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# Non-simultaneous blow-up of a reaction-diffusion system with inner absorption and coupled via nonlinear boundary flux

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**Abstract**
This paper deals with a parabolic reaction-diffusion system with a nonlinear absorption term, meanwhile the two equations of the system coupled via nonlinear boundary flux which obey different laws. Under the hypothesis condition of the initial data, we get the sufficient and necessary conditions under which there exist initial data such that non-simultaneous blow-up occurs.

**MSC:** 35B33; 35K65; 35K55

**Keywords:** reaction-diffusion system; blow-up; non-simultaneous blow-up; nonlinear absorption; nonlinear boundary flux

## 1 Introduction and main results

In this paper, we focus our attention on the non-simultaneous blow-up phenomenon of the following reaction-diffusion system with two parabolic equations coupled via nonlinear boundary flux, and one equation with a nonlinear absorption term:

$$\begin{cases} u_t = u_{xx} - \lambda u^m, \quad v_t = v_{xx}, & (x,t) \in (0,1) \times (0,T), \\ u_x(1,t) = e^{pv(1,t)} u^\alpha(1,t), \quad v_x(1,t) = u^q(1,t) e^{\beta v(1,t)}, & t \in (0,T), \\ u_x(0,t) = 0, \quad v_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in [0,1], \end{cases} \tag{1.1}$$

where the parameters $p, q, \alpha, \beta, \lambda, m \geq 0$. Assume the initial data $u_0$, $v_0$ are increasing, positive continuous functions and satisfy

$$\begin{cases} u_0'(1) = e^{pv_0(1)} u_0^\alpha(1), \\ v_0'(1) = u_0^q(1) e^{\beta v_0(1)}, \\ u_0'(0) = 0, \quad v_0'(0) = 0, \end{cases}$$

which are the compatibility conditions of the initial data. Furthermore $u_0'' - \lambda u_0^m \geq 0$, $v_0'' \geq 0$, for $x \in [0,1]$.

**Remark 1.1** Under the above assumptions, in the light of the comparison principle, we can get $u_x, v_x, u_t, v_t \geq 0$ immediately.

The nonlinear reaction-diffusion systems like (1.1) originate from the description of chemical reactions, heat transfer, *etc.*; see [1, 2]. The Neumann boundary conditions with cross-nonlinear terms can be made explicit as the boundary fluxes in heat transfer. The system (1.1) physically describes heat transfer in a mixed medium with an absorption and nonlinear boundary flux which obeys different laws, a power law and an exponential law; see [3–8]. In recent years, phenomena of non-simultaneous blow-up for nonlinear parabolic systems were widely studied [9–19].

When $\lambda = 0$, the system (1.1) without absorption term has been considered by many authors. In [7], Song gave the blow-up conditions and blow-up rates. Liu and Li [11] got the parameter regions, in which non-simultaneous blow-ups may occur.

Inspired by the above work, we will focus on the simultaneous and non-simultaneous blow-up conditions to system (1.1), and list our main results as follows.

**Theorem 1.1** *Let* $\mu = \max(\frac{m+1}{2}, 1)$, *if any of the following conditions holds*:

(1) $\alpha > \mu$,

(2) $\beta > 0$,

(3) $pq > \beta(\alpha - \mu)$,

*the solutions of* (1.1) *may blow up with proper initial data*.

If the blow-up conditions of (1.1) in Theorem 1.1 are already satisfied, actually the blow-up occurs, then we get the requirement under which simultaneous or non-simultaneous blow-up occurs.

**Theorem 1.2** *Non-simultaneous blow-up may occur if and only if*

$$\begin{cases} \alpha > q + 1, \\ \alpha > \mu \quad or \quad \alpha = \mu > 1 \end{cases} \quad or \quad \beta > p.$$

On the basis of the above theorem, we get a corollary straightforwardly, as follows.

**Corollary 1.1** *Any blow-up is simultaneous if and only if*

$$\begin{cases} \beta \leq p, \\ \alpha \leq q + 1 \quad or \quad \alpha < \mu. \end{cases}$$

We organize the rest of this paper as follows. In the next section, we address the blow-up conditions of system (1.1), and we give the proof of Theorem 1.1. In Section 3 we will consider the sufficient and necessary conditions of $u$ blowing up while $v$ keeps bounded, which will be written as a lemma. In Section 4, the sufficient and necessary conditions of $v$ blowing up while $u$ remains bounded will be researched, in order to complete the proof of Theorem 1.2.

## 2 Blow-up

In this section, we consider the blow-up condition of system (1.1). In the light of the comparison principle, we structure sub-solutions of this system, as derived from Hu and Yin [20].

*Proof of Theorem* 1.1 Under the parametric assumption, we can get

$$e^{pv} \geq v^p, \qquad e^{\beta v} \geq \left(\frac{\beta}{\beta+1}\right)^{\beta+1} \cdot v^{\beta+1}.$$

Assume $(\underline{u}, \underline{v})$ is a pair of solutions of the following system:

$$\begin{cases} \underline{u}_t = \underline{u}_{xx} - \lambda \underline{u}^m, \quad \underline{v}_t = \underline{v}_{xx}, & (x,t) \in (0,1) \times (0,T), \\ \underline{u}_x(1,t) = \underline{v}^p(1,t)\underline{u}^\alpha(1,t), & t \in (0,T), \\ \underline{v}_x(1,t) = \underline{u}^q(1,t)(\frac{\beta}{\beta+1})^{\beta+1} \cdot \underline{v}^{\beta+1}(1,t), & t \in (0,T), \\ \underline{u}_x(0,t) = 0, \quad \underline{v}_x(0,t) = 0, & t \in (0,T), \\ \underline{u}(x,0) = u_0(x), \quad \underline{v}(x,0) = v_0(x), & x \in [0,1]. \end{cases} \tag{2.1}$$

According to the conclusions of [21], let $\mu = \max(\frac{m+1}{2}, 1)$, if any of the following conditions holds:

(1) $\alpha > \mu$,

(2) $\beta > 0$,

(3) $pq > \beta(\alpha - \mu)$,

the solutions of system (2.1) may blow up with suitable initial data. It is easy to check that $(\underline{u}, \underline{v})$ is a pair sub-solution of (1.1), which indicates the solutions of (1.1) also blow up by the comparison principle. □

## 3 *u* blows up while *v* remains bounded

In the next two sections, we prove Theorem 1.2 with five lemmas. This section deals with the conditions of $u$ blowing up while $v$ remains bounded.

First of all, we consider the single equation problem

$$\begin{cases} u_t = u_{xx} - \lambda u^m, & (x,t) \in (0,1) \times (0,T), \\ u_x(1,t) = u^\alpha(1,t)e^{ph(t)}, \quad u_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,1], \end{cases} \tag{3.1}$$

where the parameters $m, p, \alpha, \lambda$ agree with system (1.1), and $h(t)$ is a continuous, bounded, and non-decreasing function, with $h(t) \geq \delta > 0$.

Now we prove a lemma, using similar steps to Theorem 1.2 in [16]. $C$ is a positive constant independent of $t$, and it may change during the proof and at different places.

**Lemma 3.1** *Assume u is a solution of* (3.1), *and one of the following conditions holds*:

(i) $\alpha > \mu$,

(ii) $\alpha = \mu > 1$ *with* $\delta > \frac{1}{2p}\log\left(\frac{\lambda}{\alpha}\right)$.

*Then u may blow up for sufficient large initial value, and furthermore*

$$u(1,t) = \max_{[0,1]} u(\cdot,t) \leq C(T-t)^{\frac{-1}{2(\alpha-1)}}, \quad 0 < t < T, \tag{3.2}$$

*where T is the blow-up time.*

*Proof* Assume $w$ is a solution of the following problem:

$$\begin{cases} w_t = w_{xx} - \lambda w^m, & (x,t) \in (0,1) \times (0,T), \\ w_x(1,t) = w^\alpha(1,t)e^{p\delta}, \qquad w_x(0,t) = 0, & t \in (0,T), \\ w(x,0) = u_0(x), & x \in [0,1]. \end{cases} \tag{3.3}$$

In the light of the comparison principle, $w \leq u$, in $(0,1) \times [0,T)$. Obviously the two parameter conditions in the lemma are, respectively, the same as

$$\text{(i)} \quad \begin{cases} m \leq 1, \\ \alpha > 1 \end{cases} \quad \text{or} \quad \begin{cases} m > 1, \\ \alpha > \frac{m+1}{2} \end{cases}$$

and

$$\text{(ii)} \quad \begin{cases} m > 1, \\ \alpha = \frac{m+1}{2}, \\ \delta > (\frac{\lambda}{\alpha})^{\frac{1}{2p}}. \end{cases}$$

From Theorem 4.2 in [22], there exist initial data $u_0$ such that $w$ blows up. Thus, $u$ blows up at a finite time $T$.

Similarly to [20, 23], we utilize the scaling method to obtain the blow-up rate estimate (3.2). Let $M(t) = \max_{[0,1]} u(\cdot,t) = u(1,t)$, for $t \in (0,T)$, and assume

$$\varphi_k(y,s) = \frac{1}{M(t)} u\big(ky+1, k^2 s + t\big), \quad [-1/k, 0] \times \big(-t/k^2, 0\big],$$

it is easy to check

$$0 \leq \varphi_k \leq 1, \qquad (\varphi_k)_s \geq 0, \qquad \varphi_k(0,0) = 1.$$

Let $k = M^{-\alpha+1}(t)$, $\varphi_k$ solves the following problem:

$$\begin{cases} (\varphi_k)_s = (\varphi_k)_{yy} - \lambda M^{m-2\alpha+1}(t)(\varphi_k)^m, & (y,s) \in (-1/k, 0) \times (-t/k^2, 0], \\ (\varphi_k)_y(0,s) = (\varphi_k)^\alpha(0,s) \cdot e^{ph(k^2 s + t)}, & s \in (-t/k^2, 0], \\ (\varphi_k)_y(-1/k, s) = 0, & s \in (-t/k^2, 0]. \end{cases} \tag{3.4}$$

Now we will prove that there exist constants $c > 0$ and $C > 0$, such that

$$c \leq (\varphi_k)_s(0,0) \leq C, \tag{3.5}$$

for every $k$ small.

First of all, we prove case (i), when $\alpha > \mu$, it easy to check $M^{m-2\alpha+1} \to 0$ as $t \to T$. For given $\{\varphi_{k_j}\}$, there exist a continuous function $\varphi$ and, maybe, a subsequence of it, which is also denoted by $\{\varphi_{k_j}\}$, such that $\varphi_{k_j} \to \varphi$, $k_j \to 0$, as $M \to \infty$.

As $h(t)$ is bounded, we conclude that $\varphi_{k_j}$ is uniformly bounded. Using the Schauder estimates

$$\|\varphi_{k_j}\|_{C^{2+\alpha,1+\alpha/2}} \leq C.$$

The upper estimate in (3.5) follows immediately.

To get the lower bound of (3.5), and prove it by contradiction, we assume that the lower bound does not hold and that there is a sequence $\varphi_{k_j}$ such that $(\varphi_{k_j})_s(0,0) \to 0$. As $\varphi_{k_j} \to \varphi$, $0 \leq \varphi \leq 1$, $\varphi(0,0) = 1$, $\varphi_s \geq 0$, $\varphi_s(0,0) = 0$, and $\varphi$ satisfies

$$\begin{cases} \varphi_s = \varphi_{yy}, & (y,s) \in (-\infty,0) \times (-\infty,0], \\ \varphi_y(0,s) = \varphi^\alpha(0,s) \cdot e^{ph(T)}, & s \in (-\infty,0]. \end{cases} \tag{3.6}$$

Set $\psi = \varphi_s$, thus $\psi$ satisfies

$$\begin{cases} \psi_s = \psi_{yy}, & (y,s) \in (-\infty,0) \times (-\infty,0], \\ \psi_y(0,s) = \alpha \cdot \varphi^{\alpha-1}(0,s) \cdot e^{ph(T)} \cdot \varphi_s(0,s) \geq 0, & s \in (-\infty,0]. \end{cases} \tag{3.7}$$

Meanwhile $(\varphi_{k_j})_s(0,0) \to \varphi_s(0,0)$, thus $\psi(0,0) = \varphi_s(0,0) = 0$, as $\psi = \varphi_s \geq 0$, hence $\psi$ has a minimum at $(0,0)$. By Hopf's lemma $\psi \equiv 0$, that is to say, $\psi$ is independent of $s$. Thus $\varphi$ satisfies the ODE problem

$$\begin{cases} \varphi_{yy} = 0, \\ \varphi_y(0) = e^{ph(T)} \geq e^{\delta p}, \\ \varphi(0) = 1, \end{cases} \tag{3.8}$$

then $\varphi(y) \geq e^{\delta p} \cdot y + 1$, which make a contradiction of the fact that $0 \leq \varphi \leq 1$, hence (3.5) holds, and (3.2) follows immediately.

Now, we prove case (ii). The proof of upper bound in (3.5) is similar, we omit it. The main difference of case (ii) is the critical case $\alpha = \frac{m+1}{2} > 1$. Thus (3.6) becomes

$$\begin{cases} \varphi_s = \varphi_{yy} - \lambda\varphi^m, & (y,s) \in (-\infty,0) \times (-\infty,0], \\ \varphi_y(0,s) = \varphi^\alpha(0,s) \cdot e^{ph(T)}, & s \in (-\infty,0]. \end{cases} \tag{3.9}$$

To obtain a contradiction, we set $\psi = \varphi_s$, thus

$$\begin{cases} \psi_s = \psi_{yy} - \lambda \cdot m \cdot \varphi^{m-1} \cdot \psi, & (y,s) \in (-\infty,0) \times (-\infty,0], \\ \psi_y(0,s) = \alpha \cdot \varphi^{\alpha-1}(0,s) \cdot e^{ph(T)} \cdot \varphi_s(0,s) \geq 0, & s \in (-\infty,0]. \end{cases}$$

By Hopf's lemma, $\psi \equiv 0$, $\varphi$ satisfies

$$\begin{cases} \varphi_{yy} - \lambda\varphi^m = 0, \\ \varphi_y(0) = e^{ph(T)} \geq e^{\delta p}, \\ \varphi(0) = 1. \end{cases} \tag{3.10}$$

The first equation in (3.10) we multiply by $\varphi_y$, and we integrate from 0 to $y$. Since $\alpha = \frac{m+1}{2}$, we derive

$$\varphi_y^2 = \varphi_y^2(0) + \frac{\lambda}{\alpha}\varphi^{2\alpha} - \frac{\lambda}{\alpha},$$

so

$$\varphi_y = \left[\varphi_y^2(0) + \frac{\lambda}{\alpha}\varphi^{2\alpha} - \frac{\lambda}{\alpha}\right]^{\frac{1}{2}},$$

where $\varphi_y \geq 0$. Furthermore

$$y = \int_0^y \varphi_y\left[\varphi_y^2(0) + \frac{\lambda}{\alpha}\varphi^{2\alpha} - \frac{\lambda}{\alpha}\right]^{-\frac{1}{2}} dy = \int_{\varphi(0)}^{\varphi(y)}\left[e^{2ph(T)} + \frac{\lambda}{\alpha}\varphi^{2\alpha} - \frac{\lambda}{\alpha}\right]^{-\frac{1}{2}} d\varphi.$$

Noticing $\delta > \frac{1}{2p}\log(\frac{\lambda}{\alpha})$, we deduce

$$y \geq -\left(e^{2\delta p} - \frac{\lambda}{\alpha}\right)^{-\frac{1}{2}},$$

which contradicts $y \in (-\infty, 0)$. We conclude (3.5) is true, and (3.2) follows. □

**Lemma 3.2** *Let $u$ be a solution of following problem*:

$$\begin{cases} u_t = u_{xx} - \lambda u^m, & (x,t) \in (0,1) \times (0,T), \\ u_x(1,t) \leq Lu^\alpha(1,t), \qquad u_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,1], \end{cases} \tag{3.11}$$

*with $\lambda > 0$, $\alpha$, $m \geq 0$. If $u$ blows up at a finite time $T$, then either $\alpha > \mu$ or $\alpha = \mu > 1$, and there exists a positive constant $C$, such that*

$$u(1,t) = \max_{[0,1]} u(\cdot,t) \geq C(T-t)^{\frac{-1}{2(\alpha-1)}}, \quad \text{as } t \to T.$$

*Proof* The proof of Lemma 3.2 is similar to Lemma 3.1 in [23], and we omit it here. □

**Lemma 3.3** *There exist initial data such that $u$ blows up while $v$ remains bounded if and only if*

$$\begin{cases} \alpha > q+1, \\ \alpha > \mu \quad or \quad \alpha = \mu > 1. \end{cases}$$

*Proof* We first prove the sufficient condition.

To find suitable initial data $(u_0, v_0)$, such that $u$ blows up while $v$ remains bounded, fix $v_0$ so that $v_0(x) \geq \delta > \frac{1}{2p}\log(\frac{\lambda}{\alpha})$, and denote $K = \max_{[0,1]} v_0 = v_0(1)$, $N = \frac{e^{2\beta K}}{K} + 3$. Assume $w(x,t)$ is a solution of (3.3), then $w(x,t)$ is a sub-solution of $u$. There must exist sufficiently large initial data $u_0$ such that $w$ blows up at a finite time $T \leq \varepsilon$, under the assumption $\delta > \frac{1}{2p}\log(\frac{\lambda}{\alpha})$.

We claim that there exists $u_0$ large enough such that $v < 2K$. If it is not true, there must exist $t_0 < T$, which is the first time $\max_{[0,1]} v(\cdot, t_0) = v(1, t_0) = 2K$. We denote the cutoff function

$$\tilde{v}(x,t) = \begin{cases} v(x,t), & (x,t) \in [0,1] \times [0,t_0], \\ 2K, & (x,t) \in [0,1] \times [t_0, T]. \end{cases} \tag{3.12}$$

Meanwhile, assume $\tilde{u}(x,t)$ solves

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} - \lambda \tilde{u}^m, & & (x,t) \in (0,1) \times (0, \widetilde{T}), \\ \tilde{u}_x(1,t) = \tilde{u}^\alpha(1,t) e^{p\tilde{v}(1,t)}, & \tilde{u}_x(0,t) = 0, & t \in (0, \widetilde{T}), \\ \tilde{u}(x,0) = u_0(x), & & x \in [0,1], \end{cases} \tag{3.13}$$

then the blow-up time of $\tilde{u}$, which is denoted by $\widetilde{T}$, is later than $T$. We conclude by Lemma 3.1

$$\tilde{u}(1,t) = \max_{[0,1]} \tilde{u}(\cdot, t) \le C(\widetilde{T} - t)^{\frac{-1}{2(\alpha-1)}}, \quad 0 < t < \widetilde{T}.$$

Thus,

$$u(1,t) = \tilde{u}(1,t) \le C(\widetilde{T} - t)^{\frac{-1}{2(\alpha-1)}} \le C(T-t)^{\frac{-1}{2(\alpha-1)}}, \quad 0 < t < t_0. \tag{3.14}$$

Let $\Gamma(x,t)$ be the fundamental solution of the heat equation in $[0,1]$, so

$$\Gamma(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{ \frac{-x^2}{4t} \right\}.$$

From the classical theory of the heat equation, we know that $\Gamma$ satisfies (see [24])

$$\begin{aligned} &\int_0^1 \Gamma(x-y, t-z)\, dy \le 1, \\ &\int_z^t \Gamma(0, t-\tau)\, d\tau = \frac{1}{\sqrt{\pi}} \sqrt{t-z}, \qquad \int_z^t \Gamma(1, t-\tau) \frac{1}{2(t-\tau)}\, d\tau \le C^* \sqrt{t-z}, \\ &\frac{\partial \Gamma}{\partial \eta_y}(x-y, t-\tau) = \frac{x-y}{2(t-\tau)} \Gamma(x-y, t-\tau), \quad x, y \in [0,1], 0 \le z < t. \end{aligned}$$

We deduce from the Green's identity of (1.1) for $v$ that

$$\begin{aligned} v(x,t) = {}& \int_0^1 \Gamma(x-y, t-z) v(y,z)\, dy + \int_z^t \frac{\partial v}{\partial x}(1,\tau) \Gamma(x-1, t-\tau)\, d\tau \\ & - \int_z^t \frac{\partial \Gamma}{\partial \eta_y}(x-1, t-\tau) v(1,\tau)\, d\tau + \int_z^t \frac{\partial \Gamma}{\partial \eta_y}(x, t-\tau) v(0,\tau)\, d\tau, \end{aligned} \tag{3.15}$$

with $0 \le z < t < T$, $0 < x < 1$. Set $z = 0$, and $x \to 1$, one derives that

$$\begin{aligned} v(1,t) = {}& \int_0^1 \Gamma(1-y, t) v(y,0)\, dy + \int_0^t u^q(1,\tau) e^{\beta v(1,\tau)} \Gamma(0, t-\tau)\, d\tau \\ & + \int_0^t v(0,\tau) \Gamma(1, t-\tau) \frac{1}{2(t-\tau)}\, d\tau. \end{aligned}$$

Combining with (3.14), we have

$$v(1,t_0) \le v_0(1) + C_0 e^{\beta v(1,t_0)} \int_0^{t_0} (t_0-\tau)^{-\frac{q}{2(\alpha-1)}-\frac{1}{2}}\,d\tau + C^* \sqrt{t_0} \cdot v(1,t_0). \tag{3.16}$$

As $q < \alpha - 1$,

$$\int_0^{t_0} (t_0-\tau)^{-\frac{q}{2(\alpha-1)}-\frac{1}{2}}\,d\tau < \frac{1}{NC_0},$$

and we can choose enough large $u_0$ such that $T$ sufficiently small, it implies $\sqrt{t_0} \le \sqrt{T} \le \frac{1}{NC^*}$. Therefore

$$\frac{N-1}{N} v(1,t_0) \le v_0(1) + \frac{1}{N} e^{\beta v(1,t_0)}.$$

Thus

$$\frac{2N-1}{N} K \le K + \frac{1}{N} e^{2K\beta},$$

so

$$N \le \frac{e^{2K\beta}}{K} + 2,$$

which is a contradiction.

Next, we prove the necessary condition.

If $u$ blows up, $v$ remains bounded. Then $v \le K$ for $(x,t) \in [0,1] \times [0,T)$, so

$$\begin{cases} u_t = u_{xx} - \lambda u^m, & & (x,t) \in (0,1) \times (0,T), \\ u_x(1,t) \le u^\alpha(1,t) e^{pK}, & u_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & & x \in [0,1]. \end{cases}$$

Thus we can conclude from Lemma 3.2 that $\alpha > \mu$ or $\alpha = \mu > 1$. Furthermore,

$$u(1,t) = \max_{[0,1]} u(\cdot,t) \ge C(T-t)^{\frac{-1}{2(\alpha-1)}}, \quad \text{as } t \to T.$$

Now, we will show the requirement of $q < \alpha - 1$. Let $x \to 1$ in (3.15), we deduce

$$v(1,t) \ge \int_z^t u^q(1,\tau) \cdot e^{\beta v(1,\tau)} \Gamma(0,t-\tau)\,d\tau \ge C_1 \int_z^t (T-\tau)^{-\frac{q}{2(\alpha-1)}-\frac{1}{2}}\,d\tau.$$

Then we get $q < \alpha - 1$, in order to ensure the boundedness of $v(1,t)$ as $t \to T$. □

## 4 *v* blows up while *u* remains bounded

This section focuses on the conditions of $v$ blowing up while $u$ remains bounded, in order to complete the proof of Theorem 1.2.

**Lemma 4.1** *There exists* $\varepsilon > 0$, *such that*

$$v(1,t) \le -\frac{1}{2\beta} \log\left[2\beta \cdot \varepsilon \cdot u_0^{2q}(1)(T-t)\right]. \tag{4.1}$$

*Proof* Let $J(x,t) = v_t - \varepsilon v_x^2$, where $\varepsilon$ is a constant to be determined. For any initial data $v_0$, we can take small $\varepsilon < \frac{\beta}{2}$, such that

$$\begin{cases} J_t - J_{xx} = 2\varepsilon \cdot v_{xx}^2 \geq 0, & (x,t) \in (0,1) \times (0,T), \\ J_x(1,t) = [q \cdot u^{q-1} \cdot u_t \cdot e^{\beta v} + (\beta - 2\varepsilon) u^q \cdot e^{\beta v} \cdot v_t](1,t) \geq 0, & t \in (0,T), \\ J_x(0,t) = 0, & t \in (0,T), \\ J(x,0) \geq 0, & x \in [0,1]. \end{cases}$$

By the comparison principle $J \geq 0$, thus $v_t(1,t) \geq \varepsilon v_x^2(1,t)$, in $t \in [0,T)$. Then

$$v_t(1,t) \cdot e^{-2\beta v(1,t)} \geq \varepsilon u^{2q}(1,t).$$

Integrate the above inequality on $(t,T)$ to obtain

$$\int_t^T e^{-2\beta v(1,\xi)} v_\xi \, d\xi \geq \int_t^T \varepsilon \cdot u^{2q}(1,\xi) \, d\xi \geq \varepsilon \cdot u_0^{2q}(1) \cdot (T-t).$$

Hence (4.1) follows. □

**Lemma 4.2** *There exist initial data such that $v$ blows up while $u$ remains bounded up to blow-up time $T$, if and only if $p < \beta$.*

*Proof* First, we prove the sufficient condition.

We denote a set of initial data as follows:

$$\begin{aligned} V = \Big\{ & u_0(x) = u_0(1) + \big(\sqrt{M^2+4} - M\big)/2 - \sqrt{1 - M\big(\sqrt{M^2+4} - M^2\big) \cdot x^2/2}, \\ & v_0(x) = v_0(1) + \big(\sqrt{N^2+4} - N\big)/2 - \sqrt{1 - N\big(\sqrt{N^2+4} - N^2\big) \cdot x^2/2}, x \in [0,1] \Big\} \end{aligned}$$

with $M = e^{pv_0(1)} u_0^\alpha(1)$, $N = u_0^q(1) e^{\beta v_0(1)}$. Set one $\varepsilon = \varepsilon_0 = \frac{\beta}{2}$, uniformly in Lemma 4.1 for any initial data in $V$.

Fix $u_0(1) = \xi_0 > 0$, with $N_0 > 2\xi_0$. Meanwhile, we can set the initial data $v_0$ large enough such that $T$ satisfies

$$N_0 \geq 2\xi_0 + \frac{2\beta}{\beta - p} \cdot \overline{C} \cdot N_0^\alpha \cdot \big(2\beta\varepsilon_0\xi_0^{2q}\big)^{-\frac{p}{2\beta}} \cdot T^{\frac{\beta-p}{2\beta}}.$$

Consider the scalar problem as follows:

$$\begin{cases} \overline{u}_t = \overline{u}_{xx} - \lambda \overline{u}^m, & (x,t) \in (0,1) \times (0,T), \\ \overline{u}_x(1,t) = N_0^\alpha (2\beta\varepsilon_0\xi_0^{2q})^{-\frac{p}{2\beta}} (T-t)^{-\frac{p}{2\beta}}, \qquad \overline{u}_x(0,t) = 0, & t \in (0,T), \\ \overline{u}(x,0) = \overline{u}_0(x), & x \in [0,1], \end{cases}$$

where $\overline{u}_0$ satisfies $\overline{u}_0'(1) = N_0^\alpha (2\beta\varepsilon_0\xi_0^{2q} \cdot T)^{-\frac{p}{2\beta}}$, $\overline{u}_0(1) = 2\xi_0$, $\overline{u}_0'(x), \overline{u}_0''(x) \geq 0$, $\overline{u}_0(x) \geq u_0(x)$.

Since $\beta > p$, by Green's identity,

$$\begin{aligned} \overline{u} &\leq 2\xi_0 + \overline{C} \cdot N_0^\alpha \cdot \big(2\beta\varepsilon_0\xi_0^{2q}\big)^{-\frac{p}{2\beta}} \cdot \int_0^t (t-\tau)^{-\frac{1}{2}} (T-\tau)^{-\frac{p}{2\beta}} \, d\tau \\ &\leq 2\xi_0 + \frac{2\beta}{\beta - p} \cdot \overline{C} \cdot N_0^\alpha \cdot \big(2\beta\varepsilon_0\xi_0^{2q}\big)^{-\frac{p}{2\beta}} \cdot T^{\frac{\beta-p}{2\beta}} \leq N_0, \end{aligned}$$

so $\overline{u}$ satisfies

$$\overline{u}_x(1,t) \geq \left(2\beta\varepsilon_0\xi_0^{2q}\right)^{-\frac{p}{2\beta}}(T-t)^{-\frac{p}{2\beta}} \cdot \overline{u}^{\alpha}(1,t).$$

It follows from Lemma 4.1 that $u$ satisfies

$$\begin{cases} u_t = u_{xx} - \lambda u^m, & (x,t) \in (0,1) \times (0,T), \\ u_x(1,t) \leq u^{\alpha}(1,t) \cdot (2\beta\varepsilon_0\xi_0^{2q})^{-\frac{p}{2\beta}}(T-t)^{-\frac{p}{2\beta}}, \qquad u_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,1]. \end{cases}$$

In the light of the comparison principle, $u \leq \overline{u} \leq N_0$, and hence $v$ blows up alone.

Next, we prove the necessary condition.

Since $u \leq C$, by Green's identity,

$$v(1,t) \geq \log\left[C(T-t)\right]^{-\frac{1}{2\beta}}, \quad t \in (0,T).$$

Then

$$\frac{1}{2}u(1,t) \geq C\int_0^t (T-\tau)^{-\frac{p}{2\beta}}(t-\tau)^{-\frac{1}{2}}\,d\tau.$$

Thus, $\beta > p$ must hold to ensure the boundedness of $u$. □

**Competing interests**
The author claims that this article has not been published elsewhere and also declares to have no competing interests.

**Author's contributions**
The author have read and approved the final manuscript.

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