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Dirichlet problems of harmonic functions

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Abstract

In this paper, a solution of the Dirichlet problem in the upper half-plane is constructed by the generalized Dirichlet integral with a fast growing continuous boundary function.

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1 Introduction and results

Let \mathbf{R} be the set of all real numbers, and let \mathbf{C} denote the complex plane with points $z = x + iy$, where $x, y \in \mathbf{R}$. The boundary and closure of an open set Ω are denoted by $\partial\Omega$ and $\overline{\Omega}$ respectively. The upper half-plane is the set $\mathbf{C}_+ := \{z = x + iy \in \mathbf{C} : y > 0\}$, whose boundary is $\partial\mathbf{C}_+ = \mathbf{R}$. Let $[d]$ denote the integer part of the positive real number d .

Given a continuous function f in $\partial\mathbf{C}_+$, we say that h is a solution of the (classical) Dirichlet problem in \mathbf{C}_+ with f , if $\Delta h = 0$ in \mathbf{C}_+ and $\lim_{z \in \mathbf{C}_+, z \rightarrow t} h(z) = f(t)$ for every $t \in \partial\mathbf{C}_+$.

The classical Poisson kernel in \mathbf{C}_+ is defined by

$$P(z, t) = \frac{y}{\pi |z - t|^2},$$

where $z = x + iy \in \mathbf{C}_+$ and $t \in \mathbf{R}$.

It is well known (see [1, 2]) that the Poisson kernel $P(z, t)$ is harmonic for $z \in \mathbf{C} - \{t\}$ and has the expansion

$$P(z, t) = \frac{1}{\pi} \operatorname{Im} \sum_{k=0}^{\infty} \frac{z^k}{t^{k+1}},$$

which converges for $|z| < |t|$. We define a modified Cauchy kernel of $z \in \mathbf{C}_+$ by

$$C_m(z, t) = \begin{cases} \frac{1}{\pi} \frac{1}{t-z} & \text{when } |t| \leq 1, \\ \frac{1}{\pi} \frac{1}{t-z} - \frac{1}{\pi} \sum_{k=0}^m \frac{z^k}{t^{k+1}} & \text{when } |t| > 1, \end{cases}$$

where m is a nonnegative integer.

To solve the Dirichlet problem in \mathbf{C}_+ , as in [3], we use the following modified Poisson kernel defined by

$$P_m(z, t) = \operatorname{Im} C_m(z, t) = \begin{cases} P(z, t) & \text{when } |t| \leq 1, \\ P(z, t) - \frac{1}{\pi} \operatorname{Im} \sum_{k=0}^m \frac{z^k}{t^{k+1}} & \text{when } |t| > 1. \end{cases}$$

We remark that the modified Poisson kernel $P_m(z, t)$ is harmonic in \mathbf{C}_+ .

Put

$$U_m(f)(z) = \int_{-\infty}^{\infty} P_m(z, t) f(t) dt,$$

where $f(t)$ is a continuous function in $\partial\mathbf{C}_+$.

We say that u is of order λ if

$$\lambda = \limsup_{r \rightarrow \infty} \frac{\log(\sup_{H \cap B(r)} |u|)}{\log r}.$$

If $\lambda < \infty$, then u is said to be of finite order. See Hayman-Kennedy [1, Definition 4.1].

In case $\lambda < \infty$, about the solution of the Dirichlet problem with continuous data in H , we refer readers to the following result, which is due to Nevanlinna (see [4–6]).

Theorem A *Let u be a nonnegative real-valued function harmonic in \mathbf{C}_+ and continuous in $\overline{\mathbf{C}_+}$. If*

$$\int_{-\infty}^{\infty} \frac{u(t)}{1+t^2} dt < \infty,$$

then there exists a nonnegative real constant d such that

$$u(z) = dy + \int_{-\infty}^{\infty} P(z, t) u(t) dt$$

for all $z = x + iy \in \mathbf{C}_+$.

Inspired by Theorem A, we consider the Dirichlet problem for harmonic functions of infinite order in \mathbf{C}_+ . To do this, we define a nondecreasing and continuously differentiable function $\rho(R) \geq 1$ on the interval $[0, +\infty)$. We assume further that

$$\varepsilon_0 = \limsup_{R \rightarrow \infty} \frac{\rho'(R) R \log R}{\rho(R)} < 1. \quad (1.1)$$

Remark For any ϵ ($0 < \epsilon < 1 - \varepsilon_0$), there exists a sufficiently large positive number R such that $r > R$, by (1.1) we have

$$\rho(r) < \rho(e)(\ln r)^{\varepsilon_0 + \epsilon}.$$

Let $F(\rho, \alpha)$ be the set of continuous functions f on $\partial\mathbf{C}_+$ such that

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1 + |t|^{\rho(|t|) + \alpha + 1}} dt < \infty, \quad (1.2)$$

where α is a positive real number.

Now we show the solution of the Dirichlet problem with continuous data in \mathbf{C}_+ . For similar results in a cone, we refer readers to the paper by Qiao (see [7, 8]). For similar results with respect to the Schrödinger operator in a half-space, we refer readers to the paper by Ren, Su and Yang (see [9–12]).

Theorem 1 If $f \in F(\rho, \alpha)$, then the integral $U_{[\rho(|t|)+\alpha]}(f)(z)$ is a solution of the Dirichlet problem in \mathbf{C}_+ with f .

The following result is obtained by putting $[\rho(|t|) + \alpha] = m$ in Theorem 1.

Corollary If f is a continuous function in $\partial\mathbf{C}_+$ satisfying

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1 + |t|^{m+2}} dt < \infty,$$

then $U_m(f)(z)$ is a solution of the Dirichlet problem in \mathbf{C}_+ with f .

Theorem 2 Let u be a real-valued function harmonic in \mathbf{C}_+ and continuous in $\overline{\mathbf{C}}_+$. If $u \in F(p, \rho, \alpha)$, then we have $u(z) = U_{[\rho(|t|)+\alpha]}(u)(z) + \text{Im } \Pi(z)$ for all $z \in \overline{\mathbf{C}}_+$, where $\Pi(z)$ is an entire function in \mathbf{C}_+ and vanishes continuously in $\partial\mathbf{C}_+$.

2 Proof of Theorem 1

By a simple calculation, we have the following inequality:

$$|C_m(z, t)| \leq M|z|^{m+1}|t|^{-m-2} \quad (2.1)$$

for any $z \in \mathbf{C}_+$ and $t \in \partial\mathbf{C}_+$ satisfying $|t| \geq \max\{1, 2|z|\}$, where M is a positive constant.

Take a number r satisfying $r > R$, where R is a sufficiently large positive number. For any ϵ ($0 < \epsilon < 1 - \epsilon_0$), from Remark we have

$$\rho(r) < \rho(e)(\ln r)^{(\epsilon_0 + \epsilon)},$$

which yields that there exists a positive constant $M(r)$ dependent only on r such that

$$k^{-\alpha/2}(2r)^{\rho(k+1)+\alpha+1} \leq M(r) \quad (2.2)$$

for any $k > k_r = [2r] + 1$.

For any $z \in \mathbf{C}_+$ and $|z| \leq r$, we have by (1.2), (2.1), (2.2), $1/p + 1/q = 1$ and Hölder's inequality

$$\begin{aligned} & M \sum_{k=k_r}^{\infty} \int_{k \leq |t| < k+1} \frac{|z|^{[\rho(|t|)+\alpha]+1}}{|t|^{[\rho(|t|)+\alpha]+2}} |f(t)| dt \\ & \leq M \sum_{k=k_r}^{\infty} \frac{r^{\rho(k+1)+\alpha+1}}{k^{\alpha/2}} \int_{k \leq |t| < k+1} \frac{|f(t)|}{|t|^{\rho(|t|)+1+p\alpha/2}} dt \\ & \leq 2MM(r) \int_{|t| \geq k_r} \frac{|f(t)|}{1 + |t|^{\rho(|t|)+1+p\alpha/2}} dt \\ & < \infty. \end{aligned}$$

Thus $U_{[\rho(|t|)+\alpha]}(f)(z)$ is finite for any $z \in \mathbf{C}_+$. Since $P_{[\rho(|t|)+\alpha]}(z, t)$ is a harmonic function of $z \in \mathbf{C}_+$ for any fixed $t \in \partial\mathbf{C}_+$, $U_{[\rho(|t|)+\alpha]}(f)(z)$ is also a harmonic function of $z \in \mathbf{C}_+$.

Now we shall prove the boundary behavior of $U_{[\rho(|t|)+\alpha]}(f)(z)$. For any fixed boundary point $t' \in \partial\mathbf{C}_+$, we can choose a number T such that $T > |t'| + 1$. Now we write

$$U_{[\rho(|t|)+\alpha]}(f)(z) = I_1(z) - I_2(z) + I_3(z),$$

where

$$I_1(z) = \int_{|t| \leq 2T} P(z, t) f(t) dt, \quad I_2(z) = \operatorname{Im} \sum_{k=0}^{[\rho(|t|+\alpha)]} \int_{1 < |t| \leq 2T} \frac{z^k}{\pi t^{k+1}} f(t) dt,$$

$$I_3(z) = \int_{|t| > 2T} P_{[\rho(|t|+\alpha)]}(z, t) f(t) dt.$$

Note that $I_1(z)$ is the Poisson integral of $u(t)\chi_{[-2T, 2T]}(t)$, where $\chi_{[-2T, 2T]}$ is the characteristic function of the interval $[-2T, 2T]$. So it tends to $f(t')$ as $z \rightarrow t'$. Clearly, $I_2(z)$ vanishes on $\partial\mathbf{C}_+$. Further, $I_3(z) = O(y)$, which tends to zero as $z \rightarrow t'$. Thus the function $U_{[\rho(|t|)+\alpha]}(f)(z)$ can be continuously extended to $\overline{\mathbf{C}}_+$ such that $U_{[\rho(|t|)+\alpha]}(f)(t') = f(t')$ for any $t' \in \partial\mathbf{C}_+$. Theorem 1 is proved.

3 Proof of Theorem 2

Consider that the function $u(z) - U_{[\rho(|t|)+\alpha]}(u)(z)$, which is harmonic in \mathbf{C}_+ , can be continuously extended to $\overline{\mathbf{C}}_+$ and vanishes in $\partial\mathbf{C}_+$.

The Schwarz reflection principle [4, p.68] applied to $u(z) - U_{[\rho(|t|)+\alpha]}(u)(z)$ shows that there exists an entire harmonic function $\Pi(z)$ in \mathbf{C}_+ satisfying $\Pi(\bar{z}) = \overline{\Pi(z)}$ such that $\operatorname{Im} \Pi(z) = u(z) - U_{[\rho(|t|)+\alpha]}(u)(z)$ for $z \in \overline{\mathbf{C}}_+$.

Thus $u(z) = U_{[\rho(|t|)+\alpha]}(u)(z) + \operatorname{Im} \Pi(z)$ for all $z \in \overline{\mathbf{C}}_+$, where $\Pi(z)$ is an entire function in \mathbf{C}_+ and vanishes continuously in $\partial\mathbf{C}_+$. Then we complete the proof of Theorem 2.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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