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Research article

Existence and nonexistence of global solutions to the Cauchy problem of the nonlinear hyperbolic equation with damping term

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Abstract: This paper concerns with the Cauchy problem for two classes of nonlinear hyperbolic equations with double damping terms. Firstly, by virtue of the Fourier transform method, we prove that the Cauchy problem of a class of high order nonlinear hyperbolic equation admits a global smooth solution $u(x,t) \in C^{\infty}((0,T];H^{\infty}(\mathbb{R})) \bigcap C([0,T];H^3(\mathbb{R})) \bigcap C^1([0,T];H^{-1}(\mathbb{R}))$ as long as initial value $u_0 \in W^{4,1}(\mathbb{R}) \bigcap H^3(\mathbb{R}), u_1 \in L^1(\mathbb{R}) \bigcap H^{-1}(\mathbb{R})$. Moreover, we give the sufficient conditions on the blow-up of the solution of a nonlinear damped hyperbolic equation with the initial value conditions in finite time and an example.

Keywords: nonlinear damped hyperbolic equation; Cauchy problem; Fourier transform; global smooth solution; blow-up
Mathematics Subject Classification: 35K70, 35K61, 35L30, 35L75

1. Introduction

In 1997, Banks et al. [1] established a class of nonlinear damped hyperbolic equation

$$w_{tt} + A_1 w + A_2 w_t + N^* g(Nw) = f(t), \tag{1.1}$$
$$w(0) = \varphi_0, w_t(0) = \varphi_1. \tag{1.2}$$

As a model it describes the motion of a neo-Hooken elastomer rod with internal damping, where $A_2 w_t$ is the exact form of the internal dynamic damping mechanisms in elastomers, A_1, A_2, N and f satisfy certain assumptions.

When $A_1 = A_2 = -\frac{\partial^2}{\partial x^2}, N = -\frac{\partial}{\partial x}$, Equation (1.1) becomes

$$u_{tt} - u_{xx} - u_{xxt} = g(u_x)_x + f. \tag{1.3}$$

The model is well known and it is been described the dynamical longitudinal vibrations of an neo-Hooken material rod, and there have been many researches on the global existence and blow-up of solutions for Equation (1.3) (see [2, 3]).

When $A_1 = \frac{\partial^4}{\partial x^4}, A_2 = -\frac{\partial^2}{\partial x^2}, N = -\frac{\partial}{\partial x}, f = 0$, Equation (1.1) becomes

$$u_{tt} - u_{xxt} + u_{xxxx} = g(u_x)_x. \tag{1.4}$$

Equation (1.4) describes the propagation of the wave in the medium with the dispersion effect, and it is connected with the equations in [4]–[10]. In [11], Yang et al. proved the well-posedness of Cauchy problem for the nonlinear beam system (1.4). When $g(s) = s^n, n \geq 2$, they proved the global existence of smooth solutions as long as initial data $\varphi \in L^2(\mathbb{R}) \bigcap H^2(\mathbb{R}), \psi \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$. In [12], Chen et al. considered Equation (1.4) with different initial boundary conditions. They proved the existence and uniqueness of the global generalized solutions and the global classical solutions with Galerkin method.

When $A_1 = A_2 = \frac{\partial^4}{\partial x^4}, N = -\frac{\partial^2}{\partial x^2}$, Equation (1.1) becomes

$$u_{tt} + u_{xxxx} + u_{xxxxt} = g(u_{xx})_{xx} + f. \tag{1.5}$$

Equation (1.5) models the vibration of a nonlinear damped beam with fixed boundary, taking account of the internal material damping. Banks et al. [13] established the existence and uniqueness of the global weak solutions to the initial boundary value problem of Equation (1.5). Later, Ackleh et al. [14] studied such system (1.5) to find the existence of weak solutions of the mixed problem in a bounded domain. In [15], Chen et al. gived the sufficient conditions of blow-up result for a nonlinear damped hyperbolic equation. Further generalizations are also given in [16]–[19] and the references therein.

When $A_1 = \frac{\partial^4}{\partial x^4}, A_2 = -\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}, N = -\frac{\partial^2}{\partial x^2}, f = 0$, Equation (1.1) becomes

$$u_{tt} - u_{xxt} + u_{xxxx} + u_{xxxxt} = g(u_{xx})_{xx}, \quad x \in \Omega, \ t > 0. \tag{1.6}$$

Recently, Yu et al. [20] established the existence and nonexistence of the global weak solutions to the initial boundary value problem of a nonlinear beam equation with double damping terms (1.6) provded that

$$-C_2|\xi|^2 - C_3 \leq G(\xi) = \int_0^{\xi} g(\tau)d\tau \leq \frac{1}{2}(1-\varepsilon)|\xi|^2 + C_1, 0 < \varepsilon < 1 \tag{1.7}$$

$$|g(\xi)| \leq \widetilde{C}_1|\xi| + \widetilde{C}_2, g'(\xi) \geq 0, \forall \xi \in \mathbb{R}. \tag{1.8}$$

Obviously, conditions (1.7) and (1.8) imply that the growth order of the nonlinear term $g(s)$ is not more than 1. The reason for the strong assumptions (1.7) and (1.8) lie in that it is very difficult to dominate the effect of the nonlinear term $g(u_{xx})_{xx}$ by standard a priori estimate technology.

When $A_1 = \frac{\partial^4}{\partial x^4}, A_2 = -\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}, N = -\frac{\partial}{\partial x}, f = 0$, Equation (1.1) becomes

$$u_{tt} - k_1 u_{xxt} + u_{xxxx} + u_{xxxxt} = g(u_x)_x, \quad x \in \mathbb{R}, \ t > 0, \tag{1.9}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \mathbb{R}, \tag{1.10}$$

where $u(x,t)$ denotes the unknown function, k_1 is a positive constant. $g(s)$ is a given nonlinear function, u_{xxxxt} denotes the strong material damping and u_{xxt} represents the internal dynamic damping.

As far as we know, there is little research on analysis of Equation (1.9) with material damping and internal dynamic damping at the same time. In this case, what happens to the existence and nonexistence of global solution to the problem (1.9)-(1.10) remain open.

When $A_1 = \frac{\partial^4}{\partial x^4} + \frac{\partial^8}{\partial x^8}, A_2 = -\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}, N = -\frac{\partial}{\partial x}, f = 0$. If $g(s) = s^n$, where $n \geq 2$ is an integer, Equation (1.1) becomes

$$u_{tt} - k_1 u_{xxt} + u_{xxxx} + u_{xxxxt} + a^2 u_{x^8} = (u_x^n)_x, \quad x \in \mathbb{R}, \ t > 0, \tag{1.11}$$

where $a^2 u_{x^8}$ is a "good" regular term. Obviously, $g(s)$ is monotone if and only if n is an odd number. For this kind of g, does problem (1.11), (1.10) have any global solutions for initial data belonging to suitably chosen functional spaces? On the other hand, if $g(s)$ is not monotone and $g'(s)$ is not bounded below, say $g(s) = s^{2m}$ (where $m \geq 1$ is an integer), does the initial value problem (1.11), (1.10) have any global solutions? The question is interesting and open.

This paper is organized as follows. In Section 2, the main results are stated. In Section 3, we prove the existence of global smooth solutions for the Cauchy problem (1.11), (1.10) by the Fourier transform method. In Section 4, using the modified concavity method, the sufficient conditions of blow-up of the solution for the Cauchy problem (1.9)–(1.10) will be given and we give an example to examine Theorem 2.3.

2. Statement of main results

For problem (1.9)–(1.10), we have the following theorem.

Theorem 2.1. *Suppose that $g \in C^2(\mathbb{R})$, $|g(s)| \leq K_1|s|^q$, $|g'(s)| \leq K_2|s|^{q-1}$ etc., where $q \geq 2$ is natural number and K_1, K_2 are positive constants. If $u_0 \in H^6(\mathbb{R})$ and $u_1 \in H^4(\mathbb{R})$, then there is a $T_1 > 0$ and the Cauchy problem* (1.9)–(1.10) *admits a local generalized solution $u(x,t)$ in $[0, T_1] \times \mathbb{R}$.*

This theorem can be proven by the method in [17], we can prove that the periodic boundary value problem admits a local generalized solution by Galerkin method and the compactness theorem. Then using the sequence of the periodic boundary value problem, we can obtain that the Cauchy problem (1.9)-(1.10) has a local generalized solution. Now, we state our main results as follows

Theorem 2.2. *Suppose that $u_0 \in W^{4,1}(\mathbb{R}) \bigcap H^3(\mathbb{R}), u_1 \in L^1(\mathbb{R}) \bigcap H^{-1}(\mathbb{R})$. If $a \neq 0$, then for any $T > 0$, Cauchy problem* (1.11), (1.10) *admits a global smooth solution $u(x,t) \in C^\infty((0,T]; H^\infty(\mathbb{R})) \bigcap C([0,T]; H^3(\mathbb{R})) \bigcap C^1([0,T]; H^{-1}(\mathbb{R}))$.*

Remark 2.1. *Theorem 2.2 shows that when excitations occur in the subspace $(W^{4,1}(\mathbb{R}) \bigcap H^3(\mathbb{R})) \times (L^1(\mathbb{R}) \bigcap H^{-1}(\mathbb{R})) \subset H^3(\mathbb{R}) \times H^{-1}(\mathbb{R})$, the orbits of the related dynamical system globally exist in the phase space $H^3(\mathbb{R}) \times H^{-1}(\mathbb{R})$.*

Theorem 2.3. *Suppose that $u_0 \in H^2(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$, $G(u_{0x}) \in L^1(\mathbb{R})$ and there exists a constant $\gamma > 0$ such that*

$$sg(s) \leq 2(1 + 2\gamma)G(s), \ s \in \mathbb{R}, \tag{2.1}$$

where $G(s) = \int_0^s g(\tau)d\tau$.

Then the solution $u(x,t)$ of the Cauchy problem (1.9)–(1.10) *blows up in finite time if one of the following conditions holds true:*

(i) $E(0) < 0$;
(ii) $E(0) = 0,\ 2(u_0, u_1) > \frac{1}{\gamma}(\|u_{0xx}\|^2 + k_1\|u_{0x}\|^2)$;
(iii) $E(0) > 0,\ (u_0, u_1) > \left[E(0)(\|u_0\|^2 + T_0k_1\|u_{0x}\|^2 + T_0\|u_{0xx}\|^2)\right]^{\frac{1}{2}}$,
where $E(0) = \|u_1\|^2 + \|u_{0xx}\|^2 + 2\int_{\mathbb{R}} G(u_{0x})dx$.

3. Existence of the global smooth solutions for the Cauchy problem (1.11), (1.10)

In this section, we establish the existence result for the Cauchy problem (1.11), (1.10) under initial value $u_0 \in W^{4,1}(\mathbb{R}) \bigcap H^3(\mathbb{R}), u_1 \in L^1(\mathbb{R}) \bigcap H^{-1}(\mathbb{R})$.

We use the following abbreviations: $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\mathbb{R})}(1 \le p < +\infty)$ denotes usual L^p norm, $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}$ denotes the L^2-inner product, and equip the Sobolev space $H^l(\mathbb{R})$ with the norm

$$\|f\|_{H^l} = (\int_{\mathbb{R}} (1 + |\xi|^2)^l |\hat{f}(\xi)|^2 d\xi)^{\frac{1}{2}},$$

for each real number l, where $\hat{f}(\xi) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-i\xi x}dx$. And we equip the space

$$W^{4,1}(\mathbb{R}) = \left\{f \in L^1(\mathbb{R}) | \frac{\partial^i f}{\partial x^i} \in L^1(\mathbb{R}), i = 1, \ldots, 4\right\}$$

with the norm

$$\|f\|_{W^{4,1}} = \sum_{i=0}^{4} \left\|\frac{\partial^i f}{\partial x^i}\right\|_{L^1}.$$

Taking the sequences $\{u_0^{(j)}\}_{j=1}^{\infty}, \{u_1^{(j)}\}_{j=1}^{\infty}$ in $C_0^{\infty}(\mathbb{R})$ such that

$$\|u_0^{(j)} - u_0\|_{W^{4,1}} + \|u_0^{(j)} - u_0\|_{H^3} \to 0,\ \|u_1^{(j)} - u_1\|_{L^1} \to 0\ \ as\ j \to \infty. \tag{3.1}$$

Using the properties of Fourier transform derivative, we obtain

$$(1 + |\xi|^4)|(\hat{u}_0^{(j)} - \hat{u}_0)(\xi)| \le C\|u_0^{(j)} - u_0\|_{W^{4,1}} \to 0\ \ as\ j \to \infty, \tag{3.2}$$

$$|(\hat{u}_1^{(j)} - \hat{u}_1)(\xi)| \le C\|u_1^{(j)} - u_1\|_{L^1} \to 0\ \ as\ j \to \infty, \tag{3.3}$$

uniformly on $\xi \in \mathbb{R}$, where C denotes positive constant. Hence

$$(1 + |\xi|^4)|\hat{u}_0^{(j)}(\xi)| \le C_0,\ |\hat{u}_1^{(j)}(\xi)| \le C_1, \xi \in \mathbb{R}, \tag{3.4}$$

where and in the sequel $C_i (i = 0, 1, \cdots)$ denotes positive constants independent of j and ξ.

Let $u^{(0)}(x,t) \in C^{\infty}((0,T]; C^{\infty}(\mathbb{R}))$ be the solution of the following linear problem

$$Lu^{(0)} \equiv u_{tt}^{(0)} - k_1 u_{xxt}^{(0)} + u_{xxxx}^{(0)} + u_{xxxxt}^{(0)} + a^2 u_{x^8}^{(0)} + b^2 u^{(0)} = 0,\ \ x \in \mathbb{R},\ t > 0, \tag{3.5}$$

$$u^{(0)}(x,0) = u_0^{(0)}(x), u_t^{(0)}(x,0) = u_1^{(0)}(x),\ \ x \in \mathbb{R}, \tag{3.6}$$

where $u_0^{(0)}(x), u_1^{(0)}(x) \in C(\mathbb{R})$, $u^{(j)}(x,t) (j = 1, 2, \ldots)$ be the solution of the following nonlinear problem

$$Lu^{(j)} = [(u_x^{(j-1)})^n]_x + b^2u^{(j-1)}, \quad x \in \mathbb{R},\ t > 0, \tag{3.7}$$

$$u^{(j)}(x,0) = u_0^{(j)}(x), u_t^{(j)}(x,0) = u_1^{(j)}(x), \quad x \in \mathbb{R}, \tag{3.8}$$

where $a^2(4 - k_1^2) = 1$ and $k_1 < 2, u_{x^m} = \frac{\partial^m u}{\partial x^m}$.

Taking the Fourier transform of (3.7)-(3.8), it follows that

$$\begin{aligned} &\hat{u}_{tt}^{(j)}(\xi,t) + (k_1\xi^2 + \xi^4)\hat{u}_t^{(j)}(\xi,t) + (a^2\xi^8 + \xi^4 + b^2)\hat{u}^{(j)}(\xi,t) \\ &\quad = i\xi\mathcal{F}[(u_x^{(j-1)})^n](\xi,t) + b^2\hat{u}^{(j-1)}(\xi,t), \\ &\hat{u}^{(j)}(\xi,0) = \hat{u}_0^{(j)}(\xi), \hat{u}_t^{(j)}(\xi,0) = \hat{u}_1^{(j)}(\xi), \quad x \in \mathbb{R}. \end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned} \hat{u}^{(j)}(\xi,t) =& e^{-\frac{1}{2}(k_1\xi^2+\xi^4)t}\Big[(cos\omega(\xi)t + \frac{1}{2}(k_1\xi^2 + \xi^4)\omega^{-1}(\xi)sin\omega(\xi)t)\hat{u}_0^{(j)}(\xi) \\ &+ \omega^{-1}(\xi)sin\omega(\xi)t \cdot \hat{u}_1^{(j)}(\xi)\Big] + \alpha(\xi,t) + \beta(\xi,t), \end{aligned} \tag{3.10}$$

where $\omega(\xi) = \Big[(4a^2 - 1)\xi^8 - 2k_1\xi^6 + (4 - k_1^2)\xi^4 + 4b^2\Big]^{1/2}$, and

$$\alpha(\xi,t) = i\xi\omega^{-1}(\xi)\int_0^t e^{-\frac{1}{2}(k_1\xi^2+\xi^4)(t-\tau)}\mathcal{F}[(u_x^{(j-1)})^n](\xi,\tau)sin\omega(\xi)(t-\tau)d\tau, \tag{3.11}$$

$$\beta(\xi,t) = b^2\omega^{-1}(\xi)\int_0^t \hat{u}^{(j-1)}(\xi,\tau)e^{-\frac{1}{2}(k_1\xi^2+\xi^4)(t-\tau)}sin\omega(\xi)(t-\tau)d\tau. \tag{3.12}$$

In order to prove Theorem 2.2, we can prove the following lemma.

Lemma 3.1. *For any $T > 0$,*

$$|\hat{u}^{(j)}(\xi,t)| \le CH^{(j)}(\xi)e^{-\frac{k_1}{2}|\xi|t}, 0 \le t \le T, j = 0, 1, \ldots, \tag{3.13}$$

where $H^{(j)}(\xi) = |\hat{u}_0^{(j)}(\xi)| + (1 + \xi^2)^{-2}|\hat{u}_1^{(j)}(\xi)| + (1 + |\xi|^4)^{-1}$, C denotes positive constant depending only on T.

Proof. From (3.4), it follows that

$$(1 + \xi^4)^{-1} \le H^{(j)}(\xi) \le C_2(1 + \xi^4)^{-1}, \xi \in \mathbb{R}. \tag{3.14}$$

Now, we prove (3.13) by induction method on j. When $j = 0, \alpha(\xi,t) = \beta(\xi,t) = 0$, and

$$e^{-\frac{1}{2}(k_1\xi^2+\xi^4)t} \le e^{-\frac{1}{2}(k_1|\xi|t-c)} \le Ce^{-\frac{k_1}{2}|\xi|t}, \tag{3.15}$$

$$\frac{1}{2}(k_1\xi^2 + \xi^4)\omega^{-1}(\xi) \le C, C_3(1 + \xi^4) \le \omega(\xi) \le C_4(1 + \xi^4). \tag{3.16}$$

From (3.14)–(3.16), it follows that (3.13) is valid.

Assume that (3.13) is valid for $0 < j \le j_1$. In order to prove that (3.13) holds true for $j = j_1 + 1$, we first prove

$$|\mathcal{F}[(u_x^{(j-1)})^n](\xi,t)| \le Ce^{-\frac{k_1}{2}|\xi|t}, t \in [0,T]. \tag{3.17}$$

In fact, when $n = 2$, noting that $|\xi - \eta| + |\eta| \geq |\xi|$, we have

$$\|(u_x^{(j-1)})(\cdot, t)\| = \|(\hat{u}_x^{(j-1)})(\cdot, t)\| \leq C\|\xi H^{(j-1)}(\xi)\| \leq C\Big(\int_{\mathbb{R}} \frac{\xi^2}{(1+|\xi|^4)^2} d\xi\Big)^{\frac{1}{2}} \leq C, \tag{3.18}$$

together with (3.13)($j \leq j_1$) and Hölder inequality, implies that

$$\begin{aligned} &|\mathcal{F}[(u_x^{(j-1)})^2](\xi, t)| \\ =& \frac{1}{\sqrt{2\pi}}\Big|(\hat{u}_x^{(j-1)} * \hat{u}_x^{(j-1)})(\xi, t)\Big| \\ \leq& Ce^{-\frac{k_1}{2}|\xi|t}\Big\|(\xi-\eta)H^{(j-1)}(\xi-\eta)\Big\|\Big\|\eta H^{(j-1)}(\eta)\Big\| \\ \leq& Ce^{-\frac{k_1}{2}|\xi|t}, t \in [0, T], \end{aligned} \tag{3.19}$$

where and in the sequel $*$ denotes convolution. Assume that (3.17) holds true for $n \leq n_1$, which implies $\Big\|(u_x^{(j-1)})^n(\cdot, t)\Big\| = \Big\|\mathcal{F}[(u_x^{(j-1)})^n](\cdot, t)\Big\| < \infty, t \in [0, T]$. When $n = n_1 + 1$, noting that

$$\int_{\mathbb{R}} \Big|\eta H^{(j)}(\eta)\Big| d\eta \leq C\int_{\mathbb{R}} \frac{|\eta|}{1+\eta^4} d\eta \leq C, \tag{3.20}$$

from (3.13)($j \leq j_1$), (3.17)($n \leq n_1$) and (3.20), we obtain

$$\begin{aligned} &|\mathcal{F}[(u_x^{(j-1)})^n](\xi, t)| \\ =& \frac{1}{\sqrt{2\pi}}\Big|(\hat{u}_x^{(j-1)} * \mathcal{F}[(u_x^{(j-1)})^{n-1}](\xi, t)\Big| \\ \leq& Ce^{-\frac{k_1}{2}\big|\xi|t}\int_{\mathbb{R}} \Big|\eta H^{(j-1)}(\eta)|d\eta \\ \leq& Ce^{-\frac{k_1}{2}|\xi|t}, t \in [0, T]. \end{aligned} \tag{3.21}$$

Therefore, (3.17) holds true.

By (3.11), (3.12), (3.13)($j \leq j_1$), (3.14) and (3.17), we see that

$$\begin{aligned} |\alpha(\xi, t)| &\leq |\xi|\omega^{-1}(\xi)\int_0^t e^{-\frac{1}{2}(k_1\xi^2+\xi^4)(t-\tau)}\Big|\mathcal{F}[(u_x^{(j-1)})^n](\xi, \tau)\Big| d\tau \\ &\leq C|\xi|\omega^{-1}(\xi)J(\xi, t). \end{aligned} \tag{3.22}$$

$$|\beta(\xi, t)| \leq b^2\omega^{-1}(\xi)(1+\xi^4)^{-1}J(\xi, t), \tag{3.23}$$

where

$$J(\xi, t) = \int_0^t e^{-\frac{1}{2}\xi^4(t-\tau)-\frac{k_1}{2}|\xi|\tau} d\tau \leq \begin{cases} \frac{2\left(e^{-\frac{k_1}{2}|\xi|t}-e^{-\frac{1}{2}\xi^4 t}\right)}{\xi^4-k_1|\xi|}, & |\xi| \geq \xi_0 \\ t, & |\xi| < \xi_0 \end{cases}$$

Taking ξ_0 large enough such that

$$e^{\frac{k_1}{2}|\xi|t}(1+\xi^4)|\alpha(\xi, t)| \quad \leq \begin{cases} \frac{2}{k_1(|\xi|^3-1)} \leq C, & |\xi| \geq \xi_0 \\ e^{\frac{k_1}{2}\xi_0 T}\xi_0 T \leq C, & |\xi| < \xi_0 \end{cases}$$

$$e^{\frac{k_1}{2}|\xi|t}(1+\xi^4)|\beta(\xi,t)| \quad \le \begin{cases} \frac{2}{(1+\xi^4)(\xi^4-k_1|\xi|)} \le C, & |\xi| \ge \xi_0 \\ e^{\frac{k_1}{2}\xi_0 T}(1+\xi_0^4)^{-1}T \le C, & |\xi| < \xi_0 \end{cases}$$

Thus, we obtain

$$|\alpha(\xi,t)| \le Ce^{-\frac{k_1}{2}|\xi|t}H^{(j)}(\xi), t \in [0,T], \tag{3.24}$$

$$|\beta(\xi,t)| \le Ce^{-\frac{k_1}{2}|\xi|t}H^{(j)}(\xi), t \in [0,T]. \tag{3.25}$$

Applying (3.16), (3.24)–(3.25) to estimate (3.10) yields (3.13). This completes the proof.
Differentiating (3.10) with respect to t, it follows that

$$\begin{aligned} \frac{\partial^m}{\partial t^m}\hat{u}^{(j)}(\xi,t) =& \Big[\frac{1}{2}(k_1\xi^2+\xi^4)\omega^{-1}(\xi)h_m(\xi,t) + g_m(\xi,t)\Big]\hat{u}_0^{(j)}(\xi) \\ &+ \omega^{-1}(\xi)h_m(\xi,t)\hat{u}_1^{(j)}(\xi) \\ &+ i\xi\omega^{-1}(\xi)\int_0^t h_m(\xi,t-\tau)\mathcal{F}[(u_x^{(j-1)})^n](\xi,\tau)d\tau \\ &+ b^2\omega^{-1}(\xi)\int_0^t h_m(\xi,t-\tau)\hat{u}^{(j-1)}(\xi,\tau)d\tau + R_m(\xi,t), \end{aligned} \tag{3.26}$$

where

$$h_m(\xi,t) = \tfrac{\partial^m}{\partial t^m}\Big(e^{-\frac{1}{2}(k_1\xi^2+\xi^4)t}sin\omega(\xi)t\Big) = \sum_{l=1}^{m} C_m^l(-1)^l\Big[\tfrac{1}{2}(k_1\xi^2+\xi^4)\Big]^l e^{-\frac{1}{2}(k_1\xi^2+\xi^4)t}\cdot$$
$$\omega^{m-l}(\xi)sin\Big[\omega(\xi)\Big(t-\tfrac{(m-l)\pi}{2}\Big)\Big],$$
$$g_m(\xi,t) = \tfrac{\partial^m}{\partial t^m}\Big(e^{-\frac{1}{2}(k_1\xi^2+\xi^4)t}cos\omega(\xi)t\Big),$$
$$R_1(\xi,t)=0, R_2(\xi,t) = \omega^{-l}(\xi)h_1(\xi,0)\Big(i\xi\mathcal{F}[(u_x^{(j-1)})^n](\xi,t) + b^2\hat{u}^{(j-1)}(\xi,t)\Big),$$
$$\vdots$$
$$R_m(\xi,t) = \omega^{-l}(\xi)h_{m-1}(\xi,0)\Big(i\xi\mathcal{F}[(u_x^{(j-1)})^n](\xi,t) + b^2\hat{u}^{(j-1)}(\xi,t) + \sum_{l=1}^{m-2}\tfrac{\partial^l}{\partial t^l}R_{m-l}(\xi,t)\Big).$$

Lemma 3.2. *For any $\delta: 0<\delta<T$,*

$$\Big|\frac{\partial^m}{\partial t^m}\hat{u}^{(j)}(\xi,t)\Big| \le A(1+\xi^4)^m H^{(j)}(\xi)e^{-\frac{k_1}{2}|\xi|t}, t\in[\delta,T], m=0,1,\dots, \tag{3.27}$$

where and in the sequel A denotes positive constants depending only on δ, m and T.
Proof. when $m=0$, (3.13) implies (3.27). Assume that (3.27) holds true for $m \le m_1$. In the following we prove that (3.27) holds true for $m = m_1+1$. Now, we need the facts

$$\Big|\frac{\partial^l}{\partial t^l}\mathcal{F}[(u_x^{(j-1)})^n](\xi,t)\Big| \le A(1+\xi^4)^l e^{-\frac{k_1}{2}|\xi|t}, t\in[\delta,T], l=0,1,\dots,m-2. \tag{3.28}$$

In fact, when $n = 2$, by (3.27)($m \leq m_1$) we obtain

$$\begin{aligned}\Big|\frac{\partial^l}{\partial t^l}\mathcal{F}[(u_x^{(j-1)})^2](\xi,t)\Big| &\leq \sum_{s=0}^{l} C_l^s \int_{\mathbb{R}} \Big||\xi-\eta|\frac{\partial^s}{\partial t^s}\hat{u}^{(j-1)}(\xi-\eta,t)\cdot|\eta|\frac{\partial^{l-s}}{\partial t^{l-s}}\hat{u}^{(j-1)}(\eta,t)\Big|d\eta \\ &\leq \sum_{s=0}^{l} C_l^s A \int_{\mathbb{R}} |\xi-\eta|(1+(\xi-\eta)^4)^s e^{-\frac{k_1}{2}|\xi-\eta|t} H^{(j-1)}(\xi-\eta) \\ &\qquad \cdot |\eta|(1+\eta^4)^{l-s} e^{-\frac{k_1}{2}|\eta|t} H^{(j-1)}(\eta)d\eta.\end{aligned} \tag{3.29}$$

Noting that

(i) When $|\eta| \leq 2|\xi|$, we have $|\xi-\eta| \leq 3|\xi|$, and

$$(1+(\xi-\eta)^4)^s(1+\eta^4)^{l-s} \leq C(1+\xi^4)^l.$$

(ii) When $|\eta| > 2|\xi|$, we have $|\xi-\eta| > |\xi|$, and

$$\begin{aligned}&e^{-\frac{k_1}{2}|\eta|t}(1+(\xi-\eta)^4)^s(1+\eta^4)^{l-s} \\ \leq\ & Ce^{-\frac{k_1}{2}|\eta|t}(1+\eta^4)^l(1+\xi^4)^s \\ \leq\ & Ct^{-4l}(1+\xi^4)^s \\ \leq\ & A(1+\xi^4)^s,\ \ t\in[\delta,T].\end{aligned}$$

Substituting above inequalities into (3.29) and using Hölder inequality, we obtain

$$\begin{aligned}&\Big|\frac{\partial^l}{\partial t^l}\mathcal{F}[(u_x^{(j-1)})^2](\xi,t)\Big| \\ \leq\ & \sum_{s=0}^{l} C_l^s A\Big(\int_{-2|\xi|}^{2|\xi|} + \int_{\mathbb{R}-[-2|\xi|,2|\xi|]}\Big)|\xi-\eta|(1+(\xi-\eta)^4)^s e^{-\frac{k_1}{2}|\xi-\eta|t} \\ & H^{(j-1)}(\xi-\eta)\cdot|\eta|(1+\eta^4)^{l-s}e^{-\frac{k_1}{2}|\eta|t}H^{(j-1)}(\eta)d\eta \\ \leq\ & \sum_{s=0}^{l} C_l^s A(1+\xi^4)^l e^{-\frac{k_1}{2}|\xi|t}\int_{\mathbb{R}}|\xi-\eta|H^{(j-1)}(\xi-\eta)\cdot|\eta|H^{(j-1)}(\eta)d\eta \\ \leq\ & \sum_{s=0}^{l} C_l^s A(1+\xi^4)^l e^{-\frac{k_1}{2}|\xi|t}\Big\||\xi-\eta|H^{(j-1)}(\xi-\eta)\Big\|\Big\||\eta|H^{(j-1)}(\eta)\Big\| \\ \leq\ & A(1+\xi^4)^l e^{-\frac{k_1}{2}|\xi|t}, t\in[\delta,T], l=0,1,\ldots,m-2.\end{aligned} \tag{3.30}$$

Assume that (3.28) holds true for $n \leq n_1$, when $n = n_1+1$, from (3.28)($n \leq n_1$), (3.27)($m \leq m_1$), it follows that

$$
\begin{aligned}
&\Big|\frac{\partial^l}{\partial t^l}\mathcal{F}[(u_x^{(j-1)})^n](\xi,t)\Big| \\
&= \Big|\frac{\partial^l}{\partial t^l}\mathcal{F}[(u_x^{(j-1)})\cdot(u_x^{(j-1)})^{n-1}](\xi,t)\Big| \\
&= \frac{1}{\sqrt{2\pi}}\Big|\frac{\partial^l}{\partial t^l} i\xi\hat{u}^{(j-1)} * \mathcal{F}[(u_x^{(j-1)})^{n-1}](\xi,t)\Big| \\
&\leq \Big|\frac{\partial^l}{\partial t^l}\int_{\mathbb{R}}(\xi-\eta)\hat{u}^{(j-1)}(\xi-\eta,t)\cdot\mathcal{F}[(u_x^{(j-1)})^{n-1}](\eta,t)d\eta\Big| \\
&\leq \sum_{s=0}^{l} C_l^s\int_{\mathbb{R}}\Big|(\xi-\eta)\frac{\partial^s}{\partial t^s}\hat{u}^{(j-1)}(\xi-\eta,t)\cdot\frac{\partial^{l-s}}{\partial t^{l-s}}\mathcal{F}[(u_x^{(j-1)})^{n-1}](\eta,t)\Big|d\eta \\
&\leq \sum_{s=0}^{l} C_l^s A\int_{\mathbb{R}}|\xi-\eta|(1+(\xi-\eta)^4)^s e^{-\frac{k_1}{2}|\xi-\eta|t}H^{(j-1)}(\xi-\eta) \\
&\qquad\cdot(1+\eta^4)^{l-s}e^{-\frac{k_1}{2}|\eta|t}d\eta \\
&\leq A(1+\xi^4)^l e^{-\frac{k_1}{2}|\xi|t},\ t\in[\delta,T],\ l=0,1,\ldots,m-2.
\end{aligned} \tag{3.31}
$$

Therefore, (3.28) holds true.

From (3.27)($m \leq m_1$) and (3.28), it follows that

$$
\begin{aligned}
&|h_i(\xi,t)|+|g_i(\xi,t)| \\
&\leq C(1+\xi^4)^i e^{-\frac{1}{2}(k_1\xi^2+\xi^4)t} \\
&\leq C(1+\xi^4)^i e^{-\frac{k_1}{2}|\xi|t},\ t\in[0,T], i=0,1,\ldots,m-2.
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
&\Big|\frac{\partial^{m-2}}{\partial t^{m-2}}R_2(\xi,t)\Big| \\
&\leq C(1+\xi^4)\omega^{-1}(\xi)\Big(\Big|\xi\frac{\partial^{m-2}}{\partial t^{m-2}}\mathcal{F}[(u_x^{(j-1)})^n](\xi,t)\Big|+\Big|\frac{\partial^{m-2}}{\partial t^{m-2}}\hat{u}^{(j-1)}(\xi,t)\Big|\Big) \\
&\leq Ae^{-\frac{k_1}{2}|\xi|t}(1+\xi^4)^{m-1}\omega^{-1}(\xi)(|\xi|+H^{(j-1)}(\xi)) \\
&\leq Ae^{-\frac{k_1}{2}|\xi|t}(1+\xi^4)^m H^{(j)}(\xi),\ t\in[\delta,T].
\end{aligned} \tag{3.33}
$$

Similarly,

$$
\Big|\frac{\partial^s}{\partial t^s}R_{m-s}(\xi,t)\Big| \leq Ae^{-\frac{k_1}{2}|\xi|t}(1+\xi^4)^m H^{(j)}(\xi),\ t\in[\delta,T],\ s=1,2,\ldots,m-3, \tag{3.34}
$$

$$
\begin{aligned}
|R_m(\xi,t)| &\leq C(1+\xi^4)^{m-1}\omega^{-1}(\xi)(|\xi\mathcal{F}[(u_x^{(j-1)})^n](\xi,t)|+\hat{u}^{(j-1)}(\xi,t)) \\
&\quad+\sum_{l=1}^{m-2}\Big|\frac{\partial^l}{\partial t^l}R_{m-l}(\xi,t)\Big| \\
&\leq Ce^{-\frac{k_1}{2}|\xi|t}(1+\xi^4)^{m-1}\omega^{-1}(\xi)(|\xi|+H^{(j-1)}(\xi)) \\
&\quad+Ce^{-\frac{k_1}{2}|\xi|t}(1+\xi^4)^m H^{(j)}(\xi) \\
&\leq Ae^{-\frac{k_1}{2}|\xi|t}(1+\xi^4)^m H^{(j)}(\xi), t\in[\delta,T].
\end{aligned} \tag{3.35}
$$

Therefore, (3.27) holds true.
Proof of Theorem 2.2.
For any $\delta : 0 < \delta < T$, we can extract a subsequence from $\{u^{(j)}\}$, still denoted by $\{u^{(j)}\}$, such that

$$u^{(j)} \to u \ in \ C^m([\delta, T]; H^l(\mathbb{R})) \ as \ j \to \infty, \tag{3.36}$$

where j and l are positive integers. Moreover, u satisfies Equation (1.11) for $t > 0$.

In fact, from Lemma 3.1 and Lemma 3.2, it deduce that

$$\begin{aligned} \left\|u^{(j)}(\cdot.t)\right\|_{H^l}^2 &= \int_{\mathbb{R}} (1+\xi^2)^l |\hat{u}^{(j)}(\xi,t)|^2 d\xi \\ &\leq C \int_{\mathbb{R}} (1+\xi^2)^l e^{-k_1|\xi|t} (H^{(j)}(\xi))^2 d\xi \\ &\leq C t^{-2l} \int_{\mathbb{R}} (H^{(j)}(\xi))^2 d\xi \leq M_1, \ t \in [\delta, T], \end{aligned} \tag{3.37}$$

$$\begin{aligned} \left\|\frac{\partial^m}{\partial t^m} u^{(j)}(\cdot.t)\right\|_{H^l}^2 &= \int_{\mathbb{R}} (1+\xi^2)^l \left|\frac{\partial^m}{\partial t^m} \hat{u}^{(j)}(\xi,t)\right|^2 d\xi \\ &\leq A \int_{\mathbb{R}} (1+\xi^2)^l (1+\xi^4)^{2m} e^{-k_1|\xi|t} (H^{(j)}(\xi))^2 d\xi \\ &\leq A t^{-2(l+4m)} \int_{\mathbb{R}} (H^{(j)}(\xi))^2 d\xi \leq M_2, \ t \in [\delta, T], \end{aligned} \tag{3.38}$$

where M_1 and M_2 are positive constants depending on δ, T, l and δ, T, l, m, respectively. Using Arzelà theorem, from (3.37)–(3.38) we arrive at (3.36) holds true.

Letting $j \to \infty$ in Equation (3.7), by the arbitrariness of δ, we obtain that u satisfies Equation (1.11) for $t > 0$.

Now we will prove

$$u^{(j)} \to u \ in \ C([0, T]; H^3(\mathbb{R})) \cap C^1([0, T]; H^{-1}(\mathbb{R})) \ as \ j \to \infty, \tag{3.39}$$

namely,

$$u(x, 0) = u_0(x) \ in \ H^3(\mathbb{R}), u_t(x, 0) = u_1(x) \ in \ H^{-1}(\mathbb{R}).$$

In fact, we extract two subsequences from $\{u_0^{(j)}\}$ and $\{u_1^{(j)}\}$, respectively, we can still denote $\{u_0^{(j)}\}$ and $\{u_1^{(j)}\}$, from (3.1), it follows that

$$\left\|u_0^{(j)} - u_0^{(j-1)}\right\|_{W^{4,1}} + \left\|u_0^{(j)} - u_0^{(j-1)}\right\|_{H^3} \leq 2^{-j}, \|u_1^{(j)} - u_1^{(j-1)}\|_{L^1} \leq 2^{-j}. \tag{3.40}$$

Furthermore, we obtain

$$\sup_{0 \leq t \leq T} \left|\hat{u}^{(j)}(\xi, t) - \hat{u}^{(j-1)}(\xi, t)\right| \leq C[2^{1-j}(H^{(j)}(\xi) + H^{(j-1)}(\xi)) + G^{(j)}(\xi)], \tag{3.41}$$

where

$$\begin{aligned} G^{(j)}(\xi) &= \left|\hat{u}_0^{(j)}(\xi) - \hat{u}_0^{(j-1)}(\xi)\right| + (1+\xi^2)^{-2} \left|\hat{u}_1^{(j)}(\xi) - \hat{u}_1^{(j-1)}(\xi)\right| \\ &\leq \frac{2^{-j}}{1+\xi^4} + \frac{2^{-j}}{(1+\xi^2)^2} \\ &\leq \frac{2^{1-j}}{1+\xi^4} \leq 2^{1-j} H^{(j)}(\xi). \end{aligned} \tag{3.42}$$

Now, we will prove (3.41) holds true by induction method. Indeed, when $j = 1$, by (3.13) we see that

$$\sup_{0\le t\le T}\left|\hat{u}_0^{(1)}(\xi,t)-\hat{u}_0^{(0)}(\xi,t)\right|\le C(H^{(1)}(\xi)+H^{(0)}(\xi)), \tag{3.43}$$

that is , (3.41) holds true.

Assume that (3.41) holds true for $j \le j_1$. When $j = j_1 + 1$, we deduce from (3.10) that

$$\begin{aligned}&\left|\hat{u}^{(j)}(\xi,t)-\hat{u}^{(j-1)}(\xi,t)\right|\\ &\le Ce^{-\frac{1}{2}(k_1\xi^2+\xi^4)t}G^{(j)}(\xi)+|\xi|\omega^{-1}(\xi)\int_0^t e^{-\frac{1}{2}(k_1\xi^2+\xi^4)(t-\tau)}P_j(\xi,\tau)d\tau\\ &\quad+b^2\omega^{-1}(\xi)\int_0^t e^{-\frac{1}{2}(k_1\xi^2+\xi^4)(t-\tau)}\left|\hat{u}^{(j-1)}(\xi,t)-\hat{u}^{(j-2)}(\xi,\tau)\right|d\tau,\end{aligned} \tag{3.44}$$

where

$$P_j(\xi,t)=\left|\mathcal{F}[(u_x^{(j-1)})^n]-\mathcal{F}[(u_x^{(j-2)})^n]\right|. \tag{3.45}$$

We claim that

$$\sup_{0\le t\le T}P_j(\xi,t)\le 2^{1-j}C. \tag{3.46}$$

In fact, when $n = 2$, we have

$$\Big(\int_{\mathbb{R}}\eta^2|G^{(j-1)}(\eta)|^2d\eta\Big)^{\frac{1}{2}}\le 2^{1-j}\Big(\int_{\mathbb{R}}\frac{\eta^2}{(1+\eta^4)^2}d\eta\Big)^{\frac{1}{2}}\le 2^{1-j}C. \tag{3.47}$$

From (3.41)($j \le j_1$), (3.13) and Hölder inequality , it deduce that

$$\begin{aligned}&\sup_{0\le t\le T}P_j(\xi,t)\\ &=\sup_{0\le t\le T}\left|(\xi\hat{u}^{(j-1)}*\xi\hat{u}^{(j-1)}-\xi\hat{u}^{(j-2)}*\xi\hat{u}^{(j-2)})(\xi,t)\right|\\ &=\sup_{0\le t\le T}\left|\int_{\mathbb{R}}\eta(\hat{u}^{(j-1)}-\hat{u}^{(j-2)})(\eta,t)\cdot(\xi-\eta)(\hat{u}^{(j-1)}+\hat{u}^{(j-2)})(\xi-\eta,t)d\eta\right|\\ &\le\int_{\mathbb{R}}|\eta|[2^{2-j}(H^{(j-1)}+H^{(j-2)})(\eta)+G^{(j-1)}(\eta)]\cdot|\xi-\eta|(H^{(j-1)}+H^{(j-2)})(\xi-\eta)d\eta\\ &\le 2^{1-j}C.\end{aligned} \tag{3.48}$$

Assume that (3.46) holds true for $n \le n_1$. When $n = n_1 + 1$, we deduce from (3.41)($j \le j_1$), (3.13), (3.17) and (3.46)($n \le n_1$) that

$$\begin{aligned}
&\sup_{0\le t\le T} P_j(\xi,t)\\
=&\sup_{0\le t\le T}\left|(\xi\hat{u}^{(j-1)} * \mathcal{F}[(u_x^{(j-1)})^{n-1}] - \xi\hat{u}^{(j-2)} * \mathcal{F}[(u_x^{(j-2)})^{n-1}])(\xi,t)\right|\\
\le&\sup_{0\le t\le T}\Big|\xi(\hat{u}^{(j-1)} - \hat{u}^{(j-2)}) * \mathcal{F}[(u_x^{(j-1)})^{n-1}]\\
&\quad + \xi\hat{u}^{(j-2)} * (\mathcal{F}[(u_x^{(j-1)})^{n-1}] - \mathcal{F}[(u_x^{(j-2)})^{n-1}])(\xi,t)\Big|\\
\le& C\int_{\mathbb{R}}|\eta|\Big[2^{2-j}\Big(H^{(j-1)} + H^{(j-2)})(\eta)\Big)\\
&\quad + G^{(j-1)}(\eta)\Big]d\eta + 2^{1-j}C\int_{\mathbb{R}}|\xi-\eta|H^{(j-2)}(\xi-\eta)d\eta\\
\le& 2^{1-j}C,
\end{aligned} \tag{3.49}$$

that is, (3.46) holds true.

It follows from (3.41)($j \le j_1$), (3.44)-(3.46) that

$$\begin{aligned}
&\left|\hat{u}^{(j)}(\xi,t) - \hat{u}^{(j-1)}(\xi,t)\right|\\
\le& G^j(\xi) + C\omega^{-1}(\xi)[2^{1-j}|\xi| + b^2(2^{2-j}(H^{(j-1)}(\xi)\\
&\quad + H^{(j-2)}(\xi)) + G^{(j-1)}(\xi))]\cdot Q(\xi,t),
\end{aligned} \tag{3.50}$$

where

$$Q(\xi,t) = \int_0^t e^{-\frac{1}{2}\xi^4(t-\tau)}d\tau \le \begin{cases} \frac{2\left(1-e^{-\frac{1}{2}\xi^4 t}\right)}{\xi^4}, & |\xi| \ge \xi_0\\ t, & |\xi| < \xi_0\end{cases}$$

By (3.42)–(3.50), we have

$$|\xi|\omega^{-1}(\xi)Q(\xi,t) \le C(1+\xi^4)^{-1} \le CH^{(j)}(\xi), \tag{3.51}$$

$$b^2\omega^{-1}(\xi)H^{(j-1)}(\xi)Q(\xi,t) \le C(1+\xi^4)^{-1} \le CH^{(j)}(\xi), \tag{3.52}$$

$$\omega^{-1}(\xi)G^{(j-1)}(\xi)Q(\xi,t) \le 2^{1-j}H^{(j)}(\xi), t \in [0,T]. \tag{3.53}$$

Substituting (3.51)–(3.53) into (3.50) yields (3.41).

It follows from (3.41)–(3.42) and (3.14) that

$$\begin{aligned}
&\sup_{0\le t\le T}\left\|u^{(j)}(\cdot,t) - u^{(j-1)}(\cdot,t)\right\|_{H^3}\\
=&\sup_{0\le t\le T}\left\|(1+\xi^2)^{\frac{3}{2}}(\hat{u}^{(j)} - \hat{u}^{(j-1)})(\xi,t))\right\|_{L^2}\\
\le& 2^{1-j}C\left\|(1+\xi^2)^{\frac{3}{2}}(H^{(j)}(\xi) + H^{(j-1)}(\xi))\right\| + \left\|(1+\xi^2)^{\frac{3}{2}}G^{(j)}(\xi)\right\|\\
\le& 2^{1-j}C.
\end{aligned} \tag{3.54}$$

Hence, we can write $\{u^{(j)}\}$ is a Cauchy sequence in $C([0,T];H^3(\mathbb{R}))$ and this implies that

$$u^{(j)} \to u \ in \ C([0,T];H^3(\mathbb{R})) \ as \ j \to \infty.$$

Similarly, by (3.51)–(3.53), we obtain

$$\begin{aligned}
&\left|\hat{u}_t^{(j)}(\xi,t)-\hat{u}_t^{(j-1)}(\xi,t)\right| \\
&\le C(1+\xi^4)\Big[G^{(j)}(\xi)e^{-\frac{1}{2}(k_1\xi^2+\xi^4)t}+|\xi|\omega^{-1}(\xi)\int_0^t e^{-\frac{1}{2}(k_1\xi^2+\xi^4)(t-\tau)}P_j(\xi,\tau)d\tau \\
&\quad + b^2\omega^{-1}(\xi)\int_0^t e^{-\frac{1}{2}(k_1\xi^2+\xi^4)(t-\tau)}|\hat{u}^{(j-1)}(\xi,\tau)-\hat{u}^{(j-2)}(\xi,\tau)|d\tau\Big] \\
&\le C(1+\xi^4)\Big[G^{(j)}(\xi)+\omega^{-1}(\xi)(2^{1-j}|\xi| \\
&\quad + b^2(2^{2-j}(H^{(j-1)}(\xi)+H^{(j-2)}(\xi))+G^{(j-1)}(\xi)))Q(\xi,t)\Big] \\
&\le C(1+\xi^4)[G^{(j)}(\xi)+2^{1-j}(H^{(j)}(\xi)+H^{(j-1)}(\xi))],\, t\in[0,T].
\end{aligned} \tag{3.55}$$

Furthermore

$$\begin{aligned}
&\sup_{0\le t\le T}\left\|u_t^{(j)}(\cdot,t)-u_t^{(j-1)}(\cdot,t)\right\|_{H^{-1}} \\
&=\sup_{0\le t\le T}\left\|(1+\xi^2)^{-1/2}(\hat{u}_t^{(j)}-\hat{u}_t^{(j-1)})(\cdot,t)\right\| \\
&\le C\left\|(1+\xi^2)^{3/2}G^{(j)}(\xi)\right\|+2^{1-j}C\left\|(1+\xi^2)^{3/2}(H^{(j)}(\xi)+H^{(j-1)}(\xi))\right\| \\
&\le C\|(1+\xi^2)^{3/2}(1+\xi^4)^{-1}\|+2^{1-j}C\|(1+\xi^2)^{3/2}(1+\xi^4)^{-1}\| \\
&\le 2^{1-j}C.
\end{aligned} \tag{3.56}$$

Hence, we can write $\{u_t^{(j)}\}$ is a Cauchy sequence in $C([0,T];H^{-1}(\mathbb{R}))$ and this implies that

$$u_t^{(j)} \to u_t \textit{ in } C([0,T];H^{-1}(\mathbb{R})) \textit{ as } j\to\infty.$$

Thus, u is a solution of problem (1.11), (1.10). This completes the proof of Theorem 2.2.

4. Nonexistence of the solution for the Cauchy problem (1.9)–(1.10)

To obtain the blow-up result of the solution to the Cauchy problem (1.9)–(1.10), we introduce the following lemma.

Lemma 4.1. ([8])*Suppose that a positive, twice-differentiable function $\theta(t)$ satisfies the inequality*

$$\theta''(t)\theta(t)-(1+\gamma)\theta'^2(t)\ge -2C_1\theta'(t)\theta(t)-C_2\theta^2(t),\ t>0,$$

where $\gamma>0$ and $C_1,C_2\ge 0$ are constants.
(1) If $C_1=C_2=0,\theta(0)>0$ and $\theta'(0)>0$, then there exists $t_1\in(0,\frac{\theta(0)}{\gamma\theta'(0)}]$ such that $\theta(t)$ tends to infinity as $t\to t_1$.
(2) If $C_1+C_2>0,\theta(0)>0$ and $\theta'(0)>-\gamma_2\gamma^{-1}\theta(0)$, then there exists $t_1>0$ such that $\theta(t)$ tends to infinity as $t\to t_1$, where t_1 is bounded above by

$$\frac{1}{2\sqrt{C_1^2+\gamma C_2}}\ln\frac{\gamma_1\theta(0)+\gamma\theta'(0)}{\gamma_2\theta(0)+\gamma\theta'(0)}$$

with $\gamma_1=-C_1+\sqrt{C_1^2+\gamma C_2}$ and $\gamma_2=-C_1-\sqrt{C_1^2+\gamma C_2}$.
Proof of Theorem 2.3.

Suppose that the maximal time of existence of the solution of the Cauchy problem (1.9)–(1.10) is infinite. The energy functional for the problem (1.9) can be defined as

$$\begin{aligned} E(t) =&\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + 2k_1 \int_0^t \|u_{xt}(\tau)\|^2 d\tau \\ &+ 2\int_0^t \|u_{xxt}(\tau)\|^2 d\tau + 2\int_{\mathbb{R}} G(u_x)dx. \end{aligned} \tag{4.1}$$

From (4.1) and (1.9), it follows that

$$\begin{aligned} \frac{d}{dt}E(t) =& \frac{d}{dt}\Big[\|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + 2k_1 \int_0^t \|u_{xt}(\tau)\|^2 d\tau \\ &+ 2\int_0^t \|u_{xxt}(\tau)\|^2 d\tau + 2\int_{\mathbb{R}} G(u_x)dx\Big] \\ =& 2\int_{\mathbb{R}}(u_{tt}u_t + u_{xxxx}u_t - k_1 u_{xxt}u_t + u_{xxxxt}u_t - g(u_x)_x u_t)dx \\ =& 0. \end{aligned} \tag{4.2}$$

Integration of (4.2) from 0 to t leads to

$$E(t) = E(0). \tag{4.3}$$

We now define

$$\begin{aligned} \phi(t) =& \|u(t)\|^2 + k_1 \int_0^t \|u_x(\tau)\|^2 d\tau + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \\ &+ k_1(T_0 - t)\|u_{0x}\|^2 + (T_0 - t)\|u_{0xx}\|^2 + \beta(t + t_0)^2, \end{aligned}$$

where $\beta \geq 0$, T_0 and t_0 are positive real numbers to be given later. Hence,

$$\begin{aligned} \phi'(t) =& 2\int_{\mathbb{R}} u(t)u_t(t)dx + k_1\|u_x(t)\|^2 + \|u_{xx}(t)\|^2 - k_1\|u_{0x}(t)\|^2 \\ &- \|u_{0xx}(t)\|^2 + 2\beta(t + t_0) \\ =& 2\Big[\int_{\mathbb{R}} u(t)u_t(t)dx + k_1 \int_0^t \int_{\mathbb{R}} u_x(\tau)u_{xt}(\tau)dxd\tau \\ &+ \int_0^t \int_{\mathbb{R}} u_{xx}(\tau)u_{xxt}(\tau)dxd\tau + \beta(t + t_0)\Big]. \end{aligned} \tag{4.4}$$

Then we have

$$\begin{aligned} &\phi''(t) \\ =& 2\Big[\|u_t(t)\|^2 + \int_{\mathbb{R}} u(t)u_{tt}(t)dx + k_1 \int_{\mathbb{R}} u_x(t)u_{xt}(t)dx + \int_{\mathbb{R}} u_{xx}(t)u_{xxt}(t)dx + \beta\Big] \\ =& 2\Big[\|u_t(t)\|^2 - \|u_{xx}(t)\|^2 - (g(u_x), u_x) + \beta\Big]. \end{aligned} \tag{4.5}$$

From (4.4), we can write

$$
\begin{aligned}
\phi'(t)^2 &= 4\Big[\int_{\mathbb{R}} u(t)u_t(t)dx + k_1\int_0^t\int_{\mathbb{R}} u_x(\tau)u_{xt}(\tau)dxd\tau + \int_0^t\int_{\mathbb{R}} u_{xx}(\tau)u_{xxt}(\tau)dxd\tau \\
&\qquad + \beta(t+t_0)\Big]^2 \\
&\le 4\Big[\|u(t)\|\|u_t(t)\| + \Big(k_1\int_0^t\|u_x(\tau)\|^2d\tau\Big)^{\frac{1}{2}}\Big(k_1\int_0^t\|u_{xt}(\tau)\|^2d\tau\Big)^{\frac{1}{2}} \\
&\qquad + \Big(\int_0^t\|u_{xx}(\tau)\|^2d\tau\Big)^{\frac{1}{2}}\Big(\int_0^t\|u_{xxt}(\tau)\|^2d\tau\Big)^{\frac{1}{2}} + \sqrt{\beta}\sqrt{\beta}(t+t_0)\Big]^2 \\
&\le 4\phi(t)\Big[\|u_t(t)\|^2 + k_1\int_0^t\|u_{xt}(\tau)\|^2d\tau + \int_0^t\|u_{xxt}(\tau)\|^2d\tau + \beta\Big].
\end{aligned}
\tag{4.6}
$$

Now, from (4.4)–(4.6), we can obtain that

$$
\begin{aligned}
&\phi''(t)\phi(t) - (1+\gamma)\phi'(t)^2 \\
&= 2\phi(t)\Big[\|u_t(t)\|^2 - \|u_{xx}(t)\|^2 - \int_{\mathbb{R}} g(u_x(t))u_x(t)dx + \beta\Big] - (1+\gamma)\phi'(t)^2 \\
&\ge 2\phi(t)\Big[\|u_t(t)\|^2 - \|u_{xx}(t)\|^2 - \int_{\mathbb{R}} g(u_x(t))u_x(t)dx + \beta \\
&\qquad - 2(1+\gamma)\Big(\|u_t(t)\|^2 + k_1\int_0^t\|u_{xt}(\tau)\|^2d\tau + \int_0^t\|u_{xxt}(\tau)\|^2d\tau + \beta\Big)\Big] \\
&\ge 2\phi(t)\psi(t),
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
\psi(t) = -\|u_{xx}(t)\|^2 &- \int_{\mathbb{R}} g(u_x(t))u_x(t)dx - (2\gamma+1)(\|u_t(t)\|^2 + \beta) \\
&- 2(1+\gamma)\int_0^t(k_1\|u_{xt}(\tau)\|^2 + \|u_{xxt}(\tau)\|^2)d\tau.
\end{aligned}
\tag{4.8}
$$

From (4.8) and (1.9), it deduces that

$$
\begin{aligned}
\psi'(t) &= -2\int_{\mathbb{R}} u_{xx}(t)u_{xxt}(t)dx - \frac{d}{dt}\int_{\mathbb{R}} g(u_x(t))u_x(t)dx \\
&\qquad - 2(1+2\gamma)\int_{\mathbb{R}} u_t(t)u_{tt}(t)dx - 2(1+\gamma)(k_1\|u_{xt}(t)\|^2 + \|u_{xxt}(t)\|^2) \\
&= -\frac{d}{dt}\int_{\mathbb{R}} g(u_x(t))u_x(t)dx + 2(1+2\gamma)\int_{\mathbb{R}} g(u_x(t))u_{xt}(t)dx \\
&\qquad + 4\gamma\int_{\mathbb{R}} u_{xxt}(t)u_{xx}(t)dx + 2\gamma\|u_{xxt}(t)\|^2 + 2\gamma k_1\|u_{xt}(t)\|^2.
\end{aligned}
\tag{4.9}
$$

Integrating (4.9) from 0 to t and making use of the assumption (2.1),

$$\begin{aligned}\psi(t) \geq \psi(0) &+ \int_{\mathbb{R}} g(u_{0x})u_{0x}dx - \int_{\mathbb{R}} g(u_x(t))u_x(t)dx \\ &+ 2(1+2\gamma)\int_{\mathbb{R}} G(u_x(t))dx - 2(1+2\gamma)\int_{\mathbb{R}} G(u_{0x})dx \\ &+ 2\gamma\|u_{xx}(t)\|^2 - 2\gamma\|u_{0xx}\|^2 \\ \geq \psi(0) &+ \int_{\mathbb{R}} g(u_{0x})u_{0x}dx - 2(1+2\gamma)\int_{\mathbb{R}} G(u_{0x})dx - 2\gamma\|u_{0xx}\|^2 \\ \geq -(1&+2\gamma)\Big(\|u_{0xx}\|^2 + \|u_1\|^2 + 2\int_{\mathbb{R}} G(u_{0x})dx + \beta\Big) \\ \geq -(1&+2\gamma)(E(0)+\beta).\end{aligned} \tag{4.10}$$

Combining this inequality with (4.7), we obtain

$$\phi''(t)\phi(t) - (1+\gamma)\phi'(t)^2 \geq -2(1+2\gamma)(E(0)+\beta)\phi(t). \tag{4.11}$$

We consider three different cases on the sign of the initial energy $E(0)$:
(1) If $E(0) < 0$, we choose $\beta = -E(0)(> 0)$. Then it follows from (4.11) that

$$\phi''(t)\phi(t) - (1+\gamma)\phi'(t)^2 \geq 0. \tag{4.12}$$

We take t_0 in a such way that

$$2(u_0, u_1) + 2\beta t_0 > \frac{1}{\gamma}(\|u_{0xx}\|^2 + k_1\|u_{0x}\|^2),$$

then $\phi(0) > 0, \phi'(0) > 0$. We choose T_0 such that

$$\frac{\|u_0\|^2 + \beta t_0^2}{2\gamma\Big[\int_{\mathbb{R}} u_0(x)u_1(x)dx + \beta t_0\Big] - \|u_{0xx}\|^2 - k_1\|u_{0x}\|^2} \leq T_0.$$

According to Lemma 4.1 (1), there exists t_1, where $0 < t_1 \leq \frac{\phi(0)}{\gamma\phi'(0)} = \frac{\|u_0\|^2 + k_1T_0\|u_{0x}\|^2 + T_0\|u_{0xx}\|^2 + \beta t_0^2}{\gamma[\int_{\mathbb{R}} 2u_0(x)u_1(x)dx + \beta t_0]}$ such that

$$\phi(t) \to \infty \ as \ t \to t_1^-$$

i.e.

$$\|u(t)\|^2 + k_1\int_0^t \|u_x(\tau)\|^2 d\tau + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \to \infty \ as \ t \to t_1^-.$$

This is a contradiction with the fact that the maximal time of the existence of the solution is infinite.
(2) If $E(0) = 0$, we choose $\beta = 0$ Then (4.11) becomes

$$\phi''(t)\phi(t) - (1+\gamma)\phi'(t)^2 \geq 0. \tag{4.13}$$

We take T_0 such that

$$\frac{\|u_0\|^2}{2\gamma\Big[\int_{\mathbb{R}} u_0(x)u_1(x)dx\Big] - \|u_{0xx}\|^2 - k_1\|u_{0x}\|^2} \le T_0.$$

By considering the assumption (ii), we obtain $\phi(0) > 0, \phi'(0) > 0$. Then according to Lemma 4.1 (1), there exists t_2, where $0 < t_2 \le \frac{\phi(0)}{\gamma\phi'(0)} = \frac{\|u_0\|^2 + k_1 T_0\|u_{0x}\|^2 + T_0\|u_{0xx}\|^2}{\gamma[\int_{\mathbb{R}} 2u_0(x)u_1(x)dx]}$, such that

$$\phi(t) \to \infty \; as \; t \to t_2^-$$

i.e.

$$\|u(t)\|^2 + k_1\int_0^t \|u_x(\tau)\|^2 d\tau + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \to \infty \; as \; t \to t_2^-.$$

This is a contradiction with the fact that the maximal time of the existence of the solution is infinite.
(3) If $E(0) > 0$, we choose $\beta = 0$, then (4.11) becomes

$$\phi''(t)\phi(t) - (1+\gamma)\phi'(t)^2 \ge -2(1+2\gamma)E(0)\phi(t). \tag{4.14}$$

We now define the auxiliary function $J(t)$ as follows:

$$J(t) = -\phi^{-\gamma}(t). \tag{4.15}$$

Now, we compute

$$J'(t) = \gamma\phi^{-\gamma-1}(t)\phi'(t). \tag{4.16}$$

and

$$\begin{aligned} J''(t) &= -\gamma(\gamma+1)\phi^{-\gamma-2}(t)\phi'(t)^2 + \gamma\phi^{-\gamma-1}(t)\phi''(t) \\ &= \gamma\phi^{-\gamma-2}(t)\Big[\phi''(t)\phi(t) - (1+\gamma)\phi'(t)^2\Big] \\ &\ge -2\gamma(1+2\gamma)E(0)\phi^{-\gamma-1}(t) \end{aligned} \tag{4.17}$$

By considering the assumption (iii), we deduce

$$J'(0) = \gamma\phi^{-\gamma-1}(0)\phi'(0) > 0.$$

It follows the continuity of $J'(t)$ that

$$J'(t) > 0 \tag{4.18}$$

for some interval near $t = 0$. Let $t^* > 0$ be a maximal time (possibly $t^* = T$) when (4.18) holds on $[0, t^*)$. Multiplying (4.17) by $2J'(t)$, we obtain

$$\begin{aligned} \frac{d}{dt}\Big[J'(t)\Big]^2 &\ge -4\gamma(2\gamma+1)E(0)\phi^{-\gamma-1}(t)J'(t) \\ &= -4\gamma^2(2\gamma+1)E(0)\phi^{-2\gamma-2}(t)\phi'(t) \\ &= 4\gamma^2 E(0)\frac{d}{dt}\Big[\phi^{-2\gamma-1}(t)\Big], \quad \forall t \in [0, t^*) \end{aligned} \tag{4.19}$$

Integrating (4.19) over $[0, t)$, we obtain

$$\begin{aligned} J'(t)^2 &\geq J'(0)^2 + 4\gamma^2 E(0)\phi^{-2\gamma-1}(t) - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0) \\ &\geq J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0). \end{aligned} \tag{4.20}$$

From the assumption (iii), we obtain

$$J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0) > 0.$$

Hence, making use of the continuity of $J'(t)$, we obtain

$$J'(t) \geq \left[J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0)\right]^{\frac{1}{2}}, \forall t \in [0, t^*). \tag{4.21}$$

By repeating the procedure, $t^\star$ is extended to T, hence (4.21) holds for all $t \geq 0$. Thus

$$J(t) \geq J(0) + \left[J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0)\right]^{\frac{1}{2}} t, \quad \forall t > 0. \tag{4.22}$$

We choose T_0 such that

$$\frac{-J(0)}{\left[J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0)\right]^{\frac{1}{2}}} \leq T_0.$$

Then there exists a finite positive number t_3, such that $J(t_3) = 0$ and $0 < t_3 \leq T^* = \frac{-J(0)}{[J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0)]^{\frac{1}{2}}}$. Thus, $\phi(t) \to \infty$ as $t \to t_3^-$. This is a contradiction with the fact that the maximal time of the existence of the solution is infinite. The theorem is proved.

Example 4.1. For the Cauchy problem (1.9)–(1.10), we take specific functions $g(s)$, $u_0(x)$ and $u_1(x)$ satisfying the conditions of Theorem 2.3. We first discuss:

(I) The case $E(0) < 0$. To this end, We take $u_0(x) = u_1(x) = e^{-x^2}$ and $g(s) = k_2 s^3$, obviously, $u_0(x) \in H^2(\mathbb{R})$, $u_1(x) \in L^2(\mathbb{R})$, $g(0) = 0$, $G(u_{0x}) \in L^1(\mathbb{R})$, $sg(s) = k_2 s^4$ and $(1 + 2\gamma)G(s) = \frac{1}{2}(1 + 2\gamma)k_2 s^4$. Thus when $\gamma = \frac{1}{2}$, $g(s)$ satisfies assumption (2.1) of the Theorem 2.3. After some simple calculation, it implies $\|u_0\|^2 = \|u_1\|^2 = \sqrt{\frac{\pi}{2}}$, $\|u_{0xx}\|^2 = 3\sqrt{\frac{\pi}{2}}$, $\int_{\mathbb{R}} G(u_{0x})dx = \frac{3k_2}{32}\sqrt{\pi}$, we choose $k_2 = -100$, then

$$E(0) = \|u_1\|^2 + \|u_{0xx}\|^2 + 2\int_{\mathbb{R}} G(u_{0x})dx = 4\sqrt{\frac{\pi}{2}} - \frac{600}{32}\sqrt{\pi}. \tag{4.23}$$

Hence, we see from (4.23) that $E(0) < 0$, then the conditions of Theorem 2.3 are satisfied. Hence there exists a $t_1 \leq \frac{\phi(0)}{\gamma\phi'(0)}$ such that $\phi(t) \to \infty$ as $t \to t_1^-$.

(II) The case $E(0) = 0$. We take $u_0(x) = e^{-x^2}, u_1(x) = 5e^{-x^2}$ and $g(s) = k_3 s^3$, then $u_0(x) \in H^2(\mathbb{R})$, $u_1(x) \in L^2(\mathbb{R})$, $G(u_{0x}) \in L^1(\mathbb{R})$, when $\gamma = \frac{1}{2}$, $g(s)$ satisfies assumption (2.1) of the Theorem 2.3, and from above relations that $\|u_0\|^2 = \sqrt{\frac{\pi}{2}}$, $\|u_1\|^2 = 25\sqrt{\frac{\pi}{2}}$, $\|u_{0x}\|^2 = \sqrt{\frac{\pi}{2}}$, $\|u_{0xx}\|^2 = 3\sqrt{\frac{\pi}{2}}$, $\int_{\mathbb{R}} G(u_{0x})dx = \frac{3k_3}{32}\sqrt{\pi}$ and when $k_1 = 1, k_3 = -\frac{448}{3\sqrt{2}}$, we obtain

$$E(0) = \|u_1\|^2 + \|u_{0xx}\|^2 + 2\int_{\mathbb{R}} G(u_{0x})dx = 25\sqrt{\frac{\pi}{2}} + 3\sqrt{\frac{\pi}{2}} - 28\sqrt{\frac{\pi}{2}} = 0, \tag{4.24}$$

$$2(u_0, u_1) - \frac{1}{\gamma}(\|u_{0xx}\|^2 + k_1\|u_{0x}\|^2) = 2\sqrt{\frac{\pi}{2}} > 0. \tag{4.25}$$

Then the conditions of Theorem 2.3 are satisfied. Hence there exists a $t_2 \leq \frac{\phi(0)}{\gamma\phi'(0)}$ such that $\phi(t) \to \infty$ as $t \to t_2^-$.

(III) The case $E(0) > 0$. Now we take $u_0(x) = e^{-x^2}$, $u_1(x) = e^{-x^2}$ and $g(s) = s^2$, then $u_0(x) \in H^2(\mathbb{R})$, $u_1(x) \in L^2(\mathbb{R})$, $G(u_{0x}) \in L^1(\mathbb{R})$, when $\gamma = \frac{1}{4}$, $g(s)$ satisfies assumption (2.1) of the Theorem 2.3. We can obtain $\|u_0\|^2 = \|u_1\|^2 = \sqrt{\frac{\pi}{2}}$, $\|u_{0xx}\|^2 = 3\sqrt{\frac{\pi}{2}}$, $\int_{\mathbb{R}} u_0u_1dx = \sqrt{\frac{\pi}{2}}$,

$$\begin{aligned} E(0) &= \|u_1\|^2 + \|u_{0xx}\|^2 + 2\int_{\mathbb{R}} G(u_{0x})dx \\ &= 4\sqrt{\frac{\pi}{2}} - 2\cdot\frac{8}{3}\int_{\mathbb{R}} x^3e^{-3x^2}dx = 4\sqrt{\frac{\pi}{2}} > 0. \end{aligned} \tag{4.26}$$

Taking T_0 small enough such that

$$(u_0, u_1) > \left[E(0)(\|u_0\|^2 + T_0\|u_{0x}\|^2 + T_0\|u_{0xx}\|^2)\right]^{\frac{1}{2}}.$$

Then the conditions of Theorem 2.3 are satisfied. Hence there exists a finite positive number t_3, such that $J(t_3) = 0$ and $0 < t_3 \leq T^* = \frac{J(0)}{[J'(0)^2 - 4\gamma^2E(0)\phi^{-2\gamma-1}(0)]^{\frac{1}{2}}}$. Thus, $\phi(t) \to \infty$ as $t \to t_3^-$.

5. Discussion

It is well known that Equation (1.11) describes the motion of the elastomer rod with internal damping. In the process of high speed movement, by the impact on damping characteristic and external excitation, the state of the elastomer rod is complicated and unpredictable at the initial velocity. Considering this situation, we choose initial data belonging to more general functional space $u_0 \in W^{4,1}(\mathbb{R}) \bigcap H^3(\mathbb{R}), u_1 \in L^1(\mathbb{R}) \bigcap H^{-1}(\mathbb{R})$. By using the L_1-based spaces instead of L_2-based ones, which are completely different from those used in [11], [18], we can still obtain the global smooth solution in the generalized space. In this paper, we just consider the problems in 1-dimensional space, but in high-dimensional space, do both Equation (1.9) and Equation (1.11) have and global solutions to the Cauchy problem or the initial boundary value problem? The question is interesting and opening.

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Conflict of Interest

The authors declare that there is no conflicts of interest in this paper.

References

1. H. T. Banks and D. S. Gilliam, *Global solvability for damped abstract nonlinear hyperbolic systems*, Differ. Integral. Equ., **10** (1997), 309–332.
2. Z. J. Yang and C. M. Song, *Blow up of solutions for a class of quasi-linear evolution equations*, Nonlinear Anal., **28** (1997), 2017–2032.
3. Z. J. Yang and G. W. Chen, *Global existence of solutions for quasi-linear wave equations with viscous damping*, J. Math. Anal. Appl., **285** (2003), 604–618.
4. Y. C. Liu and R. Z. Xu, *A class of fourth order wave equations with dissipative and nonlinear strain terms*, J. Differ. Equ., **244** (2008), 200–228.
5. G. W. Chen and Z. J. Yang, *Existence and nonexistence of global solutions for a class of nonlinear wave equations*, Math. Meth. Appl. Sci., **23** (2000), 615–631.
6. G. A. Philippin and S. V. Piro, *Lower bound for the lifespan of solutions for a class of fourth order wave equations*, Appl. Math. Lett., **50** (2015), 141–145.
7. R. Z. Xu, S. Wang, Y. B. Yang, et al. *Initial boundary value problem for a class of fourth-order wave equation with viscous damping term*, Appl. Anal., **92** (2013), 1403–1416.
8. A. Khelghati and K. Baghaei, *Blow-up phenomena for a class of fourth-order nonlinear wave equations with a viscous damping term*, Math. Meth. Appl. Sci., **41** (2018), 490–494.
9. W. F. Zhao and W. J. Liu, *A note on blow-up of solutions for a class of fourth-order wave equation with viscous damping term*, Appl. Anal., **1** (2017), 1–9.
10. K. Baghaei, *Lower bounds for the blow-up time in a superlinear hyperbolic equation with linear damping term*, Comput. Math. Appl., **73** (2017), 560–564.
11. Z. J. Yang and G. W. Chen, *Initial value problem for a nonlinear wave equation with damping term*, Acta Math. Appl. Sinica., **23** (2000), 45–54.
12. G. W. Chen and B. Lu, *The initial-boundary value problems for a class of nonlinear wave equations with damping term*, J. Math. Anal. Appl., **351** (2009), 1–15.
13. H. T. Banks and D. S. Gilliam, *Well-posedness for a one dimensional nonlinear beam*, In Computation and Control IV, Progress in Systems and Control Theory, Birkhäuser, Boston, MA. **20** (1995), 1–21.
14. A. S. Ackleh, H. T. Banks and G. A. Pinter, *A nonlinear beam equation*, Appl. Math. Lett., **15** (2002), 381–387.
15. G. W. Chen, Y. P. Wang and Z. C. Zhao, *Blow-up of solution of an initial boundary value problem for a damped nonlinear hyperbolic equation*, Appl. Math. Lett., **17** (2004), 491–497.
16. C. M. Song, *Nonexistence of global solutions of nonlinear hyperbolic equation with material damping*, Appl. Math. Mech., **27** (2006), 975–981.
17. G. W. Chen and F. Da, *Blow-up of solution of Cauchy problem for three-dimensional damped nonlinear hyperbolic equation*, Nonlinear Anal., **71** (2009), 358–372.
18. C. M. Song and Z. J. Yang, *Existence and nonexistence of global solutions to the Cauchy problem for a nonlinear beam equation*, Math. Meth. Appl. Sci., **33** (2010), 563–575.

19. G. W. Chen, R. L. Song and S. B. Wang, *Local existence and global nonexistence theorems for a damped nonlinear hyperbolic equation*, J. Math. Anal. Appl., **368** (2010), 19–31.

20. J. L. Yu, Y. D. Shang and H. F. Di, *On decay and blow-up of solutions for a nonlinear beam equation with double damping terms*, Bound.Value Probl., Accepted (2018).

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