_i\bigl(u_i(t-s,x)\bigr) - \bar{g}_i\bigl(u_i^*\bigr)\bigr|^r \, ds \Biggr] \, dx. \qquad (18)
\end{aligned}
$$

From (18) and the Young inequality, we can conclude

$$
\begin{aligned}
D^+ V(t) &\le \int_\Omega e^{2\alpha t} \sum_{i=1}^{m} w_i \Biggl[\Biggl(-rna_i^{r-1} D_i l \bigl|u_i - u_i^*\bigr|^r - rna_i^r \bigl|u_i - u_i^*\bigr|^r \\
&\quad + (r-1) \sum_{j=1}^{n} a_i^r \bigl|u_i - u_i^*\bigr|^r
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j=1}^{n} \bigl(n^r |b_{ji}|^r \bigl(L_j^f\bigr)^r \bigl|v_j - v_j^*\bigr|^r\bigr) + (r-1) \sum_{j=1}^{n} a_i^r \bigl|u_i - u_i^*\bigr|^r \\
&+ \sum_{j=1}^{n} |\tilde{b}_{ji}|^r n^r \bigl|\tilde{f}_j\bigl(v_j\bigl(t-\theta_{ji}(t),x\bigr)\bigr) - \tilde{f}_j\bigl(v_j^*\bigr)\bigr|^r \\
&+ (r-1) \sum_{j=1}^{n} \biggl(a_i^r \beta_{ji}^{-\frac{r}{r-1}} \int_{-\infty}^{t} k_{ji}(t-s)\bigl|u_i - u_i^*\bigr|^r \, ds\biggr) \\
&+ \sum_{j=1}^{n} \biggl(|\bar{b}_{ji}|^r n^r \beta_{ji}^r \int_{-\infty}^{t} k_{ji}(t-s)\bigl|\bar{f}_j\bigl(v_j(s,x)\bigr) - \bar{f}_j\bigl(v_j^*\bigr)\bigr|^r \, ds\biggr)\biggr) + 2\alpha n a_i^{r-1}\bigl|u_i - u_i^*\bigr|^r \\
&+ \sum_{j=1}^{n} |\tilde{b}_{ji}|^r n^r \frac{e^{\theta}}{1-\mu_\theta} \bigl|\tilde{f}_j\bigl(v_j(t,x)\bigr) - \tilde{f}_j\bigl(v_j^*\bigr)\bigr|^r \\
&- \sum_{j=1}^{n} |\tilde{b}_{ji}|^r n^r \bigl|\tilde{f}_j\bigl(v_j\bigl(t-\theta_{ji}(t),x\bigr)\bigr) - \tilde{f}_j\bigl(v_j^*\bigr)\bigr|^r \\
&+ \sum_{j=1}^{n} |\bar{b}_{ji}|^r n^r \beta_{ji}^r \int_{0}^{+\infty} e^{2\alpha s} k_{ji}(s)\bigl|\bar{f}_j\bigl(v_j(t,x)\bigr) - \bar{f}_j\bigl(v_j^*\bigr)\bigr|^r \, ds \\
&- \sum_{j=1}^{n} |\bar{b}_{ji}|^r n^r \beta_{ji}^r \int_{0}^{+\infty} k_{ji}(s)\bigl|\bar{f}_j\bigl(v_j(t-s,x)\bigr) - \bar{f}_j\bigl(v_j^*\bigr)\bigr|^r \, ds\biggr] \, dx \\
&+ \int_{\Omega} e^{2\alpha t} \sum_{j=1}^{n} w_{m+j} \biggl[\biggl(-rmc_j^{r-1} D_j^* l \bigl|v_j - v_j^*\bigr|^r - rmc_j^r \bigl|v_j - v_j^*\bigr|^r \\
&+ (r-1) \sum_{i=1}^{m} c_j^r \bigl|v_j - v_j^*\bigr|^r + \sum_{i=1}^{m} \bigl(|d_{ij}|^r m^r \bigl(L_i^g\bigr)^r \bigl|u_i - u_i^*\bigr|^r\bigr) \\
&+ (r-1) \sum_{i=1}^{m} \bigl(c_j^r \bigl|v_j - v_j^*\bigr|^r\bigr) + \sum_{i=1}^{m} |\tilde{d}_{ij}|^r m^r \bigl|\tilde{g}_i\bigl(u_i\bigl(t-\tau_{ij}(t),x\bigr)\bigr) - \tilde{g}_i\bigl(u_i^*\bigr)\bigr|^r \\
&+ (r-1) \sum_{i=1}^{m} \biggl(c_j^r \gamma_{ij}^{-\frac{r}{r-1}} \int_{-\infty}^{t} \bar{k}_{ij}(t-s)\bigl|v_j - v_j^*\bigr|^r \, ds\biggr) \\
&+ \sum_{i=1}^{m} \biggl(|\bar{d}_{ij}|^r m^r \gamma_{ij}^r \int_{-\infty}^{t} \bar{k}_{ij}(t-s)\bigl|\bar{g}_i\bigl(u_i(s,x)\bigr) - \bar{g}_i\bigl(u_i^*\bigr)\bigr|^r \, ds\biggr)\biggr) \\
&+ 2\alpha m c_j^{r-1}\bigl|v_j - v_j^*\bigr|^r \\
&+ \sum_{i=1}^{m} |\tilde{d}_{ij}|^r m^r \frac{e^{\tau}}{1-\mu_\tau} \bigl|\tilde{g}_i\bigl(u_i(t,x)\bigr) - \tilde{g}_i\bigl(u_i^*\bigr)\bigr|^r \\
&- \sum_{i=1}^{m} |\tilde{d}_{ij}|^r m^r \bigl|\tilde{g}_i\bigl(u_i\bigl(t-\tau_{ij}(t),x\bigr)\bigr) - \tilde{g}_i\bigl(u_i^*\bigr)\bigr|^r \\
&+ \sum_{i=1}^{m} |\bar{d}_{ij}|^r m^r \gamma_{ij}^r \int_{0}^{+\infty} e^{2\alpha s} \bar{k}_{ij}(s)\bigl|\bar{g}_i\bigl(u_i(t,x)\bigr) - \bar{g}_i\bigl(u_i^*\bigr)\bigr|^r \, ds \\
&- \sum_{i=1}^{m} |\bar{d}_{ij}|^r m^r \gamma_{ij}^r \int_{0}^{+\infty} \bar{k}_{ij}(s)\bigl|\bar{g}_i\bigl(u_i(t-s,x)\bigr) - \bar{g}_i\bigl(u_i^*\bigr)\bigr|^r \, ds\biggr] \, dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_\Omega e^{2\alpha t}\sum_{i=1}^{m}\Bigg[w_i\Bigg(-rna_i^{r-1}D_i l - rna_i^r + 2(r-1)\sum_{j=1}^{n} a_i^r + 2\alpha n a_i^{r-1} \\
&\quad + (r-1)\sum_{j=1}^{n}\bigg(a_i^r\beta_{ji}^{-\frac{r}{r-1}}\int_{-\infty}^{t}k_{ji}(t-s)\,ds\bigg)\Bigg) + \sum_{j=1}^{n} w_{m+j}m^r\bigg(|d_{ij}|^r\big(L_i^g\big)^r \\
&\quad + |\tilde{d}_{ij}|^r\frac{e^\tau}{1-\mu_\tau}\big(L_i^{\tilde{g}}\big)^r + |\bar{d}_{ij}|^r\gamma_{ij}^r\int_0^{+\infty}e^{2\alpha s}\bar{k}_{ij}(s)\big(L_i^{\bar{g}_i}\big)^r\,ds\bigg)\Bigg]\big|u_i-u_i^*\big|^r\,dx \\
&\quad + \int_\Omega e^{2\alpha t}\sum_{j=1}^{n}\Bigg[w_{m+j}\Bigg(-rmc_j^{r-1}D_j^* l - rmc_j^r + 2(r-1)\sum_{i=1}^{m}c_j^r \\
&\quad + (r-1)\sum_{i=1}^{m}\bigg(c_j^r\gamma_{ij}^{-\frac{r}{r-1}}\int_{-\infty}^{t}\bar{k}_{ij}(t-s)\,ds\bigg) + 2\alpha m c_j^{r-1}\Bigg) + \sum_{i=1}^{m} w_i n^r\bigg(|b_{ji}|^r\big(L_j^f\big)^r \\
&\quad + |\tilde{b}_{ji}|^r\frac{e^\theta}{1-\mu_\theta}\big(L_j^{\tilde{f}}\big)^r + |\bar{b}_{ji}|^r\beta_{ji}^r\big(L_j^{\bar{f}}\big)^r\int_0^{+\infty}e^{2\alpha s}k_{ji}(s)\,ds\bigg)\Bigg]\big|v_j-v_j^*\big|^r\,dx.
\end{aligned}
\tag{19}
$$

From (6), we can conclude

$$
D^+V(t) \le 0 \quad \text{and so} \quad V(t) \le V(0), \quad t \ge 0. \tag{20}
$$

Since

$$
\begin{aligned}
V(0) &= \int_\Omega \sum_{i=1}^{m} w_i\Bigg[na_i^{r-1}\big|u_i(0,x)-u_i^*\big|^r \\
&\quad + \sum_{j=1}^{n}|\tilde{b}_{ji}|^r n^r\frac{e^\theta}{1-\mu_\theta}\int_{-\theta_{ji}(t)}^{0}\big|\tilde{f}_j\big(v_j(\xi,x)\big)-\tilde{f}_j\big(v_j^*\big)\big|^r\,d\xi \\
&\quad + \sum_{j=1}^{n}|\bar{b}_{ji}|^r n^r\beta_{ji}^r\int_0^{+\infty}k_{ji}(s)\int_{-s}^{0}e^{2\alpha(s+\xi)}\big|\bar{f}_j\big(v_j(\xi,x)\big)-\bar{f}_j\big(v_j^*\big)\big|^r\,d\xi\,ds\Bigg]\,dx \\
&\quad + \int_\Omega \sum_{j=1}^{n} w_{m+j}\Bigg[mc_j^{r-1}\big|v_j(0,x)-v_j^*\big|^r \\
&\quad + \sum_{i=1}^{m}|\tilde{d}_{ij}|^r m^r\frac{e^\tau}{1-\mu_\tau}\int_{-\tau_{ij}(t)}^{0}\big|\tilde{g}_i\big(u_i(\xi,x)\big)-\tilde{g}_i\big(u_i^*\big)\big|^r\,d\xi \\
&\quad + \sum_{i=1}^{m}|\bar{d}_{ij}|^r m^r\gamma_{ij}^r\int_0^{+\infty}\bar{k}_{ij}(s)\int_{-s}^{0}e^{2\alpha(s+\xi)}\big|\bar{g}_i\big(u_i(\xi,x)\big)-\bar{g}_i\big(u_i^*\big)\big|^r\,d\xi\,ds\Bigg]\,dx \\
&\le \int_\Omega \sum_{i=1}^{m}\max_{1\le i\le m}\{w_i\}\Bigg[na_i^{r-1}\big|u_i(0,x)-u_i^*\big|^r \\
&\quad + \sum_{j=1}^{n}|\tilde{b}_{ji}|^r n^r\frac{e^\theta}{1-\mu_\theta}\big(L_j^{\tilde{f}}\big)^r\int_{-\theta_{ji}}^{0}\big|v_j(\xi,x)-v_j^*\big|^r\,d\xi \\
&\quad + \sum_{j=1}^{n}|\bar{b}_{ji}|^r n^r\big(L_j^{\bar{f}}\big)^r\beta_{ji}^r\int_0^{+\infty}k_{ji}(s)\int_{-s}^{0}e^{2\alpha(s+\xi)}\big|v_j(\xi,x)-v_j^*\big|^r\,d\xi\,ds\Bigg]\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_\Omega \sum_{j=1}^{n} \max_{1\le j\le n}\{w_{m+j}\} \Bigg[m c_j^{r-1} \big| v_j(0,x) - v_j^* \big|^r \\
&+ \sum_{i=1}^{m} |\tilde{d}_{ij}|^r \big(L_i^{\tilde{g}}\big)^r m^r \frac{e^\tau}{1-\mu_\tau} \int_{-\tau_{ij}}^{0} \big| u_i(\xi,x) - u_i^* \big|^r \, d\xi \\
&+ \sum_{j=1}^{n} |\bar{d}_{ij}|^r m^r \big(L_i^{\bar{g}}\big)^r \gamma_{ij}^r \int_0^{+\infty} \bar{k}_{ji}(s) \int_{-s}^{0} e^{2\alpha(s+\xi)} \big| u_i(\xi,x) - u_i^* \big|^r \, d\xi \, ds \Bigg] dx \\
\le{}& \Bigg\{ \max_{1\le i\le m}\{w_i\} + \max_{1\le j\le n}\{w_{m+j}\} \max_{1\le j\le n} \Bigg[\sum_{i=1}^{m} |\bar{d}_{ij}|^r m^r \big(L_i^{\bar{g}}\big)^r \gamma_{ij}^r \int_0^{+\infty} \bar{k}_{ij}(s) s e^{2\alpha s} \, ds \Bigg] \\
&+ \max_{1\le j\le n}\{w_{m+j}\} \max_{1\le j\le n} \Bigg[\sum_{i=1}^{m} |\tilde{d}_{ij}|^r \big(L_i^{\tilde{g}}\big)^r m^r \frac{e^\tau \tau}{1-\mu_\tau} \Bigg] \Bigg\} \big\| \varphi_u(s,x) - u^* \big\|^r \\
&+ \Bigg\{ \max_{1\le j\le n}\{w_{m+j}\} + \max_{1\le i\le m}\{w_i\} \max_{1\le i\le m} \Bigg[\sum_{j=1}^{n} |\bar{b}_{ji}|^r n^r \big(L_j^{\bar{f}}\big)^r \beta_{ji}^r \int_0^{+\infty} s e^{2\alpha s} k_{ji}(s) \, ds \Bigg] \\
&+ \max_{1\le i\le m}\{w_i\} \max_{1\le i\le m} \Bigg[\sum_{j=1}^{n} |\tilde{b}_{ji}|^r n^r \big(L_j^{\tilde{g}}\big)^r \frac{e^\theta \theta}{1-\mu_\theta} \Bigg] \Bigg\} \big\| \varphi_v(s,x) - v^* \big\|^r. \qquad (21)
\end{aligned}
$$

Noting that

$$
e^{2\alpha t} \Big(\min_{1\le i\le m+n} w_i \Big) \big(\| u(t,x) - u^* \| + \| v(t,x) - v^* \| \big) \le V(t), \quad t \ge 0. \qquad (22)
$$

Let

$$
\begin{aligned}
\beta = \max &\Bigg\{ \max_{1\le i\le m}\{w_i\} + \max_{1\le j\le n}\{w_{m+j}\} \max_{1\le j\le n} \Bigg[\sum_{i=1}^{m} |\bar{d}_{ij}|^r m^r \big(L_i^{\bar{g}}\big)^r \gamma_{ij}^r \int_0^{+\infty} \bar{k}_{ij}(s) s e^{2\alpha s} \, ds \Bigg] \\
&+ \max_{1\le j\le n}\{w_{m+j}\} \max_{1\le j\le n} \Bigg[\sum_{i=1}^{m} |\tilde{d}_{ij}|^r \big(L_i^{\tilde{g}}\big)^r m^r \frac{e^\tau \tau}{1-\mu_\tau} \Bigg], \\
&\max_{1\le j\le n}\{w_{m+j}\} + \max_{1\le i\le m}\{w_i\} \max_{1\le i\le m} \Bigg[\sum_{j=1}^{n} |\bar{b}_{ji}|^r n^r \big(L_j^{\bar{f}}\big)^r \beta_{ji}^r \int_0^{+\infty} s e^{2\alpha s} k_{ji}(s) \, ds \Bigg] \\
&+ \max_{1\le i\le m}\{w_i\} \max_{1\le i\le m} \Bigg[\sum_{j=1}^{n} |\tilde{b}_{ji}|^r n^r \big(L_j^{\tilde{g}}\big)^r \frac{e^\theta \theta}{1-\mu_\theta} \Bigg] \Bigg\} \Big/ \min_{1\le i\le m+n}\{w_i\}.
\end{aligned}
$$

Clearly, $\beta \ge 1$.

It follows that

$$
\| u(t,x) - u^* \| + \| v(t,x) - v^* \| \le \beta e^{-2\alpha t} \big(\| \varphi_u(s,x) - u^* \| + \| \varphi_v(s,x) - v^* \| \big),
$$

for any $t \ge 0$, where $\beta \ge 1$ is a constant. This implies that the solution of (1) is globally exponentially stable. This completes the proof of Theorem 1. □

Remark 3 In this paper, the derived sufficient condition includes diffusion terms. Unfortunately, in the proof in the previous papers [21–24], a negative integral term with gradient is left out in their deduction. This leads to the fact that those criteria are irrelevant to the diffusion term. Obviously, Lyapunov functional to construct is more general and our results expand the model in [22, 29].

When $\tilde{b}_{ji} = 0$ and $\tilde{d}_{ij} = 0$ $(i = 1, 2, \dots, m, j = 1, 2, \dots, n)$, system (1) becomes the following BAM NNs with distributed delays and reaction-diffusion terms:

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right) - p_i\big(u_i(t,x)\big) + \sum_{j=1}^{n}\big(b_{ji}f_j\big(v_j(t,x)\big)\big) \\
&\quad + \sum_{j=1}^{n}\bar{b}_{ji}\int_{-\infty}^{t} k_{ji}(t-s)\bar{f}_j\big(v_j(s,x)\big)\,ds + I_i(t), \\
\frac{\partial v_j}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D^*_{jk}\frac{\partial v_j}{\partial x_k}\right) - q_j\big(v_j(t,x)\big) + \sum_{i=1}^{m}\big(d_{ij}g_i\big(u_i(t,x)\big)\big) \\
&\quad + \sum_{i=1}^{m}\bar{d}_{ij}\int_{-\infty}^{t} \bar{k}_{ij}(t-s)\bar{g}_i\big(u_i(s,x)\big)\,ds + J_j(t).
\end{aligned}
\tag{23}
$$

For (23), we get the following result.

Corollary 1 *Let* (A1)-(A4) *be in force. If there exist* $w_i > 0$ $(i = 1, 2, \dots, n + m)$, $r \geq 2$, $\gamma_{ij} > 0$, $\beta_{ji} > 0$ *such that*

$$
\begin{aligned}
&w_i\left(-rna_i^{r-1}D_i l - rna_i^r + 2(r-1)\sum_{j=1}^{n} a_i^r + (r-1)\sum_{j=1}^{n} a_i^r \beta_{ji}^{-\frac{r}{r-1}}\right) \\
&\quad + \sum_{j=1}^{n} w_{m+j} m^r \big(|d_{ij}|^r \big(L_i^g\big)^r + |\bar{d}_{ij}|^r \gamma_{ij}^r \big(L_i^{\bar{g}}\big)^r\big) < 0
\end{aligned}
$$

and

$$
\begin{aligned}
&w_{m+j}\left(-rmc_j^{r-1}D_j^* l - rmc_j^r + 2(r-1)\sum_{i=1}^{m} c_j^r + (r-1)\sum_{i=1}^{m} c_j^r \gamma_{ij}^{-\frac{r}{r-1}}\right) \\
&\quad + \sum_{i=1}^{m} w_i n^r \big(|b_{ji}|^r \big(L_j^f\big)^r + |\bar{b}_{ji}|^r \beta_{ji}^r \big(L_j^{\bar{f}}\big)^r\big) < 0,
\end{aligned}
\tag{24}
$$

where $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $L_j^f, L_j^{\tilde{f}}, L_j^{\bar{f}}, L_i^g, L_i^{\tilde{g}}$ *and* $L_i^{\bar{g}}$ *are Lipschitz constants. Then the equilibrium point* (u^*, v^*) *of system* (1) *is unique and globally exponentially stable.*

4 Periodic solutions

In this section, we consider the stability criterion for periodic oscillatory solutions of system (1), in which external input $I_i : R^+ \to R$, $i = 1, 2, \dots, m$, and $J_j : R^+ \to R$, $j = 1, 2, \dots, n$, are continuously periodic functions with period ω, that is,

$$
I_i(t+\omega) = I_i(t), \qquad J_j(t+\omega) = J_j(t), \quad i = 1, 2, \dots, m, j = 1, 2, \dots, n.
$$

By constructing a Poincaré mapping, the existence of a unique ω-periodic solution and its stability are readily established.

Theorem 2 *Let* (A1)-(A4) *be in force. There exists only one ω-periodic solution of system* (1), *and all other solutions converge exponentially to it as $t \to +\infty$ if there exist constants $w_i > 0$ ($i = 1, 2, \dots, n + m$), $r \geq 2$, $\gamma_{ij} > 0$, $\beta_{ji} > 0$ ($i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$) such that*

$$\begin{aligned} & w_i \left(-rna_i^{r-1} D_i l - rna_i^r + 2(r-1) \sum_{j=1}^{n} a_i^r + (r-1) \sum_{j=1}^{n} a_i^r \beta_{ji}^{-\frac{r}{r-1}} \right) \\ & \quad + \sum_{j=1}^{n} w_{m+j} m^r \left(|d_{ij}|^r \left(L_i^g\right)^r + |\tilde{d}_{ij}|^r \frac{e^\tau}{1-\mu_\tau} \left(L_i^{\tilde{g}}\right)^r + |\bar{d}_{ij}|^r \gamma_{ij}^r \left(L_i^{\bar{g}}\right)^r \right) < 0 \end{aligned}$$

and

$$\begin{aligned} & w_{m+j} \left(-rmc_j^{r-1} D_j^* l - rmc_j^r + 2(r-1) \sum_{i=1}^{m} c_j^r + (r-1) \sum_{i=1}^{m} c_j^r \gamma_{ij}^{-\frac{r}{r-1}} \right) \\ & \quad + \sum_{i=1}^{m} w_i n^r \left(|b_{ji}|^r \left(L_j^f\right)^r + |\tilde{b}_{ji}|^r \frac{e^\theta}{1-\mu_\theta} \left(L_j^{\tilde{f}}\right)^r + |\bar{b}_{ji}|^r \beta_{ji}^r \left(L_j^{\bar{f}}\right)^r \right) < 0, \end{aligned} \tag{25}$$

where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, L_j^f, $L_j^{\tilde{f}}$, $L_j^{\bar{f}}$, L_i^g, $L_i^{\tilde{g}}$ and $L_i^{\bar{g}}$ are Lipschitz constants in (A3).

Proof For any $\binom{\varphi_u}{\varphi_v}, \binom{\psi_u}{\psi_v} \in C$, we denote the solutions of system (1) through $\left(\binom{0}{0}\right), \left(\binom{\varphi_u}{\varphi_v}\right)$ and $\left(\binom{0}{0}\right), \left(\binom{\psi_u}{\psi_v}\right)$ as

$$u(t, \varphi_u, x) = \left(u_1(t, \varphi_u, x), \dots, u_m(t, \varphi_u, x)\right)^T, \quad v(t, \varphi_v, x) = \left(v_1(t, \varphi_v, x), \dots, v_n(t, \varphi_v, x)\right)^T$$

and

$$u(t, \psi_u, x) = \left(u_1(t, \psi_u, x), \dots, u_m(t, \psi_u, x)\right)^T, \quad v(t, \psi_v, x) = \left(v_1(t, \psi_v, x), \dots, v_n(t, \psi_v, x)\right)^T,$$

respectively. Define

$$\begin{aligned} & u_t(\varphi_u, x) = u(t + \theta, \varphi_u, x), \quad \theta \in (-\infty, 0], t \geq 0, \\ & v_t(\varphi_v, x) = v(t + \theta, \varphi_v, x), \quad \theta \in (-\infty, 0], t \geq 0. \end{aligned}$$

Clearly, for any $t \geq 0$, $\binom{u_t(\varphi_u)}{v_t(\varphi_v)} \in C$. Now, we define

$$y_i = u_i(t, \varphi_u, x) - u_i(t, \psi_u, x), \qquad z_j = v_j(t, \varphi_v, x) - v_j(t, \psi_v, x).$$

Thus, we can obtain from system (1) that

$$\begin{aligned} \frac{\partial y_i}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k} \left(D_{ik} \frac{\partial y_i}{\partial x_k} \right) - \left(p_i\left(u_i(t, \varphi_u, x)\right) - p_i\left(u_i(t, \psi_u, x)\right) \right) \\ & \quad + \sum_{j=1}^{n} \left(b_{ji} \left(f_j\left(v_j(t, \varphi_v, x)\right) - f_j\left(v_j(t, \psi_v, x)\right) \right) \right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\left(\tilde{b}_{ji}\left(\tilde{f}_j\left(v_j\left(t-\theta_{ji}(t),\varphi_v,x\right)\right)-\tilde{f}_j\left(v_j\left(t-\theta_{ji}(t),\psi_v,x\right)\right)\right)\right)\\
&+\sum_{j=1}^{n}\left(\bar{b}_{ji}\int_{-\infty}^{t}k_{ji}(t-s)\left(\bar{f}_j\left(v_j(s,\varphi_v,x)\right)-\bar{f}_j\left(v_j(s,\psi_v,x)\right)\right)\right)ds,\\
\frac{\partial z_j}{\partial t}={}&\sum_{k=1}^{l}\frac{\partial}{\partial x_k}\left(D_{jk}^{*}\frac{\partial z_j}{\partial x_k}\right)-\left(q_j\left(v_j(t,\varphi_v,x)\right)-q_j\left(v_j(t,\psi_v,x)\right)\right)\\
&+\sum_{i=1}^{m}\left(d_{ij}\left(g_i\left(u_i(t,\varphi_u,x)\right)-g_i\left(u_i(t,\psi_u,x)\right)\right)\right)\\
&+\sum_{i=1}^{m}\left(\tilde{d}_{ij}\left(\tilde{g}_i\left(u_i\left(t-\tau_{ij}(t),\varphi_u,x\right)\right)-\tilde{g}_i\left(u_i\left(t-\tau_{ij}(t),\psi_u,x\right)\right)\right)\right)\\
&+\sum_{i=1}^{m}\left(\bar{d}_{ij}\int_{-\infty}^{t}\bar{k}_{ij}(t-s)\left(\bar{g}_i\left(u_i(s,\varphi_u,x)\right)-\bar{g}_i\left(u_i(s,\psi_u,x)\right)\right)ds\right).
\end{aligned}
$$

We consider the following Lyapunov functional:

$$
\begin{aligned}
V(t)={}&\int_{\Omega}\sum_{i=1}^{m}w_i\Bigg[na_i^{r-1}|y_i|^{r}e^{2\alpha t}\\
&+\sum_{j=1}^{n}|\tilde{b}_{ji}|^{r}n^{r}(1-\mu_\theta)\int_{t-\theta_{ji}(t)}^{t}\left|\tilde{f}_j\left(v_j(\xi,\varphi_v,x)\right)-\tilde{f}_j\left(v_j(\xi,\psi_v,x)\right)\right|^{r}d\xi\\
&+\sum_{j=1}^{n}|\bar{b}_{ji}|^{r}n^{r}\beta_{ji}^{r}\int_{0}^{+\infty}k_{ji}(s)\int_{t-s}^{t}e^{2\alpha(s+\xi)}\\
&\times\left|\bar{f}_j\left(v_j(\xi,\varphi_v,x)\right)-\bar{f}_j\left(v_j(\xi,\psi_v,x)\right)\right|^{r}d\xi\,ds\Bigg]dx\\
&+\int_{\Omega}\sum_{j=1}^{n}w_{m+j}\Bigg[mc_j^{r-1}|z_j|^{r}e^{2\alpha t}\\
&+\sum_{i=1}^{m}|\tilde{d}_{ij}|^{r}m^{r}(1-\mu_\tau)\int_{t-\tau_{ij}(t)}^{t}\left|\tilde{g}_i\left(u_i(\xi,\varphi_u,x)\right)-\tilde{g}_i\left(u_i(\xi,\psi_u,x)\right)\right|^{r}d\xi\,ds\\
&+\sum_{i=1}^{m}|\bar{d}_{ij}|^{r}m^{r}\gamma_{ij}^{r}\int_{0}^{+\infty}\bar{k}_{ij}(s)\int_{t-s}^{t}e^{2\alpha(s+\xi)}\\
&\times\left|\bar{g}_i\left(u_i(\xi,\varphi_u,x)\right)-\bar{g}_i\left(u_i(\xi,\psi_u,x)\right)\right|^{r}d\xi\,ds\Bigg]dx.
\end{aligned}
$$

By a minor modification of the proof of Theorem 1, we can easily get

$$
\begin{aligned}
&\left\|u(t,\varphi_u,x)-u(t,\psi_u,x)\right\|+\left\|v(t,\varphi_v,x)-v(t,\psi_v,x)\right\|\\
&\quad\le\beta e^{-2\alpha t}\left(\|\varphi_u-\psi_u\|+\|\varphi_v-\psi_v\|\right)
\end{aligned}
\tag{26}
$$

for $t \geq 0$,in which $\beta \geq 1$ is a constant. Now, we can choose a positive integer N such that

$$\beta e^{-\alpha N\omega} \leq \frac{1}{4}, \qquad \beta e^{-\alpha N\omega} \leq \frac{1}{4}. \tag{27}$$

Defining a Poincaré mapping $P: C \to C$ by

$$P\begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix} = \begin{pmatrix} u_\omega(\varphi_u) \\ v_\omega(\varphi_v) \end{pmatrix}, \tag{28}$$

due to the periodicity of system, we have

$$P^N\begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix} = \begin{pmatrix} u_{N\omega}(\varphi_u) \\ v_{N\omega}(\varphi_v) \end{pmatrix}. \tag{29}$$

Let $t = N\omega$, then from (26)-(29) we can derive that

$$\left|P^N\begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix} - P^N\begin{pmatrix} \psi_u \\ \psi_v \end{pmatrix}\right| \leq \frac{1}{2}\left|\begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix} - \begin{pmatrix} \psi_u \\ \psi_v \end{pmatrix}\right|,$$

which shows that P^N is a contraction mapping. Therefore, there exists a unique fixed point $\binom{\varphi_u^*}{\varphi_v^*} \in C$, namely, $P^N\binom{\varphi_u^*}{\varphi_v^*} = \binom{\varphi_u^*}{\varphi_v^*}$.

Since $P^N(P\binom{\varphi_u^*}{\varphi_v^*}) = P(P^N\binom{\varphi_u^*}{\varphi_v^*}) = P\binom{\varphi_u^*}{\varphi_v^*}$, then $P\binom{\varphi_u^*}{\varphi_v^*}$ is also a fixed point of P^N. Because of the uniqueness of a fixed point of P^N, then $P\binom{\varphi_u^*}{\varphi_v^*} = \binom{\varphi_u^*}{\varphi_v^*}$.

Let $(u(t,\varphi_u^*,x), v(t,\varphi_v^*,x))$ be the solution of system (1) through $(\binom{0}{0}, \binom{\varphi_u^*}{\varphi_v^*})$, then $(u(t+\omega,\varphi_u^*,x), v(t+\omega,\varphi_v^*,x))$ is also a solution of system (1). Clearly,

$$\begin{pmatrix} u_{t+\omega}(\varphi_u^*,x) \\ v_{t+\omega}(\varphi_v^*,x) \end{pmatrix} = \begin{pmatrix} u_t(u_\omega(\varphi_u^*)) \\ v_t(v_\omega(\varphi_v^*)) \end{pmatrix} = \begin{pmatrix} u_t(\varphi_u^*,x) \\ v_t(\varphi_v^*,x) \end{pmatrix}$$

for $t \geq 0$. Hence $(u(t+\omega,\varphi_u^*,x), v(t+\omega,\varphi_v^*,x))^T = (u(t,\varphi_u^*,x), v(t,\varphi_v^*,x))^T$ for $t \geq 0$.

This shows that $(u(t,\varphi_u^*,x), v(t,\varphi_v^*,x))^T$ is exactly one ω-periodic solution of system (1), and it is easy to see that all other solutions of system (1) converge exponentially to it as $t \to +\infty$. The proof is completed. □

5 Illustration example

In this section, a numerical example is given to illustrate the effectiveness of the obtained results.

Example 1 Consider the following system on $\Omega = \{(x_1,x_2)^T | 0 < x_k < \sqrt{0.2}\pi, k = 1,2\} \subset R^2$:

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_{ik}\frac{\partial u_i}{\partial x_k}\right) - p_i(u_i(t,x)) + \sum_{j=1}^{n}(b_{ji}f_j(v_j(t,x))) \\ &\quad + \sum_{j=1}^{n}(\tilde{b}_{ji}\tilde{f}_j(v_j(t-\theta_{ji}(t),x))) + \sum_{j=1}^{n}\bar{b}_{ji}\int_{-\infty}^{t} k_{ji}(t-s)\bar{f}_j(v_j(s,x))\,ds + I_i(t), \\ \frac{\partial v_j}{\partial t} &= \sum_{k=1}^{l} \frac{\partial}{\partial x_k}\left(D_{jk}^*\frac{\partial v_j}{\partial x_k}\right) - q_j(v_j(t,x)) + \sum_{i=1}^{m}(d_{ij}g_i(u_i(t,x))) \\ &\quad + \sum_{i=1}^{m}(\tilde{d}_{ij}\tilde{g}_i(u_i(t-\tau_{ij}(t),x))) + \sum_{i=1}^{m}\bar{d}_{ij}\int_{-\infty}^{t} \bar{k}_{ij}(t-s)\bar{g}_i(u_i(s,x))\,ds + J_j(t), \end{aligned} \tag{30}$$

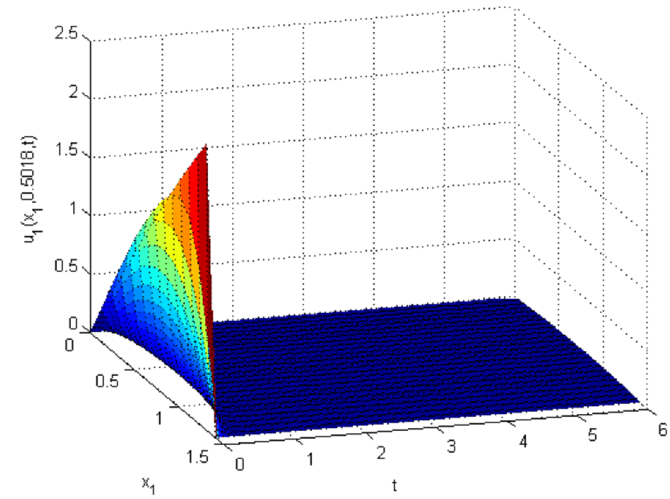

Figure 1 The surface of $u_1(x_1, 0.5018, t)$ when $x_2 = 0.5018$.

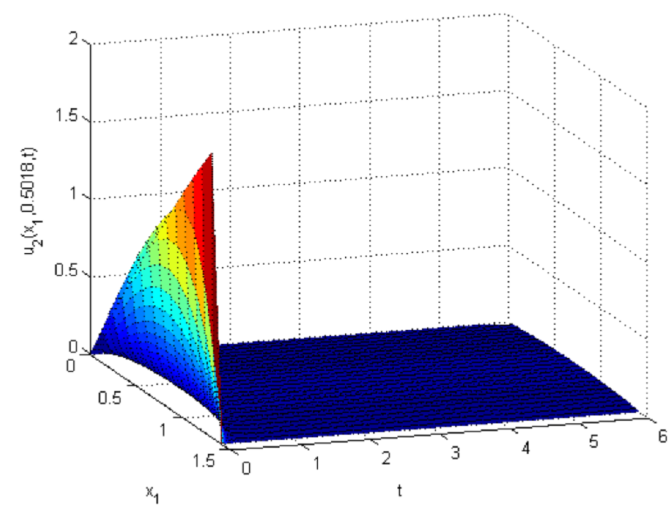

Figure 2 The surface of $u_2(x_1, 0.5018, t)$ when $x_2 = 0.5018$.

$$u_i = 0, \qquad v_j = 0, \qquad t \geq 0, \qquad x \in \partial\Omega,$$

$$u_i(s,x) = \varphi_{ui}(s,x), \qquad v_j(s,x) = \varphi_{vj}(s,x), \quad (s,x) \in (-\infty, 0] \times \Omega,$$

where $k_{ji}(t) = \bar{k}_{ij}(t) = te^{-t}$, $i,j,l = 1,2$. $f_1(\eta) = f_2(\eta) = \tilde{f}_1(\eta) = \tilde{f}_2(\eta) = \bar{f}_1(\eta) = \bar{f}_2(\eta) = g_1(\eta) = g_2(\eta) = \tilde{g}_1(\eta) = \tilde{g}_2(\eta) = \bar{g}_1(\eta) = \bar{g}_2(\eta) = \tanh(\eta)$, $n = m = l = 2$, $\lambda_1 = 2.5$, $\theta_{ji}(t) = \tau_{ij}(t) = 0.02 - 0.01\sin(2\pi t)$, $L_j^f = L_j^{\tilde{f}} = L_j^{\bar{f}} = L_i^g = L_i^{\tilde{g}} = L_i^{\bar{g}} = 1$, $i,j = 1,2$. $p_i(u_i(t,x)) = u_i(t,x)$, $q_j(v_j(t,x)) = 2v_j(t,x)$, $D_1 = D_2 = 1$, $D_1^* = D_2^* = 2$, $a_1 = a_2 = 1$, $c_1 = c_2 = 2$, $r = 2$, $\mu_\tau = \mu_\theta = 0.2$, $d_{11} = 0.5$, $d_{12} = 1$, $d_{21} = 0.5$, $d_{22} = 0.2$, $\tilde{d}_{11} = -0.1$, $\tilde{d}_{12} = 0.2$, $\tilde{d}_{21} = 0.3$, $\tilde{d}_{22} = 0.5$, $\bar{d}_{11} = 0.2$, $\bar{d}_{12} = 0.6$, $\bar{d}_{21} = 0.5$, $\bar{d}_{22} = 0.8$, $b_{11} = 0.3$, $b_{12} = 0.6$, $b_{21} = -0.5$, $b_{22} = -0.8$, $\tilde{b}_{11} = -1$, $\tilde{b}_{12} = 0.5$, $\tilde{b}_{21} = 0.3$, $\tilde{b}_{22} = 0.3$, $\bar{b}_{11} = -0.1$, $\bar{b}_{12} = 0.2$, $\bar{b}_{21} = 0.5$, $\bar{b}_{22} = 0.4$. By simple calculation with $w_1 = w_2 =$

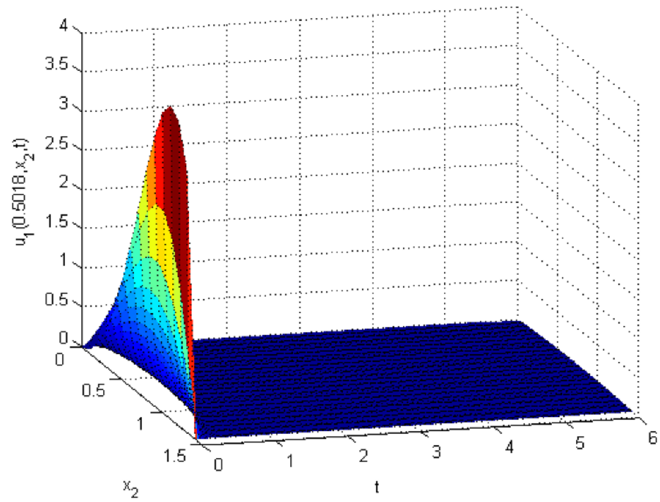

Figure 3 The surface of $u_1(0.5018, x_2, t)$ when $x_1 = 0.5018$.

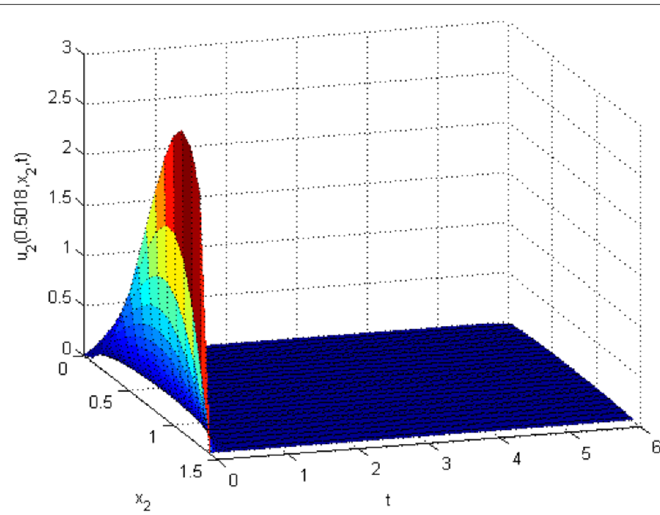

Figure 4 The surface of $u_2(0.5018, x_2, t)$ when $x_1 = 0.5018$.

$w_3 = w_4 = 1$, $\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 1$ and $\gamma_{11} = \gamma_{12} = \gamma_{21} = \gamma_{22} = 1$, we have

$$\begin{aligned}
&-rna_1^{r-1}D_1\lambda_1 - rna_1^r + 2(r-1)\sum_{j=1}^{2} a_1^r + (r-1)\sum_{j=1}^{2} a_1^r \beta_{j1}^{-\frac{r}{r-1}} \\
&\quad + \sum_{j=1}^{2} m^r \left(|d_{1j}|^r \left(L_1^g\right)^r + |\tilde{d}_{1j}|^r \frac{1}{1-\mu_\tau} \left(L_1^{\tilde{g}}\right)^r + |\bar{d}_{1j}|^r \gamma_{1j}^r \left(L_1^{\bar{g}}\right)^r \right) = -1.15 < 0,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&-rna_2^{r-1}D_2\lambda_1 - rna_2^r + 2(r-1)\sum_{j=1}^{2} a_2^r + (r-1)\sum_{j=1}^{2} a_2^r \beta_{j2}^{-\frac{r}{r-1}} \\
&\quad + \sum_{j=1}^{2} m^r \left(|d_{2j}|^r \left(L_2^g\right)^r + |\tilde{d}_{2j}|^r \frac{1}{1-\mu_\tau} \left(L_2^{\tilde{g}}\right)^r + |\bar{d}_{2j}|^r \gamma_{2j}^r \left(L_2^{\bar{g}}\right)^r \right) = -2.38 < 0,
\end{aligned} \tag{32}$$

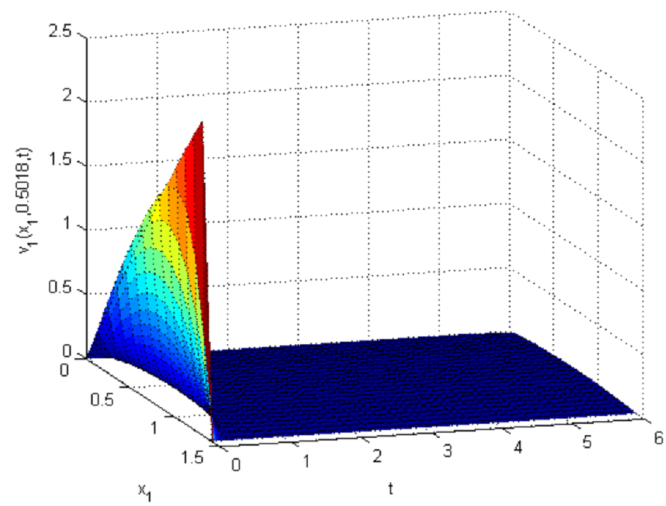

Figure 5 The surface of $v_1(x_1, 0.5018, t)$ when $x_2 = 0.5018$.

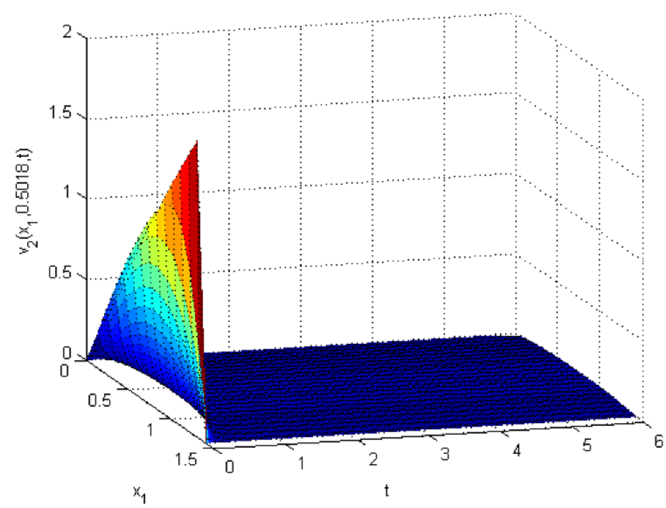

Figure 6 The surface of $v_2(x_1, 0.5018, t)$ when $x_2 = 0.5018$.

$$
\begin{aligned}
&- rmc_1^{r-1}D_1^*\lambda_1 - rmc_1^r + 2(r-1)\sum_{i=1}^{2} c_1^r + (r-1)\sum_{i=1}^{2} c_1^r \gamma_{i1}^{-\frac{r}{r-1}} \\
&\quad + \sum_{i=1}^{2} n^r \left(|b_{1i}|^r \left(L_1^f\right)^r + |\tilde{b}_{1i}|^r \frac{1}{1-\mu_\theta} \left(L_1^{\tilde{f}}\right)^r + |\bar{b}_{1i}|^r \beta_{1i}^r \left(L_1^{\bar{f}}\right)^r \right) = -19.15 < 0
\end{aligned} \tag{33}
$$

and

$$
\begin{aligned}
&- rmc_2^{r-1}D_2^*\lambda_1 - rmc_2^r + 2(r-1)\sum_{i=1}^{2} c_2^r + (r-1)\sum_{i=1}^{2} c_2^r \gamma_{i2}^{-\frac{r}{r-1}} \\
&\quad + \sum_{i=1}^{2} n^r \left(|b_{2i}|^r \left(L_2^f\right)^r + |\tilde{b}_{2i}|^r \frac{1}{1-\mu_\theta} \left(L_2^{\tilde{f}}\right)^r + |\bar{b}_{2i}|^r \beta_{2i}^r \left(L_2^{\bar{f}}\right)^r \right) = -20.98 < 0,
\end{aligned} \tag{34}
$$

that is, (6) holds.

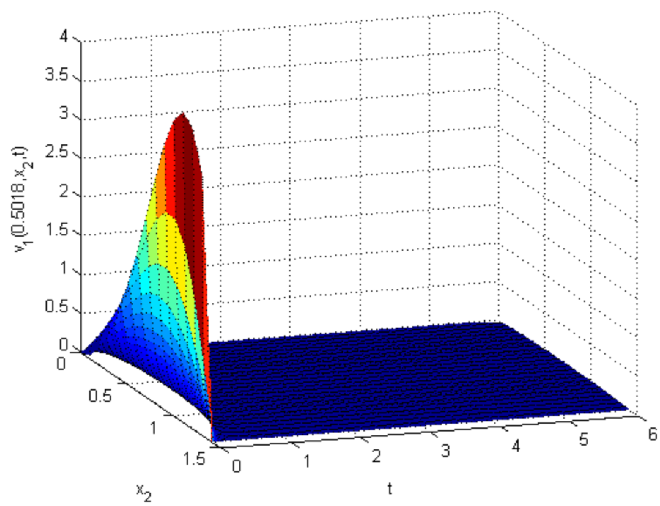

Figure 7 The surface of $v_1(0.5018, x_2, t)$ when $x_1 = 0.5018$.

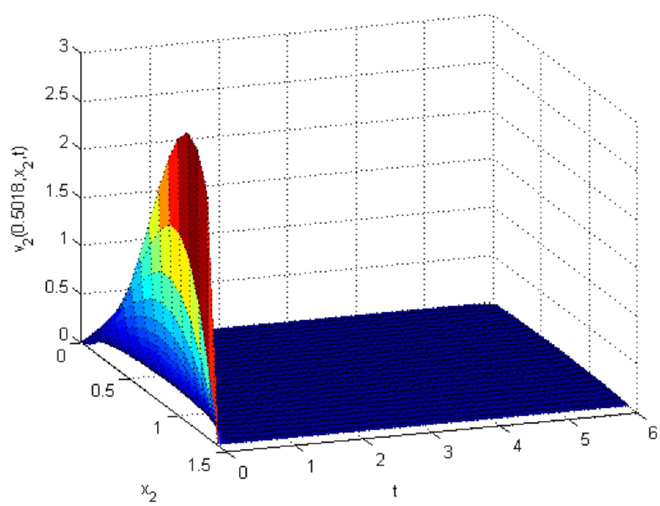

Figure 8 The surface of $v_2(0.5018, x_2, t)$ when $x_1 = 0.5018$.

The simulation results are shown in Figures 1-8. When $x_2 = 0.5018$, the states surfaces of $u(x_1, 0.5018, t)$ are shown in Figures 1-2, while $x_1 = 0.5018$, the states surfaces of $u(0.5018, x_2, t)$ are shown in Figures 3-4. When $x_2 = 0.5018$, the states surfaces of $v(x_1, 0.5018, t)$ are shown in Figures 5-6, while $x_1 = 0.5018$, the states surfaces of $v(0.5018, x_2, t)$ are shown in Figures 7-8, which illustrates that the system states in (30) converge to equilibrium solution. Therefore, it follows from Theorem 1 and the simulation study that (30) has one unique equilibrium solution which is globally exponentially stable.

Remark 4 Since $-a_1 + \frac{1}{2}\sum_{j=1}^{2}(L_j^f)^2(|b_{j1}|^2 + |\bar{b}_{j1}|^2) = 0.25 > 0$, the conditions of Corollary 3.2 in [22] and $-ra_1 + (r-1)\sum_{j=1}^{2} L_j^f(|b_{j1}| + |\bar{b}_{j1}|) + L_1^g \sum_{j=1}^{2}(|d_{1j}| + |\bar{d}_{1j}|) = 3.3 > 0$, under the conditions of Example 1, the conditions of Theorem 1 in [29] are not satisfied. However, by (31)-(34) and Theorem 1, we can derive that (30) has one unique equilibrium solution which is globally exponentially stable.

6 Conclusions

In this paper, by employing suitable Lyapunov functionals, Young's inequality and Hölder's inequality techniques, global exponential stability criteria of the equilibrium point and periodic solutions for RDNNs with mixed time delays and Dirichlet boundary conditions have been derived, respectively. The derived criteria contain and extend some previous NNs in the literature. Hence, our results have an important significance in design as well as in applications of periodic oscillatory NNs with mixed time delays. An example has been given to show the effectiveness of the obtained results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WZ designed and performed all the steps of proof in this research and also wrote the paper. JL and MC participated in the design of the study and suggested many good ideas that made this paper possible and helped to draft the first manuscript. All authors read and approved the final manuscript.

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