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Existence of solutions for a class of biharmonic equations with the Navier boundary value condition

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Abstract
In this paper, the existence of at least one nontrivial solution for a class of fourth-order elliptic equations with the Navier boundary value conditions is established by using the linking methods.

Keywords: biharmonic; Navier boundary value problems; local linking

1 Introduction

Consider the following Navier boundary value problem:

$$\begin{cases} \triangle^2 u(x) + l\triangle u = f(x,u), & \text{in } \Omega; \\ u = \triangle u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\triangle^2$ is the biharmonic operator, $l \in R$ and Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N > 4)$.

The conditions imposed on $f(x,t)$ are as follows:

(H_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and there are constants $C_1, C_2 \geq 0$ such that

$$\left|f(x,t)\right| \leq C_1 + C_2|t|^{s-1}, \quad \forall x \in \Omega, \forall t \in R, s \in \left(2, p^*\right)(N > 4),$$

where $p^* = \frac{2N}{N-4}$;

(H_2) $f(x,t) = \circ(|t|)$, $|t| \to 0$, uniformly on Ω;

(H_3) $\lim_{|t|\to\infty} \frac{f(x,t)}{t} = +\infty$ uniformly on Ω;

(H_4) There is a constant $\theta \geq 1$ such that for all $(x,t) \in \Omega \times R$ and $s \in [0,1]$,

$$\theta\left(f(x,t)t - 2F(x,t)\right) \geq \left(sf(x,st)t - 2F(x,st)\right),$$

where $F(x,t) = \int_0^t f(x,s)\,ds$;

(H_5) For some $\delta > 0$, either

$$F(x,t) \geq 0, \quad \text{for } |t| \leq \delta, x \in \Omega,$$

or

$$F(x,t) \leq 0, \quad \text{for } |t| \leq \delta, x \in \Omega.$$

This fourth-order semilinear elliptic problem has been studied by many authors. In [1], there was a survey of results obtained in this direction. In [2], Micheletti and Pistoia showed that (1.1) admits at least two solutions by a variation of linking if $f(x,u)$ is sublinear. And in [3], the authors proved that the problem (1.1) has at least three solutions by a variational reduction method and a degree argument. In [4], Zhang and Li showed that (1.1) admits at least two nontrivial solutions by the Morse theory and local linking if $f(x,u)$ is superlinear and subcritical on u. In [5], Zhang and Wei obtained the existence of infinitely many solutions for the problem (1.1) where the nonlinearity involves a combination of superlinear and asymptotically linear terms. As far as the problem (1.1) is concerned, existence results of sign-changing solutions were also obtained. See, *e.g.*, [6, 7] and the references therein.

We will use linking methods to give the existence of at least one nontrivial solution for (1.1).

Let X be a Banach space with a direct sum decomposition

$$X = X^1 \oplus X^2.$$

The function $I \in C^1(X,R)$ has a local linking at 0, with respect to (X^1,X^2) if for some $r > 0$,

$$I(u) \geq 0, \quad u \in X^1, \|u\| \leq r, \tag{1.2}$$

$$I(u) \leq 0, \quad u \in X^2, \|u\| \leq r. \tag{1.3}$$

It is clear that 0 is a critical point of I.

The notion of local linking generalizes the notions of local minimum and local maximum. When 0 is a non-degenerate critical point of a functional of class C^2 defined on a Hilbert space and $I(0) = 0$, I has local linking at 0.

The condition of local linking was introduced in [8] under stronger assumptions

$$I(u) \geq c > 0, \quad u \in X^1, \|u\| = r, \dim X^2 < \infty.$$

Under those assumptions, the existence of nontrivial critical points was proved for functionals which are

(a) bounded below [8],
(b) superquadratic [8] and
(c) asymptotically quadratic [9].

The cases (a), (b) and (c) were considered in [10] with only conditions (1.2) and (1.3), and three theorems about critical points were proved. Using these theorems, the author in [10] proved the existence of at least one nontrivial solution for the second-order elliptic boundary value problem with the Dirichlet boundary value condition. In the present paper, we also use the three theorems in [10] and the linking technique to give the existence of at least one nontrivial solution for the biharmonic problem (1.1) under a weaker condition. The main results are as follows.

Theorem 1.1 *Assume the conditions* (H_1)-(H_4) *hold. If l is an eigenvalue of* $-\triangle$ (*with the Dirichlet boundary condition*), *assume also* (H_5). *Then the problem* (1.1) *has at least one nontrivial solution.*

We also consider asymptotically quadratic functions.

Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ be the eigenvalues of $(-\triangle, H_0^1(\Omega))$. Then μ_j ($j \in N_+$) is the eigenvalue of $(\triangle^2 + l\triangle, H^2(\Omega) \cap H_0^1(\Omega))$, where $\mu_j = \lambda_j(\lambda_j - l)$. We assume that

(H_6) $f(x,u) = f_\infty u + \circ(|u|)$, $|u| \to \infty$, uniformly in Ω, and $\mu_k < f_\infty < \mu_{k+1}$.

Theorem 1.2 *Assume the conditions* (H_1), (H_6) *and one of the following conditions*:

(A_1) $\lambda_j < l < \lambda_{j+1}$, $j \neq k$;
(A_2) $\lambda_j = l < \lambda_{j+1}$, $j \neq k$, *for some* $\delta > 0$,

$$F(x,u) \geq 0, \quad \textit{for } |u| > \delta, x \in \Omega;$$

(A_3) $\lambda_j < l = \lambda_{j+1}$, $j \neq k$, *for some* $\delta > 0$,

$$F(x,u) \geq 0, \quad \textit{for } |u| \leq \delta, x \in \Omega.$$

Then the problem (1.1) *has at least one nontrivial solution.*

2 Preliminaries

Let X be a Banach space with a direct sum decomposition

$$X = X^1 \oplus X^2.$$

Consider two sequences of a subspace:

$$X_0^1 \subset X_1^1 \subset \cdots \subset X^1, \qquad X_0^2 \subset X_1^2 \subset \cdots \subset X^2$$

such that

$$X^j = \bigcup_{n \in N} X_n^j, \quad j = 1, 2.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, let $X_\alpha = X_{\alpha_1} \oplus X_{\alpha_2}$. We know that

$$\alpha \leq \beta \quad \Leftrightarrow \quad \alpha_1 \leq \beta_1, \qquad \alpha_2 \leq \beta_2.$$

A sequence $(\alpha_n) \subset N^2$ is admissible if for every $\alpha \in N^2$, there is $m \in N$ such that $n \geq m \Rightarrow \alpha_n \geq \alpha$. For every $I : X \to R$, we denote by I_α the function I restricted X_α.

Definition 2.1 Let I be locally Lipschitz on X and $c \in R$. The functional I satisfies the $(C)_c^*$ condition if every sequence (u_{α_n}) such that (α_n) is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad I(u_{\alpha_n}) \to c, \qquad (1 + \|u_{\alpha_n}\|) I'(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of I.

Definition 2.2 Let I be locally Lipschitz on X and $c \in R$. The functional I satisfies the $(C)^*$ condition if every sequence (u_{α_n}) such that (α_n) is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad \sup_n I(u_{\alpha_n}) \leq c, \qquad \bigl(1 + \|u_{\alpha_n}\|\bigr) I'(u_{\alpha_n}) \to 0$$

contains a subsequence which converges to a critical point of I.

Remark 2.1

1. The $(C)^*$ condition implies the $(C)^*_c$ condition for every $c \in R$.
2. When the $(C)^*_c$ sequence is bounded, then the sequence is a $(PS)^*_c$ sequence (see [11]).
3. Without loss of generality, we assume that the norm in X satisfies

$$\|u_1 + u_2\|^2 = \|u_1\|^2 + \|u_2\|^2, \quad u_j \in X_j, j = 1, 2.$$

Definition 2.3 Let X be a Banach space with a direct sum decomposition

$$X = X_1 \oplus X_2.$$

The function $I \in C^1(X, R)$ has a local linking at 0, with respect to (X^1, X^2), if for some $r > 0$,

$$I(u) \geq 0, \quad u \in X^1, \|u\| \leq r,$$
$$I(u) \leq 0, \quad u \in X^2, \|u\| \leq r.$$

Lemma 2.1 (see [10]) *Suppose that $I \in C^1(X, R)$ satisfies the following assumptions*:

(B_1) *I has a local linking at* 0 *and $X^1 \neq \{0\}$*;
(B_2) *I satisfies $(PS)^*$*;
(B_3) *I maps bounded sets into bounded sets*;
(B_4) *for every $m \in N$, $I(u) \to -\infty$, $\|u\| \to \infty$, $u \in X = X^1_m \oplus X^2$. Then I has at least two critical points.*

Remark 2.2 Assume I satisfies the $(C)^*_c$ condition. Then this theorem still holds.

Let X be a real Hilbert space and let $I \in C^1(X, R)$. The gradient of I has the form

$$\nabla I(u) = Au + B(u),$$

where A is a bounded self-adjoint operator, 0 is not the essential spectrum of A, and B is a nonlinear compact mapping.

We assume that there exist an orthogonal decomposition,

$$X = X_1 + X_2,$$

and two sequences of finite-dimensional subspaces,

$$X^1_0 \subset X^1_1 \subset X^1_1 \subset \cdots \subset X^1, \qquad X^2_0 \subset X^2_1 \subset \cdots \subset X^2,$$

such that

$$X^j = \overline{\bigcup_{n\in N} X_n^j}, \quad j = 1,2,$$
$$AX_n^j \subset X_n^j, \quad j = 1,2, n \in N.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, we denote by X_α the space

$$X_\alpha^1 \oplus X_\alpha^2,$$

by $p_\alpha : X \to X_\alpha$ the orthogonal projector onto X_α, and by $M^-(L)$ the Morse index of a self-adjoint operator L.

Lemma 2.2 (see [10]) *I satisfies the following assumptions*:

(i) *I has a local linking at* 0 *with respect to* (X^1, X^2);

(ii) *there exists a compact self-adjoint operator* B_∞ *such that*

$$B(u) = B_\infty(u) + \circ(\|u\|), \quad \|u\| \to \infty;$$

(iii) $A + B_\infty$ *is invertible*;

(iv) *for infinitely many multiple-indices* $\alpha := (n,n)$,

$$M^-\big((A + P_\alpha B_\infty)|_{X_\alpha}\big) \neq \dim X_n^2.$$

Then I has at least two critical points.

3 The proof of main results

Proof of Theorem 1.1 (1) We shall apply Lemma 2.1 to the functional

$$I(u) = \frac{1}{2}\int_\Omega \big(|\triangle u|^2 - l|\nabla u|^2\big)\,dx - \int_\Omega F(x,u)\,dx$$

defined on $X = H_0^1(\Omega) \cap H^2(\Omega)$. We consider only the case $l = \lambda_j$, and

$$F(x,u) \le 0, \quad \text{for } |u| \le \delta, x \in \Omega. \tag{3.1}$$

Then other case is similar and simple.

Let X^2 be the finite dimensional space spanned by the eigenfunctions corresponding to negative eigenvalues of $-\triangle^2 + l\triangle$ and let X^1 be its orthogonal complement in X. Choose a Hilbertian basis e_n $(n \ge 0)$ for X and define

$$X_n^1 = \operatorname{span}(e_0, e_1, \ldots, e_n), \quad n \in N;$$
$$X_n^2 = X^2, \quad n \in N;$$
$$X^1 = \overline{\bigcup_{n\in N} X_n^1}.$$

By the condition (H_1) and Sobolev inequalities, it is easy to see that the functional I belongs to $C^1(X,R)$ and maps bounded sets to bounded sets.

(2) We claim that I has a local linking at 0 with respect to (X^1,X^2). Decompose X^1 into $V+W$ when $V=\ker(-\triangle^2+l\triangle)$, $W=(X^2+V)^\perp$. Also, set $u=v+w$, $u\in X^1$, $v\in V$, $w\in W$. By the equivalence of norm in the finite-dimensional space, there exists $C>0$ such that

$$\|v\|_\infty \le C\|v\|_X, \quad \forall v\in V. \tag{3.2}$$

It follows from (H_1) and (H_2) that for any $\epsilon>0$, there exists C_ϵ such that

$$\bigl|F(x,u)\bigr| \le \epsilon u^2 + C_\epsilon|u|^s. \tag{3.3}$$

Hence, we obtain

$$\begin{aligned} I(u) &\le \frac{1}{2}\int_\Omega \bigl(|\triangle u|^2 - l|\nabla u|^2\bigr)\,dx + \epsilon\int_\Omega u^2\,dx + c\|u\|_X^{s+1} \\ &\le -m\|u\|^2 + \epsilon\int_\Omega u^2\,dx + c^*\|u\|_X^{s+1}, \end{aligned}$$

where $m>0$, c^* is a constant and hence, for $r>0$ small enough,

$$I(u)\le 0, \quad u\in X^2, \|u\|_X\le r.$$

Let $u=v+w\in X^1$ be such that $\|u\|_X\le r_1=\frac{\delta}{2C}$ and let

$$\begin{aligned} &\Omega_1 = \left\{x\in\Omega : \bigl|w(x)\bigr| \le \frac{\delta}{2}\right\}, \\ &\Omega_2 = \Omega\setminus\Omega_1. \end{aligned}$$

From (3.2), we have

$$\bigl|v(x)\bigr| \le \|v\|_\infty \le C\|v\| \le \frac{\delta}{2}$$

for all $\|u\|\le r_1$ and $x\in\Omega$. On the one hand, one has $|u(x)|\le|v(x)|+|w(x)|\le\|v\|_\infty+\frac{\delta}{2}\le\delta$ for all $x\in\Omega_1$. Hence, from (H_5), we obtain

$$\int_{\Omega_1} F(x,u)\,dx \le 0.$$

On the other hand, we have

$$\bigl|u(x)\bigr| \le \bigl|v(x)\bigr| + \bigl|w(x)\bigr| \le \frac{\delta}{2} + \bigl|w(x)\bigr| \le 2\bigl|w(x)\bigr|$$

for all $x\in\Omega_2$. It follows from (3.3) that

$$F(x,u) \le \epsilon u^2 + C_\epsilon|u|^{s+1} \le 4\epsilon w^2 + 2^{s+1}C_\epsilon|w|^{s+1}$$

for all $x \in \Omega_2$ and all $u \in X_1$ with $\|u\| \leq r_1$, which implies that

$$\begin{aligned}\int_\Omega F(x,u)\,dx &\leq 4\epsilon \int_{\Omega_2} w^2\,dx + \int_{\Omega_2} 2^{s+1} C_\epsilon |w|^{s+1}\,dx \\ &\leq 4(C_3)^2\epsilon\|w\|^2 + (2C_3)^{\lambda+1}C_\epsilon\|w\|^{s+1},\end{aligned}$$

where C_3 is a constant. Hence, there exist positive constants C^{**}, C_4 and C_5 such that

$$\begin{aligned}I(u) &= \frac{1}{2}\|w\|^2 - \frac{1}{2}\int_\Omega l|\nabla w|^2\,dx - \int_{\Omega_2} F(x,u)\,dx - \int_{\Omega_1} F(x,u)\,dx \\ &\geq C^{**}\|w\|^2 - 4(C_3)^2\epsilon\|w\|^2 - (2C_3)^{\lambda+1}C_\epsilon\|w\|^{s+1} - \int_{\Omega_1} G(x,u)\,dx \\ &\geq C_4\|w\|^2 - C_5\|w\|^{s+1}\end{aligned}$$

for all $u \in X^1$ with $\|u\| \leq r_1$, which implies that

$$I(u) \geq 0, \quad \forall u \in X^1 \text{ with } \|u\| \leq r$$

for $0 < r$ small enough.

(3) We claim that I satisfies $(C)_c^*$. Consider a sequence (u_{α_n}) such that (u_{α_n}) is admissible and

$$u_{\alpha_n} \in X_{\alpha_n}, \quad I(u_{\alpha_n}) \to c, \qquad \bigl(1 + \|u_{\alpha_n}\|\bigr)I'(u_{\alpha_n}) \to 0 \tag{3.4}$$

and

$$\lim_{n\to\infty} \int_\Omega \left(\frac{1}{2} f(x,u_{\alpha_n})u_{\alpha_n} - F(x,u_{\alpha_n})\right) dx = c. \tag{3.5}$$

Let $w_{\alpha_n} = \|u_{\alpha_n}\|^{-1}u_{\alpha_n}$. Up to a subsequence, we have

$$w_{\alpha_n} \rightharpoonup w \quad \text{in } X, \qquad w_{\alpha_n} \to w \quad \text{in } L^2, \qquad w_{\alpha_n}(x) \to w(x) \quad \text{a.e. } x \in \Omega.$$

If $w = 0$, we choose a sequence $\{t_n\} \subset [0,1]$ such that

$$I(t_n u_{\alpha_n}) = \max_{t\in[0,1]} I(tu_{\alpha_n}).$$

For any $m > 0$, let $v_{\alpha_n} = 2\sqrt{m}w_{\alpha_n}$. By the Sobolev imbedded theory, we have

$$\lim_{n\to\infty} \int_\Omega F(x,v_{\alpha_n})\,dx = 0.$$

So, for n large enough, $2\sqrt{m}\|u_{\alpha_n}\|^{-1} \in (0,1)$, and combining Ehrling-Nirenberg-Gagliardo inequality, we have

$$I(t_n u_{\alpha_n}) \geq I(v_{\alpha_n}) \geq m - \epsilon \geq \frac{m}{2}, \tag{3.6}$$

where ϵ is a small enough constant.

That is, $I(t_n u_{\alpha_n}) \to \infty$. Now, $I(0) = 0$, $I(u_{\alpha_n}) \to c$, we know that $t_n \in [0,1]$ and

$$\begin{aligned}
&\int_\Omega \left(\left|\triangle(t_n u_{\alpha_n})\right|^2 - l\left|\nabla(t_n u_{\alpha_n})\right|^2\right) dx - \int_\Omega f(x, t_n u_{\alpha_n}) t_n u_{\alpha_n}\, dx \\
&\quad = t_n \frac{d}{dt}\bigg|_{t=t_n} I(t u_{\alpha_n}) = 0.
\end{aligned} \tag{3.7}$$

Therefore, using (H_3), we have

$$\begin{aligned}
&\int_\Omega \frac{1}{2} f(x, u_{\alpha_n}) u_{\alpha_n} - F(x, u_{\alpha_n})\, dx \\
&\quad \geq \frac{1}{\theta} \int_\Omega \left(\frac{1}{2} f(x, t_n u_{\alpha_n}) t_n u_{\alpha_n} - F(x, t_n u_{\alpha_n})\right) dx \to +\infty.
\end{aligned}$$

This contradicts (3.5).

If $w \neq 0$, then the set $\ominus = \{x \in \Omega : w(x) \neq 0\}$ has a positive Lebesgue measure. For $x \in \ominus$, we have $|u_{\alpha_n}(x)| \to \infty$. Hence, by (H_3), we have

$$\frac{f(x, u_{\alpha_n}(x)) u_{\alpha_n}(x)}{|u_{\alpha_n}(x)|^2} \left|w_{\alpha_n}(x)\right|^2 dx \to \infty. \tag{3.8}$$

From (3.4), we obtain

$$1 - \circ(1) \geq \left(\int_{w \neq 0} + \int_{w=0}\right) \frac{f(x, u_{\alpha_n}(x)) u_{\alpha_n}(x)}{|u_{\alpha_n}(x)|^2} \left|w_{\alpha_n}(x)\right|^2 dx. \tag{3.9}$$

By (3.8), the right-hand side of (3.9) $\to +\infty$. This is a contradiction.

In any case, we obtain a contradiction. Therefore, $\{u_{\alpha_n}\}$ is bounded.

Finally, we claim that for every $m \in N$,

$$I(u) \to -\infty \quad \text{as } \|u\| \to \infty, u \in X_m^1 \oplus X^2.$$

By (H_2) and (H_3), there exists large enough M such that

$$F(x,t) \geq Mt^2 - C_6, \quad x \in \Omega, t \in \mathbb{R}.$$

So, for any $u \in X_m^1 \oplus X^2$, we have

$$\begin{aligned}
I(tu) &= \frac{1}{2} t^2 \int_\Omega \left(|\triangle u|^2 - l|\nabla u|^2\right) dx - \int_\Omega F(x, tu)\, dx \\
&\leq \frac{1}{2} t^2 \int_\Omega \left(|\triangle u|^2 - l|\nabla u|^2\right) dx - Mt^2 \int_\Omega u^2\, dx + C_6|\Omega| \to -\infty \quad \text{as } t \to +\infty.
\end{aligned}$$

Hence, our claim holds. □

Proof of Theorem 1.2 We omit the proof which depends on Lemma 2.2 and is similar to the preceding one. □

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author read and approved the final manuscript.

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