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Global behavior for a diffusive predator-prey system with Holling type II functional response

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Abstract

A strongly coupled self- and cross-diffusion predator-prey system with Holling type II functional response is considered. Using the energy estimate, Sobolev embedding theorem and bootstrap arguments, the global existence of non-negative classical solutions to this system in which the space dimension is not more than five is obtained.

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1 Introduction

In this paper, we consider the global existence of non-negative classical solutions to the following diffusion predator-prey system with Holling type II functional response:

$$\begin{cases} u_t = \Delta(d_1 u + \alpha_{11} u^2) + \alpha u(1 - \frac{u}{K}) - \frac{\beta m u v}{1+amu}, & x \in \Omega, t > 0, \\ v_t = \Delta(d_2 v + \alpha_{21} u v + \alpha_{22} v^2) - rv + \frac{c\beta m u v}{1+amu}, & x \in \Omega, t > 0, \\ \partial_\eta u(x,t) = \partial_\eta v(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded region in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$; η is the outward normal on $\partial\Omega$, $\partial_\eta = \partial/\partial\eta$; $u_0(x)$ and $v_0(x)$ are non-negative smooth functions and are not identically zero; u and v denote the population densities of predator and prey, respectively; α, β, r, a, K, m and c are positive constants, and $m \in (0,1]$; d_1 and d_2 are the diffusion rates of the two species; α_{ij} $(i,j = 1,2)$ are given non-negative constants, α_{11} and α_{22} are self-diffusion rates; α_{21} is the cross-diffusion rate. It means that the diffusion is from one species of high-density areas to the other species of low-density areas. See [1, 2] for more details on the ecological backgrounds of this system.

Obviously, the non-negative equilibrium solutions of system (1.1) are $(K,0)$ and $(u_*, v_*) = (\frac{r}{m(c\beta-ar)}, \frac{\alpha cKm(c\beta-ar)-\alpha cr}{Km^2(c\beta-ar)^2})$. For the reaction-diffusion problem of system (1.1), *i.e.*, $\alpha_{ii} = 0$ $(i = 1,2)$, the global attraction, persistence and stability of non-negative equilibrium solutions are studied in [3]. The main result can be summarized as follows:

(1) If $m < \frac{r}{K(c\beta-ar)}$, a semi-trivial solution $(K,0)$ is globally asymptotically stable;

(2) If $\frac{r}{K(c\beta-ar)} < m \le \frac{r}{K(c\beta-ar)} + \frac{1}{Ka}$ and $c\beta > ar$, a unique positive constant solution (u_*, v_*) is globally asymptotically stable;

(3) If $\frac{r}{K(c\beta-ar)} < m < \frac{r}{K(c\beta-ar)} + \frac{c\beta}{Ka(c\beta-ar)}$ and $c\beta > ar$, a positive constant solution(u_*, v_*) is locally asymptotically stable.

In view of the study of dynamic behavior of a predator-prey reaction-diffusion system with Holling type II functional response, a natural problem is what the global behavior for a predator-prey cross-diffusion system (1.1) is. To the best of our knowledge, the existing results are very few. In this paper, we consider the space dimension to be less than six, and initial function $u_0(x)$ and $v_0(x)$ under some smooth conditions, using the energy estimate, Sobolev embedding theorem and bootstrap arguments, we consider the global existence of non-negative classical solutions for system (1.1).

We denote $Q_T = \Omega \times (0,T)$. $u \in W_q^{2,1}(Q_T)$ means that u, u_{x_i}, $u_{x_i x_j}$ $(i,j = 1,\dots,n)$ and u_t are in $L^q(Q_T)$. $\|u\|_{L^{p,q}(Q_T)} = [\int_0^T(\int_\Omega |u(x,t)|^p \,\mathrm{d}x)^{\frac{q}{p}} \,\mathrm{d}t]^{\frac{1}{q}}$. $\|u\|_{V_2(Q_T)} = \sup_{0\le t\le T}\|u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)}$.

2 Auxiliary results

Lemma 2.1 *Let* (u, v) *be the solution of* (1.1). *There exists a positive constant* $M_0(\ge 1)$ *such that*

$$0 \le u \le M_0, \qquad 0 \le v, \quad \forall t \ge 0. \tag{2.1}$$

Proof Firstly, the existence of local solutions for system (1.1) is given in [4–6]. Roughly speaking, if $u_0, v_0 \in W_p^1(\Omega)$, $p > 1$, there exists the maximum $T \le +\infty$ such that system (1.1) admits a unique non-negative solution

$$u, v \in C\big([0,T), W_p^1(\Omega)\big) \cap C^\infty\big((0,T), C^\infty(\Omega)\big). \tag{2.2}$$

If

$$\sup\big\{\|u(\cdot,t)\|_{w_p^1(\Omega)}, \|v(\cdot,t)\|_{w_p^1(\Omega)} : 0 < t < T\big\} < \infty,$$

then $T = +\infty$.

Choose $M_0 = \max\{K, \|u_0\|_{L^\infty(\Omega)}\}$. By use of the maximum principle, the non-negative solution of system (1.1) can be derived from the maximum principle, *i.e.*, $u, v \ge 0$ for all $t \ge 0$. This completes the proof of Lemma 2.1. □

Lemma 2.2 *Let* $X = (d_1 + \alpha_{11}u)u$, $u \in L^\infty(Q_T)$ *for the solution to the following equation*:

$$u_t = \Delta\big[(d_1 + \alpha_{11}u)u\big] + \alpha u\left(1 - \frac{u}{K}\right) - \frac{\beta muv}{1 + amu}, \quad (x,t) \in \Omega \times (0,T),$$
$$\partial_\eta u = 0, \quad (x,t) \in \partial\Omega \times (0,T),$$
$$u(x,0) = u_0(x) \ge 0, \quad x \in \Omega,$$

where d_1, α_{11} *are positive constants and* $0 \le u \in L^2(Q_T)$. *Then there exists a positive constant* $C(T)$, *depending on* $\|u_0\|_{W_2^1(\Omega)}$ *and* $\|u_0\|_{L^\infty(\Omega)}$, *such that*

$$\|X\|_{W_2^{2,1}(Q_T)} \le C(T). \tag{2.3}$$

Furthermore,

$$\nabla X \in V_2(Q_T), \qquad \nabla u \in L^{\frac{2(n+2)}{n}}(Q_T). \tag{2.4}$$

Proof From $X = (d_1 + \alpha_{11}u)u$, it is easy to find that

$$X_t = (d_1 + 2\alpha_{11}u)\Delta X + C_1 - C_2 v, \tag{2.5}$$

where $C_1 = d_1\alpha u + (2\alpha_{11}\alpha - \frac{d_1\alpha}{K})u^2 - \frac{2\alpha_{11}\alpha}{K}u^3$ and $C_2 = \frac{\beta mu}{1+amu}(d_1 + 2\alpha_{11}u)$. C_1 and C_2 are bounded in Q_T from (2.1). Multiplying (2.5) by $-\Delta X$ and integrating by parts over Q_t yields

$$\begin{aligned}&\frac{1}{2}\int_\Omega \left|\nabla X(x,t)\right|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega \left|\nabla X(x,0)\right|^2 \mathrm{d}x + d_1\int_{Q_t} |\Delta X|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le \int_{Q_T} |C_1 + C_2 v||\Delta X| \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.6}$$

Using the Hölder inequality and Young inequality to estimate the right-hand side of (2.6), we have

$$\begin{aligned}\|C_1 + C_2 v\|_{L^2(Q_T)}\|\Delta X\|_{L^2(Q_T)} &\le m_1\big(1 + \|v\|_{L^2(Q_T)}\big)\|\Delta X\|_{L^2(Q_T)} \\ &\le \frac{m_1^2(1+M_3)^2}{2d_1} + \frac{d_1}{2}\|\Delta X\|_{L^2(Q_T)}^2\end{aligned} \tag{2.7}$$

with some $m_1 > 0$. Substituting (2.7) into (2.6), we obtain

$$\sup_{0\le t\le T}\int_\Omega \left|\nabla X(x,t)\right|^2 \mathrm{d}x + d_1\int_{Q_t} |\Delta X|^2 \,\mathrm{d}x\,\mathrm{d}t \le m_2,$$

where m_2 depends on $\|u_0\|_{W_2^1(\Omega)}$ and $\|u_0\|_{L^\infty(\Omega)}$. So, we know $\nabla X \in V_2(Q_T)$. Since $X \in L^2(Q_T)$, it follows from the elliptic regularity estimate [7, Lemma 2.3] that

$$\int_{Q_T} |X_{x_i x_j}|^2 \,\mathrm{d}x\,\mathrm{d}t \le m_3, \quad i,j = 1,\dots,n.$$

From (2.5), we have $X_t \in L^2(Q_T)$. Hence, $\|X\|_{W_2^{2,1}(Q_T)} \le C(T)$. Moreover, (2.4) can be obtained by use of the Sobolev embedding theorem. □

Lemma 2.3 *Assume that* $w \in W_P^{2,1}(Q_T) \cap C^{2,1}(\bar{\Omega} \times [0,T))$ *is a bounded function satisfying*

$$w_t \le a(x,t,w)\Delta w + f(x,t) \quad \textit{in } Q_T$$

with the boundary condition $\frac{\partial w}{\partial \nu} \le 0$ *on* ∂_{Q_T}, *where* $f \in L^P(Q_T)$. *Then* ∇W *is in* $L^{2p}(Q_T)$.

The proof of the above lemma can be found in [8, Proposition 2.1].

The following result can be derived from Lemma 2.3 and Lemma 2.4 of [9].

Lemma 2.4 *Let $p > 1$, $\tilde{p} = 2 + \frac{4p}{n(p+1)}$. If*

$$\sup_{0\le t\le T} \|w\|_{L^{\frac{2p}{p+1}}(\Omega)} + \|\nabla w\|_{L^2(Q_T)} < \infty,$$

and there exist positive constants $\beta \in (0,1)$ and C_T such that $\int_\Omega |w(\cdot,t)|^\beta \,\mathrm{d}x \le C_T$ $(\forall t \le T)$, there exists a positive constant M', independent of w but possibly depending on n, Ω, p, β and C_T, such that

$$\|w\|_{L^{\tilde{p}}(Q_T)} \le M'\left\{1 + \left(\sup_{0\le t\le T} \big\|w(t)\big\|_{L^{\frac{2p}{p+1}}(\Omega)}\right)^{\frac{4p}{n(p+1)\tilde{p}}} \|\nabla w\|_{L^2(Q_T)}^{\frac{2}{\tilde{p}}}\right\}.$$

Finally, one proposes some standard embedding results which are important to obtain the L^∞ and $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$ normal estimates of the solution for (1.1).

Lemma 2.5 *There exists a constant $C_3(T)$ such that $\|\nabla u\|_{L^4}(Q_T) \le C_3(T)$.*

Proof Let $\delta = \alpha_{11}/d_1$, $X = (1 + \delta u)u$. By Lemma 2.1, u is bounded. Therefore, X is also bounded. By Lemma 2.2, we have $X \in W_2^{2,1}(Q_T)$. Moreover, X satisfies

$$\begin{aligned} X_t &\le d_1(1 + 2\delta u)\Delta X + \alpha u(1 + 2\delta u) \\ &= \sqrt{d_1^2 + 4\delta d_1 X}\,\Delta X + \alpha u(1 + 2\delta u). \end{aligned}$$

By Lemma 2.3 with $p = 2$, $a(x,t,\xi) = \sqrt{d_1^2 + 4\delta d_1\xi}$, $f(x,t) = \alpha u(x,t)(1 + 2\delta u(x,t))$, we obtain the desired result. □

Lemma 2.6 *Let $\Omega \subset R^n$ be a fixed bounded domain and $\partial\Omega \subset C^2$. Then for all $u \in W_q^{2,1}(Q_T)$ with $q \ge 1$, one has*
(1) $\|\nabla u\|_{L^p(Q_T)} \le C\|u\|_{W_q^{2,1}(Q_T)}$, $\forall 1 \le p \le \frac{(n+2)q}{n+2-q}$, $q < n + 2$;
(2) $\|\nabla u\|_{L^p(Q_T)} \le C\|u\|_{W_q^{2,1}(Q_T)}$, $\forall 1 \le p \le \infty$, $q = n + 2$;
(3) $\|\nabla u\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C\|u\|_{W_q^{2,1}(Q_T)}$, $\forall 1 - \frac{n+2}{q} \le \alpha \le 1$, $q > n + 2$,
where C is a positive constant dependent on q, n, Ω, T.

3 The existence of classical solutions

The main result about the global existence of non-negative classical solutions for the cross-diffusion system (1.1) is given as follows.

Theorem 3.1 *Assume that $u_0 > 0$ and $v_0 > 0$ satisfy homogeneous Neumann boundary conditions and belong to $C^{2+\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. Then system (1.1) has a unique non-negative solution $(u,v) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega} \times [0,\infty))$ if the space dimension is $n \le 5$.*

Proof When $n = 1$, the proof is similar to the methods of [10–12]. So, we just give the proof of Theorem 3.1 for $n = 2,3,4,5$. The proof is divided into three parts.

(i) L^1-, L^2-estimate and L^q-estimate for v.

Firstly, integrating the first equation of (1.1) over Ω, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\,\mathrm{d}x &= \int_\Omega u\left(\alpha-\frac{\alpha u}{K}-\frac{\beta m v}{1+amu}\right)\mathrm{d}x \le \int_\Omega \alpha u\,\mathrm{d}x-\frac{\alpha}{K}\int_\Omega u^2\,\mathrm{d}x\\ &\le \alpha\int_\Omega u\,\mathrm{d}x-\frac{\alpha}{K|\Omega|}\left(\int_\Omega u\,\mathrm{d}x\right)^2.\end{aligned}$$

Thus, for all $t\ge 0$, we can obtain

$$\int_\Omega u\,\mathrm{d}x\le M_1',$$

where $M_1'=\max\{K|\Omega|,\int_\Omega u_0\,\mathrm{d}x\}$.

Furthermore,

$$\|u\|_{L^1(Q_T)}\le\int_0^T M_1'\,\mathrm{d}t\triangleq M_1. \tag{3.1}$$

Secondly, linear combination of the second and first equations of (1.1) and integrating over Ω yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega (cu+v)\,\mathrm{d}x &= c\int_\Omega \alpha u\,\mathrm{d}x-\frac{c\alpha}{K}\int_\Omega u^2\,\mathrm{d}x-r\int_\Omega v\,\mathrm{d}x\\ &\le c(r+\alpha)\int_\Omega u\,\mathrm{d}x-\frac{c\alpha}{K|\Omega|}\left(\int_\Omega u\,\mathrm{d}x\right)^2-r\int_\Omega (cu+v)\,\mathrm{d}x\\ &\le c(r+\alpha)M_1'-r\int_\Omega (cu+v)\,\mathrm{d}x.\end{aligned}$$

So, we get

$$\int_\Omega (cu+v)\,\mathrm{d}x\le\max\left\{\frac{c(r+\alpha)M_1}{r},\int_\Omega\big(cu_0(x)+v_0(x)\big)\,\mathrm{d}x\right\}\triangleq M_2',\quad \forall t\ge 0. \tag{3.2}$$

Further,

$$\|v\|_{L^1(Q_T)}\le\int_0^T M_1'\,\mathrm{d}t\triangleq M_2. \tag{3.3}$$

Then multiplying both sides of the second equation of system (1.1) by v and integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v^2\,\mathrm{d}x &= -\int_\Omega \nabla v\nabla\left[d_2 v+\alpha_{21}uv+\alpha_{22}v^2\right]\mathrm{d}x-r\int_\Omega v^2\,\mathrm{d}x+cm\beta\int_\Omega\frac{uv^2}{1+amu}\,\mathrm{d}x\\ &\le -d_2\int_\Omega|\nabla v|^2\,\mathrm{d}x-\alpha_{21}\int_\Omega v\nabla v\nabla u\,\mathrm{d}x-2\alpha_{22}\int_\Omega v|\nabla v|^2\,\mathrm{d}x+\frac{c\beta}{a}\int_\Omega v^2\,\mathrm{d}x.\end{aligned}$$

Integrating the above expression in $[0,1]$ yields

$$\begin{aligned}&\int_\Omega v^2(x,t)\,\mathrm{d}x-\int_\Omega v_0^2(x)\,\mathrm{d}x+2\int_{Q_t}(d_2+\alpha_{21}u+2\alpha_{22}v)|\nabla v|^2\,\mathrm{d}x\,\mathrm{d}t+r\int_{Q_t}v^2\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\le -2\alpha_{21}\int_{Q_t}\nabla u\cdot v\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t+\frac{c\beta}{a}\int_{Q_t}v^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.4}$$

Estimating the first term on the right-hand side of (3.4),

$$\begin{aligned}\left|\int_{Q_t} \nabla u \cdot v \cdot \nabla v \, dx \, dt\right| &= \left|\int_{Q_t} \nabla u \cdot v^{\frac{1}{2}} \cdot \nabla v \cdot v^{\frac{1}{2}} \, dx \, dt\right| \\ &\le \varepsilon_1 \int_{Q_t} v|\nabla u|^2 \, dx \, dt + \frac{1}{4\varepsilon_1} \int_{Q_t} v \cdot |\nabla v|^2 \, dx \, dt \\ &\le \varepsilon_2 \int_{Q_t} v^2 \, dx \, dt + \frac{1}{4\varepsilon_2} \int_{Q_t} |\nabla u|^4 \, dx \, dt \\ &\quad + \frac{1}{4\varepsilon_1} \int_{Q_t} v \cdot |\nabla v|^2 \, dx \, dt. \end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4) yields

$$\begin{aligned}&\int_\Omega v^2(x,t) \, dx - \int_\Omega v_0^2(x) \, dx + 2\int_\Omega (d_2 + \alpha_{21} u + 2\alpha_{22} v)|\nabla v|^2 \, dx \, dt + r\int_{Q_t} v^2 \, dx \, dt \\ &\quad \le \left(2\alpha_{21}\varepsilon_2 + \frac{c\beta}{a}\right)\int_{Q_t} v^2 \, dx \, dt + \frac{\alpha_{21}}{2\varepsilon_2}\int_{Q_t} |\nabla u|^4 \, dx \, dt + \frac{\alpha_{21}}{2\varepsilon_1}\int_{Q_t} v|\nabla v|^2 \, dx \, dt. \end{aligned} \tag{3.6}$$

Select $\varepsilon_1 > \frac{\alpha_{21}}{8\alpha_{22}}$ and denote $C_4 = 4\alpha_{22} - \frac{\alpha_{21}}{2\varepsilon_1}$. Notice the positive equilibrium point of (1.1) exists under condition $c\beta > ar$, then

$$\begin{aligned}&\int_\Omega v^2(x,t) \, dx + 2d_2 \int_{Q_t} |\nabla v|^2 \, dx \, dt + 2\alpha_{21}\|u\|_\infty \int_{Q_t} |\nabla v|^2 \, dx \, dt + C_4 \int_{Q_t} v|\nabla v|^2 \, dx \, dt \\ &\quad \le \int_\Omega v_0^2(x) \, dx + \left(2\alpha_{21}\varepsilon_2 - r + \frac{c\beta}{a}\right)\int_{Q_t} v^2 \, dx \, dt + \frac{\alpha_{21}}{2\varepsilon_2}\int_{Q_t} |\nabla u|^4 \, dx \, dt. \end{aligned} \tag{3.7}$$

By Lemma 2.5, $\|\nabla u\|_{L^4}(Q_T) \le C_3(T)$. Integrating the above inequality and using the Gronwall inequality, we get

$$\sup_{0<t<T} \int_\Omega v^2 \, dx \le C(T).$$

Hence, there exists a positive constant M_3' such that $\int_\Omega v^2 \, dx \le M_3'$. Furthermore, we have

$$\|v\|_{L^2(Q_T)} \le \int_0^T M_2' \, dt \triangleq M_3. \tag{3.8}$$

Secondly, multiplying both sides of the second equation of system (1.1) by qv^{q-1} $(q > 1)$ and integrating over Ω, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega v^q \, dx &\le -\frac{4(q-1)d_2}{q}\int_\Omega \left|\nabla\left(v^{\frac{q}{2}}\right)\right|^2 dx - \frac{8q(q-1)\alpha_{22}}{(q+1)^2}\int_\Omega \left|\nabla\left(v^{\frac{q+1}{2}}\right)\right|^2 dx \\ &\quad - q(q-1)\alpha_{21}\int_\Omega v^{q-1}\nabla u \cdot \nabla v \, dx + q\int_\Omega v^q\left(-r + \frac{c\beta mu}{1 + amu}\right) dx. \end{aligned}$$

Integrating the above equation over $[0,t]$ $(t \le T)$, it is clear that

$$\begin{aligned}
&\int_\Omega v^q(x,t)\,\mathrm{d}x + \frac{4(q-1)d_2}{q}\int_{Q_t}\left|\nabla\left(v^{\frac{q}{2}}\right)\right|^2 \mathrm{d}x\,\mathrm{d}t + \frac{8q(q-1)\alpha_{22}}{(q+1)^2}\int_{Q_t}\left|\nabla\left(v^{\frac{q+1}{2}}\right)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_\Omega v_0^q(x)\,\mathrm{d}x - q(q-1)\alpha_{21}\int_{Q_t} v^{q-1}\nabla u\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t + \frac{c\beta q}{a}\int_{Q_t} v^q\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.9)
\end{aligned}$$

By Lemma 2.2, it can be found that $\nabla u \in L^{\frac{2(n+2)}{n}}(Q_T)$. According to the Hölder inequality and Young inequality, we get

$$\begin{aligned}
&-q(q-1)\alpha_{21}\int_{Q_t} v^{q-1}\nabla u\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \frac{2q(q-1)\alpha_{21}}{q+1}\left|\int_{Q_t} v^{\frac{q-1}{2}}\nabla\left(v^{\frac{q+1}{2}}\right)\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \le \frac{2q(q-1)\alpha_{21}}{q+1}\|\nabla u\|_{L^{\frac{2(n+2)}{n}}(Q_T)}\left\|v^{\frac{q-1}{2}}\right\|_{L^{n+2}(Q_T)}\left\|\nabla\left(v^{\frac{q+1}{2}}\right)\right\|_{L^2(Q_T)} \\
&\quad \le C_3\left\|\nabla\left(v^{\frac{q+1}{2}}\right)\right\|_{L^2(Q_T)}\left\|v^{\frac{q-1}{2}}\right\|_{L^{n+2}(Q_T)} \\
&\quad \le \frac{C_3\varepsilon}{2}\left\|\nabla\left(v^{\frac{q+1}{2}}\right)\right\|^2_{L^2(Q_T)} + \frac{C_3}{2\varepsilon}\left\|v^{\frac{q-1}{2}}\right\|^2_{L^{n+2}(Q_T)}. \qquad (3.10)
\end{aligned}$$

Choose an appropriate number ε satisfying $\frac{C_3\varepsilon}{2} \le \frac{8q(q-1)\alpha_{22}}{(q+1)^2}$. Substituting (3.10) into (3.9) and taking $\bar{v} = v^{\frac{q+1}{2}}$, we have

$$\begin{aligned}
&\int_\Omega \bar{v}^{\frac{2q}{q+1}}(x,t)\,\mathrm{d}x + \int_{Q_t}|\nabla\bar{v}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_\Omega v_0^q\,\mathrm{d}x + \frac{C_3}{2\varepsilon}\|\bar{v}\|^{\frac{2(q-1)}{q+1}}_{L^{\frac{(q-1)(n+2)}{q+1}}(Q_T)} + \frac{c\beta q}{a}\|\bar{v}\|^{\frac{2q}{q+1}}_{L^{\frac{2q}{q+1}}(Q_T)} \\
&\quad \le C_4\left(1 + \|\bar{v}\|^{\frac{2(q-1)}{q+1}}_{L^{\frac{(q-1)(n+2)}{q+1}}(Q_T)} + \|\bar{v}\|^{\frac{2q}{q+1}}_{L^{\frac{2q}{q+1}}(Q_T)}\right). \qquad (3.11)
\end{aligned}$$

Let

$$E \equiv \sup_{0<t<T}\int_\Omega \bar{v}^{\frac{2q}{q+1}}(x,t)\,\mathrm{d}x + \int_{Q_T}|\nabla\bar{v}|^2\,\mathrm{d}x\,\mathrm{d}t.$$

From (3.11), we know

$$E \le C_4\left(1 + \|\bar{v}\|^{\frac{2(q-1)}{q+1}}_{L^{\frac{(q-1)(n+2)}{q+1}}(Q_T)} + \|\bar{v}\|^{\frac{2q}{q+1}}_{L^{\frac{2q}{q+1}}(Q_T)}\right).$$

When $q < \frac{n(n+4)}{n^2-4}$, it is easy to find that $\frac{2q}{q+1} < 2 < \tilde{q}$ and $\frac{(q-1)(n+2)}{q+1} < \tilde{q} = 2 + \frac{4q}{n(q+1)}$. So,

$$E \le C_5\left(1 + \|\bar{v}\|^{\frac{2(q-1)}{q+1}}_{L^{\tilde{q}}(Q_T)} + \|\bar{v}\|^{\frac{2q}{q+1}}_{L^{\tilde{q}}(Q_T)}\right). \qquad (3.12)$$

Set $\beta = \frac{2}{q+1} \in (0,1)$. It follows from the $L^1(\Omega)$-estimate for v that

$$\|\bar{v}\|_{L^\beta(\Omega)} = \left(\int_\Omega \left|\bar{v}(\cdot,t)\right|^\beta \,\mathrm{d}x\right)^{\frac{1}{\beta}} = \|v\|_{L^1(\Omega)}^{\frac{1}{\beta}} \le M_2'^{\frac{1}{\beta}}, \quad \forall t \le T.$$

By Lemma 2.4 and (3.12), we know

$$\begin{aligned}
E &\le C_5\Big[1 + \Big(M' + M' \sup_{0<t<T} \left\|\bar{v}(\cdot,t)\right\|_{L^{\frac{2q}{q+1}}(\Omega)}^{\frac{4q}{n(q+1)\tilde{q}}} \|\nabla \bar{v}\|_{L^2(Q_T)}^{\frac{2}{\tilde{q}}}\Big)^{\frac{2(q-1)}{q+1}} \\
&\quad + \Big(M' + M' \sup_{0<t<T} \left\|\bar{v}(\cdot,t)\right\|_{L^{\frac{2q}{q+1}}(\Omega)}^{\frac{4q}{n(q+1)\tilde{q}}} \|\nabla \bar{v}\|_{L^2(Q_T)}^{\frac{2}{\tilde{q}}}\Big)^{\frac{2q}{q+1}}\Big] \\
&\le C_6\Big[1 + \Big(\sup_{0<t<T} \left\|\bar{v}(\cdot,t)\right\|_{L^{\frac{2q}{q+1}}(\Omega)}^{\frac{2q}{q+1}}\Big)^{\frac{4(q-1)}{n(q+1)\tilde{q}}} \left(\|\nabla \bar{v}\|_{L^2(Q_T)}^2\right)^{\frac{2(q-1)}{(q+1)\tilde{q}}} \\
&\quad + \Big(\sup_{0<t<T} \left\|\bar{v}(\cdot,t)\right\|_{L^{\frac{2q}{q+1}}(\Omega)}^{\frac{2q}{q+1}}\Big)^{\frac{4(q-1)}{n(q+1)\tilde{q}}} \left(\|\nabla \bar{v}\|_{L^2(Q_T)}^2\right)^{\frac{2q}{(q+1)\tilde{q}}}\Big] \\
&\le C_6\left(1 + E^{\frac{4(q-1)}{n(q+1)\tilde{q}}} E^{\frac{2(q-1)}{(q+1)\tilde{q}}} + E^{\frac{4q}{n(q+1)\tilde{q}}} E^{\frac{2q}{(q+1)\tilde{q}}}\right).
\end{aligned} \tag{3.13}$$

Obviously, $\frac{4(q-1)}{n(q+1)\tilde{q}} + \frac{2(q-1)}{(q+1)\tilde{q}} \in (0,1)$ and $\frac{4q}{n(q+1)\tilde{q}} + \frac{2q}{(q+1)\tilde{q}} \in (0,1)$. It is easy to know that E is bounded by use of reduction to absurdity. Since $q < \frac{n(n+4)}{n^2-4}$, $\frac{(q+1)\tilde{q}}{2} \in (1, \frac{2(n+1)}{n-2})$. So, $\|\bar{v}\|_{L^{\tilde{q}}(Q_T)}$ is bounded, *i.e.*, $v \in L^{\frac{(q+1)\tilde{q}}{2}}(Q_T)$. Denote $\frac{(q+1)\tilde{q}}{2}$ still as q. So,

$$v \in L^q(Q_T), \quad \forall q \in \left(1, \frac{2(n+1)}{n-2}\right). \tag{3.14}$$

Finally, when $n = 2,3,4,5$, $(n^2-4)q < n^2 + 4n$ with $q = 2$. For $n \le 5$, taking $q = 2$ in (3.9), it follows from (3.8) that there exists a positive constant M_4 such that

$$\|v\|_{V_2(Q_T)} \le M_4. \tag{3.15}$$

By embedding theorem, we get

$$\|v\|_{L^{\frac{2(n+2)}{n}}(Q_T)} \le M_4.$$

(ii) L^∞-estimate for v.

The second equation of system (1.1) can be written as the following divergence form:

$$\frac{\partial v}{\partial t} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial v}{\partial x_j}\right) + \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(a_i(x,t)v\right) + v\left(-r + \frac{c\beta m u}{1 + a m u}\right), \tag{3.16}$$

where $a_{ij}(x,t) = (d_2 + \alpha_{21}u + 2\alpha_{22}v)\delta_{ij}$, $a_i(x,t) = \alpha_{21}u_{x_j}$ and δ_{ij} is the Kronecker sign.

In order to apply the maximum principle [13] to (3.16), we need to prove the following conditions:

(1) $\|v\|_{V_2(Q_T)}$ is bounded;
(2) $\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \ge \nu \sum_{i=1}^n \xi_i^2$;

(3) $\|\sum_{i=1}^{n} a_i^2(x,t), v(-r+\frac{c\beta mu}{1+amu})\|_{L^{p,r}(Q_T)} \le \mu_1$,
where v, μ_1 are positive constants and

$$\frac{1}{r}+\frac{n}{2p}=1-\chi, \quad 0<\chi<1, p\in\left[\frac{n}{2(1-\chi)},+\infty\right), r\in\left[\frac{1}{1-\chi},+\infty\right), n\ge 2. \tag{3.17}$$

Next, we will show that the above conditions (1) to (3) are satisfied for (3.16). When $n \le 5$, it is easy to find that the condition (1) is satisfied by use of (3.15). Since $\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ge d_3|\xi|^2$ for all $\xi \in \mathbb{R}^n$, the condition (2) is verified. In view of the condition (3), we take appropriate q and r. Rewrite the first equation of system (1.1) as

$$u_t = \nabla\cdot\left[(d_1+2\alpha_{11}u)\nabla u\right]+\alpha u\left(1-\frac{u}{K}\right)-\frac{\beta muv}{1+amu}. \tag{3.18}$$

When $n = 2,3,4,5$, $\frac{n+2}{2} < \frac{2(n+1)}{n-2}$, it is clear that $d_1 + 2\alpha_{11}u$ has an upper bound over $\overline{Q}_T$ by Lemma 2.1. Set

$$q\in\left(\frac{n+2}{2},\frac{2(n+1)}{n-2}\right).$$

From (3.14), we have $\alpha u(1-\frac{u}{K})-\frac{\beta muv}{1+amu} \in L^q(Q_T)$. Therefore, all conditions of the Hölder continuity theorem [5, Theorem 10.1] hold for (3.18). Hence,

$$u\in C^{\beta,\frac{\beta}{2}}(\overline{Q}_T), \quad \beta\in(0,1). \tag{3.19}$$

We will discuss (2.5) which is the corresponding form of (3.18). It follows from (2.1) and (3.14) that $C_1 - C_2 v \in L^q(Q_T)$, $\forall q \in (\frac{n+2}{2}, \frac{2(n+1)}{n-2})$. From (3.19), we obtain $d_1 + \alpha_{11}u \in C^{\beta,\frac{\beta}{2}}(\overline{Q}_T)$. Thus, according to the parabolic regularity result of [5, pp.341-342, Theorem 9.1], we can conclude that

$$X\in W_q^{2,1}(Q_T), \quad \forall q\in\left(\frac{n+2}{2},\frac{2(n+1)}{n-2}\right), \tag{3.20}$$

which implies that $\nabla X \in L^{\frac{(n+2)q}{n+2-q}}(Q_T)$ by Lemma 2.6.

Since $X = (d_1 + 2\alpha_{11}u)u$, we have $\nabla u = (d_1 + 2\alpha_{11}u)^{-1}\nabla X$, *i.e.*, $\nabla u \in L^{\frac{(n+2)q}{n+2-q}}(Q_T)$. It means that $|\nabla u|^2, |\nabla v|^2 \in L^{\frac{(n+2)q}{2(n+2-q)}}(Q_T)$. So, $\sum_{i=1}^{n} a_i^2(x,t) \in L^{\frac{(n+2)q}{2(n+2-q)}}(Q_T)$. From (2.1) and (3.14), $v(-r+\frac{c\beta mu}{1+amu}) \in L^q(Q_T)$.

Then the condition (3) and (3.17) are satisfied by choosing $p = r = \frac{(n+2)p}{2(n+2-p)}$. According to the maximum principle [13, p.181, Theorem 7.1], we can conclude that $v \in L^\infty(Q_T)$. From (2.1), there exists a positive constant M_5 such that

$$\|u\|_{L^\infty(Q_T)}, \|v\|_{L^\infty(Q_T)} \le M_5, \quad \forall T>0. \tag{3.21}$$

Therefore, the global solution to the problem (1.1) exists.

(iii) The existence of classical solutions.

Under the conditions of Theorem 3.1, we consider above global solutions (u,v) to be classical. By (3.20) and Lemma 2.6, we know $\nabla X \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$, $\forall\alpha \in (0,1)$. It follows from Lemma 3.3 in [13] that $X \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}_T)$. Since $X = (d_1 + \alpha_{11}u)u$, we have $u = \frac{-d_1\sqrt{d_1^2+4\alpha_{11}X}}{2\alpha_{11}}$.

So,

$$u \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}_T), \quad \forall \alpha \in (0,1). \tag{3.22}$$

Rewrite the second equation of system (1.1) as

$$v_t = \nabla \cdot \left[(d_2 + \alpha_{21}u + 2\alpha_{22}v)\nabla v + \alpha_{21}\nabla uv\right] + v\left(-r + \frac{c\beta mu}{1+amu}\right).$$

Therefore, we can conclude that $v(-r + \frac{c\beta mu}{1+amu}) \in L^\infty(Q_T)$, u, v, ∇u and ∇v are all bounded. By the Schauder estimate [13], there exists $\alpha^* \in (0,1)$ such that

$$v \in C^{\alpha^*,\frac{\alpha^*}{2}}(\overline{Q}_T). \tag{3.23}$$

Furthermore, by the Schauder estimate, we obtain

$$u \in C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{Q}_T), \quad \sigma = \min\{\alpha, \alpha^*\}. \tag{3.24}$$

Next, the regularity of v will be discussed. Set $\bar{v} = (d_2 + \alpha_{21}u + \alpha_{22}v)v$. So, $\bar{v}$ satisfies

$$\bar{v}_t = (d_2 + \alpha_{21}u + 2\alpha_{22}v)\Delta\bar{v} + f(x,t), \tag{3.25}$$

where $f(x,t) = (d_2 + \alpha_{21}u + 2\alpha_{22}v)v(-r + \frac{c\beta mu}{1+amu}) + \alpha_{21}u_t v$. According to (3.22) to (3.24), we have $d_2 + \alpha_{21}u + 2\alpha_{22}v, f(x,t) \in C^{\sigma,\frac{\sigma}{2}}(\overline{Q}_T)$. Applying the Schauder estimate to (3.25), we know

$$\bar{v} \in C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{Q}_T).$$

From $v = \frac{-(d_3+\alpha_{21}u)+\sqrt{(d_2+\alpha_{21}u)^2+4\alpha_{22}\bar{v}}}{2\alpha_{22}}$, we can see

$$v \in C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{Q}_T), \quad \sigma = \min\{\alpha, \alpha^*\}. \tag{3.26}$$

Combining (3.24) and (3.26), we get

$$u, v \in C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{Q}_T).$$

Therefore, the result of Theorem 3.1 can be obtained for $\alpha < \alpha^*$, namely $\sigma = \alpha$. When $\alpha > \alpha^*$, namely $\sigma < \alpha$, we have $C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{Q}_T) \hookrightarrow C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$. (3.24) and (3.26) are obtained by repeating the above bootstrap argument and the Schauder estimate. This completes the proof of Theorem 3.1. □

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The work was realized by the author.

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