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Exponential dichotomy on time scales and admissibility of the pair $(C_{\mathrm{rd}}^b(\mathbb{T}^+,X),L^p(\mathbb{T}^+,X))$

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Abstract

In this paper, we study a relation between exponential dichotomy on time scales and admissibility of the pair $(C_{\mathrm{rd}}^b(\mathbb{T}^+,X),L^p(\mathbb{T}^+,X))$ for an evolution family on time scales. We establish a sufficient criterion for the existence of exponential dichotomy on time scales in terms of the admissibility of the pair $(C_{\mathrm{rd}}^b(\mathbb{T}^+,X),L^p(\mathbb{T}^+,X))$ for the evolution family. Conversely, with the help of exponential dichotomy on time scales, we give the admissibility of the pair $(C_{\mathrm{rd}}^b(\mathbb{T}^+,X),L^p(\mathbb{T}^+,X))$ for an input-output equation on time scales.

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Keywords: time scales; exponential dichotomy; admissibility

1 Introduction

The concept of exponential dichotomies was first introduced by Perron in 1930 [1] to study the conditional stability of the linear differential equations and the existence of bounded solutions of the nonlinear differential equations. Then Li [2] established the corresponding analogous concept for the linear difference equations. The theory of exponential dichotomies has been playing an important role in the study of the theory of differential equations and difference equations (see [3–5]). An interesting and challenging problem is to establish necessary and sufficient criteria for the existence of exponential dichotomies. Among the many methods and tools, the admissibility techniques or input-output methods have been extensively applying to study the existence of exponential dichotomies for differential equations and difference equations [6–15].

It is well known that the theory of dynamic equations on time scales provides a unifying structure for the study of differential equations in the continuous case and difference equations in the discrete case and has tremendous potential for applications in mathematical models of real processes and phenomena [16–19]. In recent years, the theory of exponential dichotomies on time scales for the linear dynamic equations on time scales extends the idea of hyperbolicity from autonomous dynamic equations on time scales to explicitly nonautonomous ones and has progressed greatly [20–30]. In view of the important role of the admissibility techniques or input-output methods in the study of the exponential dichotomy on differential equations and difference equations, it is natural for us to ask whether the admissibility techniques or input-output methods can be applied to deal with problems of exponential dichotomies on time scales for an evolution family on time scales.

Motivated by the results of admissibility and exponential dichotomy for differential equations and difference equations in [6–15], in this paper, we establish a relation between exponential dichotomy on time scales and admissibility of the pair $(C^b_{\mathrm{rd}}(\mathbb{T}^+, X), L^p(\mathbb{T}^+, X))$ for an evolution family on time scales. The paper is organized as follows. In the next section, we present some basic information concerning exponential dichotomies on time scales and admissibility for an evolution family. In Section 3, we construct an equivalent relation between exponential dichotomy on time scales and the admissibility of the pair $(C^b_{\mathrm{rd}}(\mathbb{T}^+, X), L^p(\mathbb{T}^+, X))$ for the evolution family on time scales. Our result extends related results known for differential equations and difference equations on the half-line to more general time scales.

2 Preliminaries and basic definitions

In this section, we first introduce the following concepts related to the notion of time scales, which can be found in [16, 17, 30]. A time scale $\mathbb{T}$ is defined as a nonempty closed subset of the real numbers. Define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the graininess function $\mu(t) = \sigma(t) - t$ for any $t \in \mathbb{T}$. In the following discussion, the time scale $\mathbb{T}$ is assumed to be unbounded above and below. Let $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ be the set of rd-continuous functions $g : \mathbb{T} \to \mathbb{R}$ and $\mathcal{R}^+(\mathbb{T}, \mathbb{R}) := \{g \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}) : 1 + \mu(t)g(t) > 0, t \in \mathbb{T}\}$ be the space of positively regressive functions. We define the exponential function on time scales by

$$e_\varphi(t,s) = \exp\left\{\int_s^t \zeta_{\mu(\tau)}\big(\varphi(\tau)\big)\Delta\tau\right\} \quad \text{with } \zeta_h(z) = \begin{cases} z & \text{if } h = 0, \\ \mathrm{Log}(1+hz)/h & \text{if } h \neq 0, \end{cases}$$

for any $\varphi \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$ and $s, t \in \mathbb{T}$, where Log is the principal logarithm. Define

$$\begin{aligned} &(\varphi \oplus \psi)(t) := \varphi(t) + \psi(t) + \mu(t)\varphi(t)\psi(t), \\ &\ominus\varphi := -\frac{\varphi(t)}{1 + \mu(t)\varphi(t)}, \\ &(\omega \odot \varphi)(t) := \lim_{h \searrow \mu(t)} \frac{(1 + h\varphi(t))^\omega - 1}{h} \end{aligned}$$

for a given $\omega \in \mathbb{R}^+$ and for any $t \in \mathbb{T}$, $\varphi, \psi \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$. Let

$$\begin{aligned} &\mathbb{T}^+ = \mathbb{T} \cap [0, +\infty), \quad \kappa = \min\{t \in \mathbb{T}^+\}, \qquad [t,s]_{\mathbb{T}^+} = [t,s] \cap \mathbb{T}^+, \quad t, s \in \mathbb{T}^+, \\ &[\varphi]^* := \sup_{t \in \mathbb{T}^+}\big(\varphi(t)\big), \qquad [\varphi]_* := \inf_{t \in \mathbb{T}^+}\big(\varphi(t)\big) \end{aligned}$$

for any bounded function $\varphi \in C_{\mathrm{rd}}(\mathbb{T}^+, \mathbb{R})$.

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{B}(X)$ be the space of bounded linear operators defined on X. Now we give some definitions on time scales.

Definition 2.1 $\{U(t,s)\}_{t \geq s} \subset \mathcal{B}(X)$ is said to be an evolution family on a time scale $\mathbb{T}^+$ if

(i) $U(t,t) = \mathrm{id}$ for every $t \in \mathbb{T}^+$ and $U(t,\tau)U(\tau,s) = U(t,s)$ for any $t \geq \tau \geq s \geq \kappa$;

(ii) for each $s \in \mathbb{T}^+$ and any $x \in X$, $U(\cdot, s)x$ is rd-continuous on $[s, \infty)_{\mathbb{T}^+}$ for the first variable and $U(s, \cdot)x$ is rd-continuous on $[\kappa, s]_{\mathbb{T}^+}$ for the second variable.

Remark 2.1 In the general case, an evolution family $\{U(t,s)\}_{t\geq s}$ is related to an evolution operator of a linear dynamic equation on time scales.

Definition 2.2 An evolution family $\{U(t,s)\}_{t,s\in\mathbb{T}^+}$ is said to be exponential growth on a time scale $\mathbb{T}^+$ if there exist positive constants L and ρ such that

$$\|U(t,s)\| \leq Le_\rho(t,s), \quad t \geq s, t,s \in \mathbb{T}^+. \tag{2.1}$$

Definition 2.3 An evolution family $\{U(t,s)\}_{t,s\in\mathbb{T}^+}$ is said to admit an exponential dichotomy on a time scale $\mathbb{T}^+$ if there exist projections $\{P(t)\}_{t\in\mathbb{T}^+}$ such that $U(t,s)P(s) = P(t)U(t,s)$ for any $t \geq s \geq \kappa$ and $U(t,s)|_{\operatorname{Ker} P(s)} : \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$ is an isomorphism for any $t \geq s$, $t,s \in \mathbb{T}^+$ and there exist a constant $K > 0$ and $\alpha \in C_{\mathrm{rd}}(\mathbb{T}^+,\mathbb{R})$ with $[\alpha]_* > 0$ such that

(i) $\|U(t,s)x\| \leq Ke_{\ominus\alpha}(t,s)\|x\|$ for all $x \in \operatorname{Range} P(s)$ and any $t \geq s$, $t,s \in \mathbb{T}^+$;
(ii) $\|U(t,s)y\| \geq \frac{1}{K}e_\alpha(t,s)\|y\|$ for all $y \in \operatorname{Ker} P(s)$ and any $t \geq s$, $t,s \in \mathbb{T}^+$.

Remark 2.2 The exponential function on time scales can display different forms when we choose different time scales. For example, when $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, we have $e_{\ominus\alpha}(t,s) = e^{-\alpha(t-s)}$ or $e_{\ominus\alpha}(t,s) = (1/(1+\alpha))^{t-s}$ if α is a constant. Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $e_{\ominus\alpha}(t,s) = \prod_{\tau\in[s,t)}[1/(1+(q-1)\alpha\tau)]$. More examples for the exponential function on different time scales can be found in [16]. This shows that the exponential dichotomy on time scales is more general and unifies the notions of exponential dichotomies on the continuous and discrete case. On the other hand, we have

$$e_\alpha(t,s) \leq e^{\alpha(t-s)}, \qquad e^{-\alpha(t-s)} \leq e_{\ominus\alpha}(t,s) \tag{2.2}$$

for any $t \geq s$ and any time scale $\mathbb{T}$ (see (3.3) in [29]).

We let

$$C^b_{\mathrm{rd}}(\mathbb{T}^+,X) := \Bigl\{u \in C_{\mathrm{rd}}(\mathbb{T}^+,X) \,|\, \|u\|_\infty := \sup_{t\in\mathbb{T}^+}\|u(t)\| < \infty\Bigr\}$$

and

$$L^p(\mathbb{T}^+,X) := \biggl\{f \Bigm| f : \mathbb{T}^+ \to X \text{ is a Bochner measurable function with } \|f\|_p := \biggl(\int_\kappa^\infty \|f(t)\|^p \Delta t\biggr)^{1/p} < \infty\biggr\}$$

for $p > 1$. It is not difficult to show that $(C^b_{\mathrm{rd}}(\mathbb{T}^+,X), \|\cdot\|_\infty)$ and $(L^p(\mathbb{T}^+,X), \|\cdot\|_p)$ are both Banach spaces (see [31]). We consider the integral equation on time scales

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\Delta\tau, \quad t \geq s, t,s \in \mathbb{T}^+, \tag{2.3}$$

where $f \in L^p(\mathbb{T}^+,X)$ and $u \in C^b_{\mathrm{rd}}(\mathbb{T}^+,X)$.

Definition 2.4 The pair $(C^b_{\mathrm{rd}}(\mathbb{T}^+,X), L^p(\mathbb{T}^+,X))$ is said to be admissible for an evolution family $\{U(t,s)\}_{t,s\in\mathbb{T}^+}$ if for every $f \in L^p(\mathbb{T}^+,X)$ there exists a function $u \in C^b_{\mathrm{rd}}(\mathbb{T}^+,X)$ such that the pair (u,f) satisfies (2.3). We say that $L^p(\mathbb{T}^+,X)$ is the input space and $C^b_{\mathrm{rd}}(\mathbb{T}^+,X)$ is the output space.

We easily show that a pair (u,f) satisfies (2.3) if and only if (u,f) satisfies

$$u(t) = U(t,\kappa)u(\kappa) + \int_\kappa^t U(t,\tau)f(\tau)\Delta\tau, \quad t \geq \kappa, t \in \mathbb{T}^+. \tag{2.4}$$

In fact, if (2.4) holds, then for each $s \geq \kappa$

$$u(s) = U(s,\kappa)u(\kappa) + \int_\kappa^s U(s,\tau)f(\tau)\Delta\tau$$

and

$$\begin{aligned} U(t,s)u(s) &= U(t,s)U(s,\kappa)u(\kappa) + \int_\kappa^s U(t,s)U(s,\tau)f(\tau)\Delta\tau \\ &= U(t,\kappa)u(\kappa) + \int_\kappa^s U(t,\tau)f(\tau)\Delta\tau \\ &= u(t) - \int_s^t U(t,\tau)f(\tau)\Delta\tau \end{aligned}$$

for any $t \geq s$, $t \in \mathbb{T}^+$.

3 Main result

In this section, we establish a relation between exponential dichotomy on time scales and admissibility of the pair $(C^b_{\mathrm{rd}}(\mathbb{T}^+,X), L^p(\mathbb{T}^+,X))$ for an evolution family on time scales. Let the linear subspace $E_\kappa := \{x \in X | U(\cdot,\kappa)x \in C^b_{\mathrm{rd}}(\mathbb{T}^+,X)\}$. Now we state our main result.

Theorem 3.1 *Assume that an evolution family $U(t,s)_{t\geq s}$ admits an exponential growth on a time scale $\mathbb{T}^+$ with $[u]^* < \infty$. Then the pair $(C^b_{\mathrm{rd}}(\mathbb{T}^+,X), L^p(\mathbb{T}^+,X))$ is admissible for the evolution family $U(t,s)_{t\geq s}$ on the time scale $\mathbb{T}^+$ and E_κ is closed and complemented in X if and only if $U(t,s)_{t\geq s}$ admits an exponential dichotomy on the time scale $\mathbb{T}^+$.*

The proof of Theorem 3.1 is nontrivial, we shall divide it into several steps and assume that the conditions in Theorem 3.1 are always satisfied. We first establish some auxiliary results. If E_κ is closed and complemented in X, then there is a closed linear subspace F_κ such that $X = E_\kappa \oplus F_\kappa$. We define the linear subspace $C^{b,F_\kappa}_{\mathrm{rd}}(\mathbb{T}^+,X) := \{u \in C^b_{\mathrm{rd}}(\mathbb{T}^+,X) : u(\kappa) \in F_\kappa\}$. Using similar arguments to that of Lemma 2.1 in [15], we conclude that if the pair $(C^b_{\mathrm{rd}}(\mathbb{T}^+,X), L^p(\mathbb{T}^+,X))$ is admissible, then for every $f \in L^p(\mathbb{T}^+,X)$ there exists a unique function $\bar{u} \in C^{b,F_\kappa}_{\mathrm{rd}}(\mathbb{T}^+,X)$ such that the pair $(\bar{u},f)$ satisfies (2.3). Therefore, we can define the input-output operator $J : L^p(\mathbb{T}^+,X) \to C^{b,F_\kappa}_{\mathrm{rd}}(\mathbb{T}^+,X)$ by $J(f) = \bar{u}$, where the pair $(\bar{u},f)$ satisfies (2.3).

Lemma 3.1 *The operator J is bounded.*

Proof According to the closed graph theorem, we only need to prove that J is closed. We assume that $\{f_n\}_{n\in\mathbb{N}} \subset L^p(\mathbb{T}^+,X)$, $f \in L^p(\mathbb{T}^+,X)$, and $f_n \to f$ in $L^p(\mathbb{T}^+,X)$ as $n\to\infty$ and there exists a function $\bar{u} \in C_{\mathrm{rd}}^{b,F_\kappa}(\mathbb{T}^+,X)$ such that $\bar{u}_n = J(f_n) \to \bar{u}$ in $C_{\mathrm{rd}}^{b,F_\kappa}(\mathbb{T}^+,X)$ as $n\to\infty$. It follows from (2.4) that

$$\bar{u}_n(t) = U(t,\kappa)\bar{u}_n(\kappa) + \int_\kappa^t U(t,\tau)f_n(\tau)\Delta\tau, \quad t\in\mathbb{T}^+. \tag{3.1}$$

On the other hand, by (2.1) and the Hölder inequality on time scales, we have

$$\begin{aligned}\left\|\int_\kappa^t U(t,\tau)\big(f_n(\tau)-f(\tau)\big)\Delta\tau\right\| &\le \int_\kappa^t \big\|U(t,\tau)\big\|\big\|f_n(\tau)-f(\tau)\big\|\Delta\tau \\ &\le L\int_\kappa^t e_\rho(t,\tau)\big\|f_n(\tau)-f(\tau)\big\|\Delta\tau \\ &\le L\left(\int_\kappa^t e_{q\odot\rho}(t,\tau)\Delta\tau\right)^{1/q}\left(\int_\kappa^t \big\|f_n(\tau)-f(\tau)\big\|^p\Delta\tau\right)^{1/p} \\ &= L\left(\int_\kappa^t \big[1/\ominus(q\odot\rho)\big]e^\Delta_{\ominus(q\odot\rho)}(\tau,t)\Delta\tau\right)^{1/q}\|f_n-f\|_p \\ &\le \frac{1+[(q\odot\rho)\mu]^*}{[q\odot\rho]_*}e_\rho(t,\kappa)\|f_n-f\|_p\end{aligned}$$

for each $t\in\mathbb{T}^+$, where $1/q+1/p=1$. Then $\int_\kappa^t U(t,\tau)f_n(\tau)\Delta\tau \to \int_\kappa^t U(t,\tau)f(\tau)\Delta\tau$ since $f_n\to f$ in $L^p(\mathbb{T}^+,X)$ as $n\to\infty$. Combining with (3.1) gives

$$\bar{u}(t) = U(t,\kappa)\bar{u}(\kappa) + \int_\kappa^t U(t,\tau)f(\tau)\Delta\tau$$

since $\bar{u}_n \to \bar{u}$ in $C_{\mathrm{rd}}^{b,F_\kappa}(\mathbb{T}^+,X)$ as $n\to\infty$. This implies that $J(f)=\bar{u}$. The proof is completed. □

For each given $s\in\mathbb{T}^+$, we let

$$E_s := \left\{x\in X : \sup_{t\ge s}\big\|U(t,s)x\big\| < \infty\right\}, \qquad F_s := U(s,\kappa)F_\kappa. \tag{3.2}$$

Lemma 3.2 *If the pair* $(C_{\mathrm{rd}}^b(\mathbb{T}^+,X), L^p(\mathbb{T}^+,X))$ *is admissible for the evolution family* $U(t,s)_{t\ge s}$ *on the time scale* $\mathbb{T}^+$, *then the subspace* E_s *is closed for every* $s\in\mathbb{T}^+$ *and there is a positive constant* K_1 *and* $\alpha\in C_{\mathrm{rd}}(\mathbb{T}^+,\mathbb{R})$ *with* $[\alpha]_*>0$ *such that*

$$\big\|U(t,s)x\big\| \le K_1 e_{\ominus\alpha}(t,s)\|x\| \tag{3.3}$$

for any $x\in E_s$ *and* $t\ge s$.

Proof Let

$$\gamma := \frac{1}{\|J\|}\left(\frac{[p\odot\beta]_*}{1+[(p\odot\beta)\mu]^*}\right)^{1/p}, \quad 0<\beta<\gamma, \qquad \alpha=\gamma\ominus\beta, \tag{3.4}$$

where J is defined in Lemma 3.1. A direct calculation gives $[\gamma \ominus \beta]_* > 0$. For each given $s \in \mathbb{T}^+$ and any $x \in E_s \setminus \{0\}$, we let

$$d_{s,x} = \sup\{t \in \mathbb{T}^+ | U(t,s)x \neq 0, t \geq s\}$$

and

$$\eta_s = \inf\{t \in \mathbb{T}^+ | t \geq s+1\}. \tag{3.5}$$

Next we consider two different cases.

The first case is $d_{s,x} > \eta_s$. We let $f_t : \mathbb{T}^+ \to X$ by

$$f_t(r) = \chi_{[s,t)_{\mathbb{T}^+}}(r) e_\beta(r,s) \frac{U(r,s)x}{\|U(r,s)x\|}$$

and $u_t : \mathbb{T}^+ \to X$ by

$$u_t(r) = \int_\kappa^r \frac{\chi_{[s,t)_{\mathbb{T}^+}}(\tau)}{\|U(\tau,s)x\|} e_\beta(\tau,s) \Delta\tau \, U(r,s)x$$

for every $t \in (s, d_{s,x})_{\mathbb{T}^+}$. Then

$$\begin{aligned} \|f_t\|_p &= \left(\int_\kappa^\infty \|f_t(r)\|^p \Delta r\right)^{1/p} = \left(\int_s^t e_{p\odot\beta}(r,s)\Delta r\right)^{1/p} \\ &< \left(1/[p\odot\beta]_*\right)^{1/p} e_\beta(t,s) < \infty \end{aligned} \tag{3.6}$$

and $\sup_{r\in[t,\infty)_{\mathbb{T}^+}} \|u_t(r)\| < \infty$ since $u_t(r) = \int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|} \Delta\tau \, U(r,s)x$ for $r \geq t$ and $\sup_{r\in[s,\infty)_{\mathbb{T}^+}} \|U(r,s)x\| < \infty$ for $x \in E_s$. This implies that $f_t \in L^p(\mathbb{T}^+, X)$ and $u_t \in C^b_{\mathrm{rd}}(\mathbb{T}^+, X)$ for every $t \in \mathbb{T}^+$. Direct calculation shows that the pair (u_t, f_t) satisfies (2.3). Therefore, we have $u_t = J(f_t)$ and

$$\|u_t\|_\infty \leq \|J\| \|f_t\|_p \tag{3.7}$$

since $u_t(\kappa) = 0 \in F_\kappa$. Noting that $t \in (s, d_{s,x})_{\mathbb{T}^+}$ is arbitrary, by (3.7), (3.6) and (3.4), we have

$$\int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau \leq \frac{\|J\|}{([p\odot\beta]_*)^{1/p}} \frac{e_\beta(t,s)}{\|U(t,s)x\|} \leq \frac{1}{\gamma} \frac{e_\beta(t,s)}{\|U(t,s)x\|} \tag{3.8}$$

for any $t \in (s, d_{s,x})_{\mathbb{T}^+}$ since $\|u_t(t)\| \leq \|u_t\|_\infty$. On the other hand, it follows from (3.8) that

$$\begin{aligned} &\left(e_{\ominus\gamma}(t,\kappa) \int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau\right)^\Delta \\ &= e^\Delta_{\ominus\gamma}(t,\kappa) \int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau + e_{\ominus\gamma}(\sigma(t),\kappa)\left(\int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau\right)^\Delta \\ &= (\ominus\gamma) e_{\ominus\gamma}(t,\kappa) \int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau + e_{\ominus\gamma}(t,\kappa)\left(1+\mu(t)(\ominus\gamma)\right) \frac{e_\beta(t,s)}{\|U(t,s)x\|} \\ &= -\frac{\gamma e_{\ominus\gamma}(t,\kappa)}{1+\mu(t)\gamma} \int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau + \left(\frac{e_{\ominus\gamma}(t,\kappa)}{1+\mu(t)\gamma}\right) \frac{e_\beta(t,s)}{\|U(t,s)x\|} \end{aligned}$$

$$
\begin{aligned}
&= \frac{e_{\ominus\gamma}(t,\kappa)}{1+\mu(t)\gamma}\left(-\gamma\int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau + \frac{e_\beta(t,s)}{\|U(t,s)x\|}\right) \\
&\geq 0
\end{aligned}
$$

for any $t \in (s, d_{s,x})_{\mathbb{T}^+}$, which implies that $e_{\ominus\gamma}(t,\kappa)\int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau$ is nondecreasing on $(s, d_{s,x})_{\mathbb{T}^+}$. Then

$$
e_{\ominus\gamma}(t,\kappa)\int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau \geq e_{\ominus\gamma}(\eta_s,\kappa)\int_s^{\eta_s} \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau \tag{3.9}
$$

for any $t \in [\eta_s, d_{s,x})_{\mathbb{T}^+}$. By (2.1), we have

$$
\|U(t,s)x\| \leq Le_\rho(t,s)\|x\| \leq Le_\rho(\eta_s,s)\|x\| \tag{3.10}
$$

for any $t \in [s, \eta_s]_{\mathbb{T}^+}$. It follows from (3.8), (3.9), and (3.10) that

$$
\begin{aligned}
\frac{1}{Le_\rho(\eta_s,s)\|x\|} &\leq \int_s^{\eta_s} \frac{1}{Le_\rho(\eta_s,s)\|x\|}\Delta\tau \leq \int_s^{\eta_s} \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau \\
&\leq e_{\ominus\gamma}(t,\eta_s)\int_s^t \frac{e_\beta(\tau,s)}{\|U(\tau,s)x\|}\Delta\tau \leq \frac{e_{\ominus\gamma}(t,\eta_s)}{\gamma}\frac{e_\beta(t,s)}{\|U(t,s)x\|}
\end{aligned}
$$

for any $t \in [\eta_s, d_{s,x})_{\mathbb{T}^+}$. Then

$$
\begin{aligned}
\|U(t,s)x\| &\leq \frac{L}{\gamma}e_\rho(\eta_s,s)e_{\ominus\gamma}(t,\eta_s)e_\beta(t,s)\|x\| \\
&= \frac{L}{\gamma}e_\rho(\eta_s,s)e_{\ominus\gamma}(s,\eta_s)e_{\beta\ominus\gamma}(t,s)\|x\| \\
&= \frac{L}{\gamma}e_{\rho\oplus\gamma}(\eta_s,s)e_{\ominus\alpha}(t,s)\|x\|
\end{aligned}
$$

for any $t \in [\eta_s, d_{s,x})_{\mathbb{T}^+}$. To obtain the conclusion, we need to show that $\delta(s) := e_{\rho\oplus\gamma}(\eta_s,s)$ is bounded for any $s \in \mathbb{T}^+$. For the definition of η_s (see (3.5)), there are the following three different cases:

Case 1. $s+1 \in \mathbb{T}^+$. We have $\eta_s = \inf\{t \in \mathbb{T}^+ | t \geq s+1\} = s+1 < s+1+[\mu]^*$.

Case 2. $s+1 \notin \mathbb{T}^+$ and $(s,s+1] \cap \mathbb{T}^+ \neq \emptyset$. Let $t^* = \max\{t \in [s,s+1]_{\mathbb{T}^+}\}$. We have $\sigma(t^*) > t^*$. In fact, if $\sigma(t^*) = t^*$, then t^* is a right-dense point, which implies that there is a point $t^{**} > t^*$ and $t^{**} \in [s,s+1]_{\mathbb{T}^+}$. This is a contradiction. By the definition of t^*, we get $\eta_s = \sigma(t^*)$ and $\eta_s \leq s+1+\sigma(t^*)-t^* \leq s+1+[\mu]^*$.

Case 3. $(s,s+1] \cap \mathbb{T}^+ = \emptyset$. We have $\eta_s = \sigma(s) > s$ and $\eta_s \leq s+\sigma(s)-s \leq s+1+[\mu]^*$.

In view of the above discussion and (2.2), we have

$$
\delta(s) \leq e_\rho(\eta_s,s)e_\gamma(\eta_s,s) \leq e^{\rho(\eta_s-s)}e^{\gamma(\eta_s-s)} \leq e^{(\rho+\gamma)(\eta_s-s)} \leq e^{(\rho+\gamma)(1+[\mu]^*)} := L_1
$$

for any $s \in \mathbb{T}^+$. Then

$$
\|U(t,s)x\| \leq (LL_1/\gamma)e_{\ominus\alpha}(t,s)\|x\| \tag{3.11}
$$

for all $t \in [\eta_s, d_{s,x})_{\mathbb{T}^+}$. Moreover, by (2.1), we get

$$\begin{aligned} \|U(t,s)x\| &\le Le_\rho(t,s)\|x\| = Le_\rho(t,s)e_\alpha(t,s)e_{\ominus\alpha}(t,s)\|x\| \\ &\le Le_\rho(t,s)e_\gamma(t,s)e_{\ominus\alpha}(t,s)\|x\| \le Le_\rho(\eta_s,s)e_\gamma(\eta_s,s)e_{\ominus\alpha}(t,s)\|x\| \\ &\le LL_1 e_{\ominus\alpha}(t,s)\|x\| \end{aligned} \tag{3.12}$$

for all $t \in [s, \eta_s]_{\mathbb{T}^+}$. It follows from (3.11) and (3.12) that

$$\|U(t,s)x\| \le K_1 e_{\ominus\alpha}(t,s)\|x\| \tag{3.13}$$

for all $t \in [s, d_{s,x})_{\mathbb{T}^+}$, where $K_1 = \max\{LL_1, (LL_1/\gamma)\}$.

The second case is $s \le d_{s,x} \le \eta_s$. It follows from (2.1) and (3.13) that

$$\|U(t,s)x\| \le Le_\rho(t,s)\|x\| \le Le_\rho(\eta_s,s)e_\gamma(\eta_s,s)e_{\ominus\alpha}(t,s)\|x\| \le K_1 e_{\ominus\alpha}(t,s)\|x\| \tag{3.14}$$

for all $t \in [s, d_{s,x}]_{\mathbb{T}^+}$.

Based on (3.13), (3.14), and the definition of $d_{s,x}$, we conclude that (3.3) holds. Let $s \in \mathbb{T}^+$ and $\{x_n\}_{n\in\mathbb{N}} \subset E_s$ with $x_n \to x$ as $n \to \infty$. Combining with (3.3) gives $\|U(t,s)x_n\| \le K_1\|x_n\|$ for any $n \in \mathbb{N}$ and any $t \ge s$. Thus, we get $\|U(t,s)x\| \le K_1\|x\|$ for any $t \ge s$. This implies that $x \in E_s$ and E_s is closed. The proof is completed. □

Lemma 3.3 *If the pair* $(C^b_{\mathrm{rd}}(\mathbb{T}^+, X), L^p(\mathbb{T}^+, X))$ *is admissible for the evolution family* $U(t,s)_{t\ge s}$ *on the time scale* $\mathbb{T}^+$, *then the subspace* F_s *is closed for every* $s \in \mathbb{T}^+$ *and there is a positive constant* K_2 *and* $\alpha \in C_{\mathrm{rd}}(\mathbb{T}^+, \mathbb{R})$ *with* $[\alpha]_* > 0$ *such that*

$$K_2\|U(t,s)y\| \ge e_\alpha(t,s)\|y\| \tag{3.15}$$

for any $y \in F_s$ *and* $t \ge s$. *Moreover,* $U(t,s)|_{F_s} : F_s \to F_t$ *is an isomorphism for any* $t \ge s$, $t, s \in \mathbb{T}^+$.

Proof Let β, γ be positive constants and α be a rd-continuous function defined in (3.4). For $y \in F_\kappa \setminus \{0\}$, we have $U(t,\kappa)y \ne 0$ for any $t \in \mathbb{T}^+$. In fact, if there is $\bar{t} \in \mathbb{T}^+$ such that $U(\bar{t},\kappa)y = 0$, then $U(t,\kappa)y = U(t,\bar{t})U(\bar{t},\kappa)y = 0$ for any $t \ge \bar{t}$ and $y \in E_\kappa$. This means that $y \in E_\kappa \cap F_\kappa$ and $y = 0$. This is a contradiction to $y \in F_\kappa \setminus \{0\}$. For each $t \in \mathbb{T}^+$, we choose $\{\tau^t_n\}_{n\in\mathbb{N}} \subset \mathbb{T}^+$ such that $t < \tau^t_1 < \tau^t_2 < \cdots < \tau^t_n < \cdots$ and $\tau^t_n \to \infty$ as $n \to \infty$. We define $f_{\tau^t_n} : \mathbb{T}^+ \to X$ by

$$f_{\tau^t_n}(s) = -\chi_{[t,\tau^t_n]_{\mathbb{T}^+}} e_{\ominus\beta}(s,\kappa)\frac{U(s,\kappa)y}{\|U(s,\kappa)y\|}$$

and $u_{\tau^t_n} : \mathbb{T}^+ \to X$ by

$$u_{\tau^t_n}(s) = \int_s^\infty \frac{\chi_{[t,\tau^t_n]_{\mathbb{T}^+}}(\tau)e_{\ominus\beta}(\tau,\kappa)}{\|U(\tau,\kappa)y\|}\Delta\tau\, U(s,\kappa)y.$$

It follows that

$$\|f_{\tau^t_n}\|_p \le \left(\int_t^\infty e_{\ominus(p\odot\beta)}(s,\kappa)\Delta s\right)^{1/p} \le \left(\frac{1 + [(p\odot\beta)\mu]^*}{[p\odot\beta]_*}\right)^{1/p} e_{\ominus\beta}(t,\kappa) < \infty. \tag{3.16}$$

Moreover, $u_{\tau_n^t}$ is rd-continuous,

$$u_{\tau_n^t}(\kappa) = \left(\int_t^{\tau_n^t} \frac{e_{\ominus\beta}(\tau,\kappa)}{\|U(\tau,\kappa)y\|}\Delta\tau\right)y \in F_\kappa$$

and $u_{\tau_n^t}(s) = 0$ for $s \geq \tau_n^t$, which implies that $u_{\tau_n^t} \in C_{\mathrm{rd}}^{b,F_\kappa}(\mathbb{T}^+, X)$. Then $u_{\tau_n^t} = J(f_{\tau_n^t})$ and $\|u_{\tau_n^t}\|_\infty \leq \|J\|\|f_{\tau_n^t}\|_p$ for any $n \in \mathbb{N}$ since it is easy to show that the pair $(u_{\tau_n^t}, f_{\tau_n^t})$ satisfies (2.3). It follows from $\|u_{\tau_n^t}(t)\| \leq \|u_{\tau_n^t}\|_\infty$ and (3.16) that

$$\int_t^{\tau_n^t} \frac{e_{\ominus\beta}(\tau,\kappa)}{\|U(\tau,\kappa)y\|}\Delta\tau \left\|U(t,\kappa)y\right\| \leq \|J\|\left(\frac{1 + [(p \odot \beta)\mu]^*}{[p \odot \beta]_*}\right)^{1/p} e_{\ominus\beta}(t,k) = \frac{1}{\gamma} e_{\ominus\beta}(t,\kappa)$$

for any $n \in \mathbb{N}$. This also reads

$$\gamma \int_t^{\infty} \frac{e_{\ominus\beta}(\tau,\kappa)}{\|U(\tau,\kappa)y\|}\Delta\tau \leq \frac{e_{\ominus\beta}(t,\kappa)}{\|U(t,\kappa)y\|} \tag{3.17}$$

as $n \to \infty$. It follows from (3.17) that

$$\begin{aligned}
&\left(e_\gamma(t,\kappa)\int_t^\infty \frac{e_{\ominus\beta}(\tau,\kappa)}{\|u(\tau,\kappa)y\|}\Delta\tau\right)^\Delta \\
&\quad = e_\gamma^\Delta(t,\kappa)\int_t^\infty \frac{e_{\ominus\beta}(\tau,\kappa)}{\|u(\tau,\kappa)y\|}\Delta\tau + e_\gamma\big(\sigma(t),\kappa\big)\left(\int_t^\infty \frac{e_{\ominus\beta}(\tau,\kappa)}{\|u(\tau,\kappa)y\|}\Delta\tau\right)^\Delta \\
&\quad = \gamma e_\gamma(t,\kappa)\int_t^\infty \frac{e_{\ominus\beta}(\tau,\kappa)}{\|u(\tau,\kappa)y\|}\Delta\tau - \big(1 + \mu(t)\gamma\big)e_\gamma(t,\kappa)\frac{e_{\ominus\beta}(t,\kappa)}{\|U(t,\kappa)y\|} \\
&\quad \leq e_\gamma(t,\kappa)\left(\gamma\int_t^\infty \frac{e_{\ominus\beta}(\tau,\kappa)}{\|u(\tau,\kappa)y\|}\Delta\tau - \frac{e_{\ominus\beta}(t,\kappa)}{\|U(t,\kappa)y\|}\right) \leq 0.
\end{aligned}$$

Thus, we get

$$e_\gamma(t,\kappa)\int_t^\infty \frac{e_{\ominus\beta}(\tau,\kappa)}{\|u(\tau,\kappa)y\|}\Delta\tau \leq e_\gamma(s,\kappa)\int_s^\infty \frac{e_{\ominus\beta}(\tau,\kappa)}{\|U(\tau,\kappa)y\|}\Delta\tau \tag{3.18}$$

for any $t \geq s$, $t, s \in \mathbb{T}^+$. Combining with (3.17) gives

$$\gamma e_\gamma(t,s)\int_t^\infty \frac{e_{\ominus\beta}(\tau,\kappa)}{\|u(\tau,\kappa)y\|}\Delta\tau \leq \frac{e_{\ominus\beta}(s,\kappa)}{\|U(s,\kappa)y\|} \tag{3.19}$$

for any $t \geq s$, $t, s \in \mathbb{T}^+$. On the other hand, due to (2.1), it is sufficient to have

$$\left\|U(\tau,\kappa)y\right\| = \left\|U(\tau,t)U(t,\kappa)y\right\| \leq Le_\rho(\tau,t)\left\|U(t,\kappa)y\right\|$$

for any $\tau \geq t$, $\tau, t \in \mathbb{T}^+$. This implies that

$$\begin{aligned}
\int_t^\infty \frac{e_{\ominus\beta}(\tau,\kappa)}{\|u(\tau,\kappa)y\|}\Delta\tau &= e_{\ominus\beta}(t,\kappa)\int_t^\infty \frac{e_{\ominus\beta}(\tau,t)}{\|u(\tau,\kappa)y\|}\Delta\tau \geq \frac{e_{\ominus\beta}(t,\kappa)}{L\|U(t,\kappa)y\|}\int_t^\infty e_{\ominus(\beta\oplus\rho)}(\tau,t)\Delta\tau \\
&\geq \frac{(1 + [(\beta\oplus\rho)\mu]_*)e_{\ominus\beta}(t,\kappa)}{L[\beta\oplus\rho]^*\|U(t,\kappa)y\|}.
\end{aligned} \tag{3.20}$$

By (3.19) and (3.20), we have

$$e_{\gamma\ominus\beta}(t,s)\|U(s,\kappa)y\| \leq \frac{L[\beta\oplus\rho]^*}{\gamma(1+[(\beta\oplus\rho)\mu]_*)}\|U(t,\kappa)y\| \tag{3.21}$$

for any $t \geq s$, $t,s \in \mathbb{T}^+$. Together with $F_s = U(s,\kappa)F_\kappa$, $K_2\|U(t,s)y\| \geq e_\alpha(t,s)\|y\|$ holds for any $y \in F_s$ and $t \geq s$, where $K_2 = (L[\beta\oplus\rho]^*/\gamma(1+[(\beta\oplus\rho)\mu]_*))$.

We easily conclude that the subspace F_s is closed for every $s \in \mathbb{T}^+$ since $F_s = U(s,\kappa)F_\kappa$ and F_κ is closed. It follows from $F_t = U(t,\kappa)F_\kappa = U(t,s)F_s$ and (3.15) that $U(t,s)|_{F_s} : F_s \to F_t$ is well defined and bijection $t \geq s$, $t,s \in \mathbb{T}^+$. The proof is completed. □

We are now at the right position to establish Theorem 3.1.

Proof of Theorem 3.1 (Sufficiency). If $U(t,s)_{t\geq s}$ admits an exponential growth and an exponential dichotomy on the time scale $\mathbb{T}^+$, then

$$\|x+y\| \geq \frac{1}{L}e_{\ominus\rho}(t,s)\|U(t,s)(x+y)\| \geq \frac{1}{L}e_{\ominus\rho}(t,s)\left(\frac{1}{K}e_\alpha(t,s) - Ke_{\ominus\alpha}(t,s)\right)$$

for any $t \geq s$ and $x \in \operatorname{Range} P(s)$, $y \in \operatorname{Ker} P(s)$ with $\|x\| = \|y\| = 1$. This shows that there is a positive constant $\hat{c}$ such that

$$\begin{aligned}\hat{c} &\leq \inf_{s\in\mathbb{T}^+}\bigl\{\|x+y\| \,|\, x \in \operatorname{Range} P(s), y \in \operatorname{Ker} P(s), \|x\| = 1, \|y\| = 1\bigr\}\\ &\leq \left\|\frac{P(s)z}{\|P(s)z\|} + \frac{\mathrm{id} - P(s)z}{\|\mathrm{id} - P(s)z\|}\right\| \leq \frac{2\|z\|}{\|P(s)z\|}\end{aligned}$$

for any $z \in X$, which implies that $\|P(s)\| \leq 2/\hat{c} := c$ for any $s \in \mathbb{T}^+$. For every $f \in L^p(\mathbb{T}^+,X)$, we let

$$u(t) = \int_\kappa^t U(t,\tau)P(\tau)f(\tau)\Delta\tau - \int_t^\infty U(t,\tau)\bigl(\mathrm{id} - P(\tau)\bigr)f(\tau)\Delta\tau.$$

It follows from (i) and (ii) in Definition 2.3 that

$$\begin{aligned}\|u(t)\| &\leq Kc\int_\kappa^t e_{\ominus\alpha}(t,\tau)\|f(\tau)\|\Delta\tau + K(1+c)\int_t^\infty e_{\ominus\alpha}(\tau,t)\|f(\tau)\|\Delta\tau\\ &\leq \left(\frac{Kc}{[q\odot\alpha]_*}\right)^{1/q}\|f\|_p + \left(\frac{1+[(q\odot\alpha)\mu]^*}{[q\odot\alpha]_*}\right)^{1/q}\|f\|_p\end{aligned}$$

for any $t \in \mathbb{T}^+$, where $1/q + 1/p = 1$. Then $u \in C_{\mathrm{rd}}(\mathbb{T}^+,\mathbb{R})$. A direct calculation gives the pair (u,f) that satisfies (2.4). Thus, the pair $(C^b_{\mathrm{rd}}(\mathbb{T}^+,X), L^p(\mathbb{T}^+,X))$ is admissible for the evolution family $U(t,s)_{t\geq s}$ on the time scale $\mathbb{T}^+$. In view of (i) and (ii) in Definition 2.3, for any $x \in E_\kappa$, we have $\sup_{t\in\mathbb{T}^+}\|U(t,\kappa)x\| < \infty$ and

$$\begin{aligned}\frac{e_\alpha(t,\kappa)}{K}\|\bigl(\mathrm{id} - P(\kappa)\bigr)x\| &\leq \|U(t,\kappa)\bigl(\mathrm{id} - P(\kappa)\bigr)x\|\\ &\leq \sup_{t\in\mathbb{T}^+}\|U(t,\kappa)x\| + Ke_{\ominus\alpha}(t,\kappa)\|P(\kappa)x\|\\ &\leq \sup_{t\in\mathbb{T}^+}\|U(t,\kappa)x\| + Kc\|x\| < \infty\end{aligned}$$

for any $t \in \mathbb{T}^+$. Therefore, $(\mathrm{id} - P(\kappa))x = 0$ and $x \in \operatorname{Range} P(\kappa)$. On the other hand, it is clear that $\operatorname{Range} P(\kappa) \subset E_\kappa$. This means that $E_\kappa = \operatorname{Range} P(\kappa)$ is closed and complemented in X.

(Necessity). By Lemmas 3.2 and 3.3, if the pair $(\mathrm{C}^b_{\mathrm{rd}}(\mathbb{T}^+, X), L^p(\mathbb{T}^+, X))$ is admissible for the evolution family $U(t,s)_{t\geq s}$ on the time scale $\mathbb{T}^+$, then E_s and F_s (see (3.2)) are both closed linear subspaces for every $s \in \mathbb{T}^+$. Let $P(s)$ be the projection satisfying $P(s)(X) = E_s$ for every $s \in \mathbb{T}^+$. To obtain the conclusions, we need to prove $(I - P(s))(X) = F_s$. If $z \in E_s \cap F_s$ for every $s \in \mathbb{T}^+$, then there is $\hat{z} \in F_\kappa$ such that $U(s,\kappa)\hat{z} = z$. By $U(t,\kappa)\hat{z} = U(t,s)U(s,\kappa)\hat{z} = U(t,s)x \in \mathrm{C}^b_{\mathrm{rd}}(\mathbb{T}^+, X)$, we get $\hat{z} \in E_\kappa \cap F_\kappa = \{0\}$ and $z = U(s,\kappa)\hat{z} = 0$. Thus, $E_s \cap F_s = \{0\}$. For any $z \in X$, we have $f(t) := \chi_{[s,\eta_s)_{\mathbb{T}^+}} U(t,s)z \in L^p(\mathbb{T}^+, X)$ and there exists $u \in \mathrm{C}^{b.F_\kappa}_{\mathrm{rd}}(\mathbb{T}^+, X)$ such that

$$\begin{aligned} u(t) &= J(f) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\Delta\tau \\ &\geq U(t,s)u(s) + \int_s^{\eta_s} U(t,\tau)f(\tau)\Delta\tau \\ &\geq U(t,s)\big(u(s) + z\big) \end{aligned}$$

for any $t \geq \eta_s$, where η_s can be found in (3.5). Then we get $u(s) + z \in E_s$. This implies together with the fact that $u(s) \in F_s$ since $u(\kappa) \in F_\kappa$ that $z = u(s) + z - u(s) \in E_s + F_s$. Combining with $E_s \cap F_s = \{0\}$ gives $X = E_s \oplus F_s$. This means that $(I - P(s))(X) = F_s$ is well defined. Hence, we have $U(t,s)P(s) = P(t)U(t,s)$, $\operatorname{Range} P(s) = E_s$ and $\operatorname{Ker} P(s) = F(s)$. It follows from Lemma 3.2 and Lemma 3.3 that $U(t,s)_{t\geq s}$ admits an exponential dichotomy on the time scale $\mathbb{T}^+$, where $K = \max\{K_1, K_2\}$ and β, γ, α can be found in (3.4). □

Remark 3.1 Our result extends related results known for differential equations [15] and difference equations [12] on the half-line to more general time scales.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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