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Fixed point theorems for the functions having monotone property or comparable property in the product spaces

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Abstract
The main aim of this paper is to study and establish some new coincidence point and common fixed point theorems in the product space of mixed-monotonically complete quasi-ordered metric space. Especially, we shall study the fixed points of functions having the monotone property or the comparable property in the product space of quasi-ordered metric space. An interesting application is to investigate the existence and uniqueness of a solution for the system of integral equations.

MSC: 47H10; 54H25

Keywords: function of contractive factor; coincidence point; system of integral equations

1 Introduction

The existence of coincidence point has been studied in [1–4] and the references therein. Also, the existence of common fixed point has been studied in [5–15] and the references therein. In this paper, we shall introduce the concept of mixed-monotonically complete quasi-ordered metric space, and establish some new coincidence point and common fixed point theorems in the product space of those quasi-ordered metric spaces. We shall also present the interesting applications to the existence and uniqueness of solution for system of integral equations.

In Section 2, we shall derive the coincidence point theorems in the product space of mixed-monotonically complete quasi-ordered metric space. In Section 3, we shall study the fixed point theorems for the functions having mixed-monotone property in the product space of monotonically complete quasi-ordered metric space. Also, in Section 4, the fixed point theorems for the functions having the comparable property in the product space of mixed-monotonically complete quasi-ordered metric space will be derived. Finally, in Section 5, we shall present the interesting application to investigate the existence and uniqueness of solutions for the system of integral equations.

2 Coincidence point theorems in product spaces

Let X be a nonempty set. We consider the product set

$$X^m = \underbrace{X \times \cdots \times X}_{m\ \text{times}}.$$

The element of X^m is represented by the vectorial notation $\mathbf{x} = (x^{(1)}, \ldots, x^{(m)})$, where $x^{(i)} \in X$ for $i = 1, \ldots, m$. We also consider the function $\mathbf{F} : X^m \to X^m$ defined by

$$\mathbf{F}(\mathbf{x}) = \bigl(F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_m(\mathbf{x})\bigr),$$

where $F_k : X^m \to X$ for all $k = 1, 2, \ldots, m$. The vectorial element $\widehat{\mathbf{x}} = (\widehat{x}^{(1)}, \widehat{x}^{(2)}, \ldots, \widehat{x}^{(m)}) \in X^m$ is a *fixed point* of $\mathbf{F}$ if and only if $\mathbf{F}(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$; that is,

$$F_k\bigl(\widehat{x}^{(1)}, \widehat{x}^{(2)}, \ldots, \widehat{x}^{(m)}\bigr) = \widehat{x}^{(k)}$$

for all $k = 1, 2, \ldots, m$.

Definition 2.1 Let X be a nonempty set. Consider the functions $\mathbf{F} : X^m \to X^m$ and $\mathbf{f} : X^m \to X^m$ by $\mathbf{F} = (F_1, F_2, \ldots, F_k)$ and $\mathbf{f} = (f_1, f_2, \ldots, f_k)$, where $F_k : X^m \to X$ and $f_k : X^m \to X$ for $k = 1, 2, \ldots, m$.

- The element $\widehat{\mathbf{x}} \in X^m$ is a *coincidence point* of $\mathbf{F}$ and $\mathbf{f}$ if and only if $\mathbf{F}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}})$, *i.e.*, $F_k(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{x}})$ for all $k = 1, 2, \ldots, m$.
- The element $\widehat{\mathbf{x}}$ is a *common fixed point* of $\mathbf{F}$ and $\mathbf{f}$ if and only if $\mathbf{F}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$, *i.e.*, $F_k(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{x}}) = \widehat{x}^{(k)}$ for all $k = 1, 2, \ldots, m$.
- The functions $\mathbf{F}$ and $\mathbf{f}$ are said to be *commutative* if and only if $\mathbf{f}(\mathbf{F}(\mathbf{x})) = \mathbf{F}(\mathbf{f}(\mathbf{x}))$ for all $\mathbf{x} \in X^m$.

Let '$\preceq$' be a binary relation defined on X. We say that the binary relation '$\preceq$' is a quasi-order (pre-order or pseudo-order) if and only if it is reflexive and transitive. In this case, $(X, \preceq)$ is called a *quasi-ordered set*.

For any $\mathbf{x}, \mathbf{y} \in X^m$, we say that $\mathbf{x}$ and $\mathbf{y}$ are *$\preceq$-mixed comparable* if and only if, for each $k = 1, \ldots, m$, one has either $x^{(k)} \preceq y^{(k)}$ or $y^{(k)} \preceq x^{(k)}$. Let I be a subset of $\{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, m\} \setminus I$. In this case, we say that I and J are the *disjoint pair* of $\{1, 2, \ldots, m\}$. We can define a binary relation on X^m as follows:

$$\mathbf{x} \preccurlyeq_I \mathbf{y} \quad \text{if and only if} \quad x^{(k)} \preceq y^{(k)} \quad \text{for } k \in I \quad \text{and} \quad y^{(k)} \preceq x^{(k)} \quad \text{for } k \in J. \tag{1}$$

It is obvious that $(X^m, \preccurlyeq_I)$ is a quasi-ordered set that depends on I. We also have

$$\mathbf{x} \preccurlyeq_I \mathbf{y} \quad \text{if and only if} \quad \mathbf{y} \preccurlyeq_J \mathbf{x}. \tag{2}$$

We need to mention that I or J is allowed to be empty set.

Remark 2.1 For any $\mathbf{x}, \mathbf{y} \in X^m$, we have the following observations.

- If $\mathbf{x} \preccurlyeq_I \mathbf{y}$ for some disjoint pair I and J of $\{1, \ldots, m\}$, then $\mathbf{x}$ and $\mathbf{y}$ are $\preceq$-mixed comparable.
- If $\mathbf{x}$ and $\mathbf{y}$ are $\preceq$-mixed comparable, then there exists a disjoint pair I and J of $\{1, \ldots, m\}$ such that $\mathbf{x} \preccurlyeq_I \mathbf{y}$.

Definition 2.2 Let I and J be a disjoint pair of $\{1, 2, \ldots, m\}$. Given a quasi-ordered set $(X, \preceq)$, we consider the quasi-ordered set $(X^m, \preccurlyeq_I)$ defined in (1).

- The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be a *mixed* $\preceq$*-monotone sequence* if and only if $x_n \preceq x_{n+1}$ or $x_{n+1} \preceq x_n$ (*i.e.*, x_n and x_{n+1} are comparable with respect to '$\preceq$') for all $n \in \mathbb{N}$.
- The sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m is said to be a *mixed* $\preceq$*-monotone sequence* if and only if each sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ in X is a mixed $\preceq$-monotone sequence for all $k = 1,\ldots,m$.
- The sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m is said to be a *mixed* $\preccurlyeq_I$*-monotone sequence* if and only if $\mathbf{x}_n \preccurlyeq_I \mathbf{x}_{n+1}$ or $\mathbf{x}_{n+1} \preccurlyeq_I \mathbf{x}_n$ (*i.e.*, $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ are comparable with respect to '$\preccurlyeq_I$') for all $n \in \mathbb{N}$.

Remark 2.2 Let I and J be a disjoint pair of $\{1,2,\ldots,m\}$. We have the following observations.

(a) $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m is a mixed $\preccurlyeq_I$-monotone sequence if and only if it is a mixed $\preccurlyeq_J$-monotone sequence.

(b) If $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m is a mixed $\preccurlyeq_I$-monotone sequence, then it is also a mixed $\preceq$-monotone sequence; that is, each sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ in X is a mixed $\preceq$-monotone sequence for all $k = 1,\ldots,m$.

(c) If $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m is a mixed $\preceq$-monotone sequence, then, given any $n \in \mathbb{N}$, there exists a disjoint pair of I_n and J_n (which depends on n) of $\{1,\ldots,m\}$ such that $\mathbf{x}_n \preccurlyeq_{I_n} \mathbf{x}_{n+1}$ or $\mathbf{x}_{n+1} \preccurlyeq_{I_n} \mathbf{x}_n$.

(d) $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m is a mixed $\preceq$-monotone sequence if and only if, for each $n \in \mathbb{N}$, $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ are $\preceq$-mixed comparable.

Definition 2.3 Let I and J be a disjoint pair of $\{1,2,\ldots,m\}$. Given a quasi-ordered set $(X,\preceq)$, we also consider the quasi-ordered set $(X^m,\preccurlyeq_I)$ defined in (1), and the function $\mathbf{f}:(X^m,\preccurlyeq_I) \to (X^m,\preccurlyeq_I)$.

- The function $\mathbf{f}$ is said to have the *sequentially mixed* $\preceq$*-monotone property* if and only if, given any mixed $\preceq$-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m, $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is also a mixed $\preceq$-monotone sequence.
- The function $\mathbf{f}$ is said to have the *sequentially mixed* $\preccurlyeq_I$*-monotone property* if and only if, given any mixed $\preccurlyeq_I$-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m, $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is also a mixed $\preccurlyeq_I$-monotone sequence.

It is obvious that the identity function on X^m has the sequentially mixed $\preccurlyeq_I$-monotone and $\preceq$-monotone property.

Let X be a nonempty set. We consider the functions $\mathbf{F}:X^m \to X^m$ and $\mathbf{f}:X^m \to X^m$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$, where $\mathbf{F}^p(\mathbf{x}) = \mathbf{F}(\mathbf{F}^{p-1}(\mathbf{x}))$ for any $\mathbf{x} \in X^m$. Therefore, we have $F_k^p(\mathbf{x}) = F_k(\mathbf{F}^{p-1}(\mathbf{x}))$ for $k = 1,\ldots,m$. Given an initial element $\mathbf{x}_0 = (x_0^{(1)},x_0^{(2)},\ldots,x_0^{(m)}) \in X^m$, where $x_0^{(k)} \in X$ for $k = 1,\ldots,m$, since $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$, there exists $\mathbf{x}_1 \in X^m$ such that $\mathbf{f}(\mathbf{x}_1) = \mathbf{F}^p(\mathbf{x}_0)$. Similarly, there also exists $\mathbf{x}_2 \in X^m$ such that $\mathbf{f}(\mathbf{x}_2) = \mathbf{F}^p(\mathbf{x}_1)$. Continuing this process, we can construct a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ such that

$$\mathbf{f}(\mathbf{x}_n) = \mathbf{F}^p(\mathbf{x}_{n-1}) \tag{3}$$

for all $n \in \mathbb{N}$; that is,

$$f_k(\mathbf{x}_n) = f_k\bigl(x_n^{(1)},\ldots,x_n^{(k)},\ldots,x_n^{(m)}\bigr) = F_k^p\bigl(x_{n-1}^{(1)},\ldots,x_{n-1}^{(k)},\ldots,x_{n-1}^{(m)}\bigr) = F_k^p(\mathbf{x}_{n-1})$$

for all $k = 1,\ldots,m$. We introduce the concepts of mixed-monotone seed elements as follows.

(A) The initial element $\mathbf{x}_0$ is said to be a *mixed $\preceq$-monotone seed element* of X^m if and only if the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (3) is a mixed $\preceq$-monotone sequence; that is, each sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ in X is a mixed $\preceq$-monotone sequence for $k = 1,\ldots,m$.

(B) Given a disjoint pair I and J of $\{1,2,\ldots,m\}$, we say that the initial element $\mathbf{x}_0$ is a *mixed $\preccurlyeq_I$-monotone seed element* of X^m if and only if the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (3) is a mixed $\preccurlyeq_I$-monotone sequence.

From observation (b) of Remark 2.2, it follows that if $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element, then it is also a mixed $\preceq$-monotone seed element.

Example 2.1 Suppose that the initial element $\mathbf{x}_0$ can generate a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ such that, for each $k = 1,\ldots,m$, the generated sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ is either $\preceq$-increasing or $\preceq$-decreasing. In this case, we define the disjoint pair I and J of $\{1,2,\ldots,m\}$ as follows:

$$I = \bigl\{k : \text{the sequence } \bigl\{x_n^{(k)}\bigr\}_{n\in\mathbb{N}} \text{ is } \preceq\text{-increasing}\bigr\} \quad \text{and} \quad J = \{1,2,\ldots,m\} \setminus I. \tag{4}$$

It means that if $k \in J$, then the sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ is $\preceq$-decreasing. Therefore, the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ satisfies $\mathbf{x}_n \preccurlyeq_I \mathbf{x}_{n+1}$ for any $n \in \mathbb{N}$. In this case, the initial element $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element with the disjoint pair I and J defined in (4).

Definition 2.4 Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order '$\preceq$'. We say that $(X,d,\preceq)$ is *mixed-monotonically complete* if and only if each mixed $\preceq$-monotone Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is convergent.

It is obvious that if the quasi-ordered metric space $(X,d,\preceq)$ is complete, then it is also mixed-monotonically complete. However, the converse is not necessarily true.

For the metric space (X,d), we consider the product metric space $(X^m,\eth)$ in which the metric $\eth$ is defined by

$$\eth(\mathbf{x},\mathbf{y}) = \max_{k=1,\ldots,m}\bigl\{d\bigl(x^{(k)},y^{(k)}\bigr)\bigr\} \tag{5}$$

or

$$\eth(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{m} d\bigl(x^{(k)},y^{(k)}\bigr). \tag{6}$$

Remark 2.3 We can check that the product metric $\eth$ defined in (5) or (6) satisfies the following concepts.

- Given a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m, the following statement holds:

$$\eth(\mathbf{x}_n,\widehat{\mathbf{x}}) \to 0 \quad \text{if and only if} \quad d\bigl(x_n^{(k)},\widehat{x}^{(k)}\bigr) \to 0 \quad \text{for all } k = 1,\ldots,m.$$

- Given any $\epsilon > 0$, there exists a positive constant $\mathfrak{k} > 0$ (which depends on ϵ) such that the following statement holds:

$$\eth(\mathbf{x},\mathbf{y}) < \epsilon \quad \text{if and only if} \quad d\bigl(x^{(k)},y^{(k)}\bigr) < \mathfrak{k}\cdot\epsilon \quad \text{for all } k = 1,\ldots,m.$$

Mizoguchi and Takahashi [16, 17] considered the mapping $\varphi : [0,\infty) \to [0,1)$ that satisfies the following condition:

$$\limsup_{x\to c+} \varphi(x) < 1 \quad \text{for all } c \in [0,\infty), \tag{7}$$

in the contractive inequality, and generalized the Nadler fixed point theorem as shown in [18]. Suzuki [19] also gave a simple proof of the theorem obtained by Mizoguchi and Takahashi [16]. In this paper, we consider the following definition.

Definition 2.5 We say that $\varphi : [0,\infty) \to [0,1)$ is a *function of contractive factor* if and only if, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have

$$0 \le \sup_n \varphi(x_n) < 1. \tag{8}$$

Using the routine arguments, we can show that the function $\varphi : [0,\infty) \to [0,1)$ satisfies (7) if and only if φ is a function of the contractive factor. Throughout this paper, we shall assume that the mapping φ satisfies (8) in order to prove the various types of coincidence and common fixed point theorems in the product space.

Let (X,d) be a metric space, and let $\mathbf{F} : (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ be a function defined on $(X^m,\mathfrak{d})$ into itself. If $\mathbf{F}$ is continuous at $\widehat{\mathbf{x}} \in X^m$, then, given $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbf{x} \in X^m$ with $\mathfrak{d}(\widehat{\mathbf{x}},\mathbf{x}) < \delta$ implies $\mathfrak{d}(\mathbf{F}(\widehat{\mathbf{x}}),\mathbf{F}(\mathbf{x})) < \epsilon$. From Remark 2.3, we see that $\mathbf{F}$ is continuous at $\widehat{\mathbf{x}} \in X^m$ if and only if each F_k is continuous at $\widehat{\mathbf{x}}$ for $k = 1,\ldots,m$. Next, we propose another concept of continuity.

Definition 2.6 Let (X,d) be a metric space, and let $(X^m,\mathfrak{d})$ be the corresponding product metric space. Let $\mathbf{F} : (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ and $\mathbf{f} : (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ be functions defined on $(X^m,\mathfrak{d})$ into itself. We say that $\mathbf{F}$ is *continuous with respect to* $\mathbf{f}$ at $\widehat{\mathbf{x}} \in X^m$ if and only if, given any $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbf{x} \in X^m$ with $\mathfrak{d}(\widehat{\mathbf{x}},\mathbf{f}(\mathbf{x})) < \delta$ implies $\mathfrak{d}(\mathbf{F}(\widehat{\mathbf{x}}),\mathbf{F}(\mathbf{x})) < \epsilon$. We say that $\mathbf{F}$ is continuous with respect to $\mathbf{f}$ on X^m if and only if it is continuous with respect to $\mathbf{f}$ at each $\widehat{\mathbf{x}} \in X^m$.

It is obvious that if the function $\mathbf{F}$ is continuous at $\widehat{\mathbf{x}}$ with respect to the identity function, then it is also continuous at $\widehat{\mathbf{x}}$.

Proposition 2.1 *The function* $\mathbf{F}$ *is continuous with respect to* $\mathbf{f}$ *at* $\widehat{\mathbf{x}} \in X^m$ *if and only if, given any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $\mathbf{x} \in X^m$ *with* $d(\widehat{x}^{(k)},f_k(\mathbf{x})) < \delta$ *for all* $k = 1,\ldots,m$ *imply* $d(F_k(\widehat{\mathbf{x}}),F_k(\mathbf{x})) < \epsilon$ *for all* $k = 1,\ldots,m$.

Theorem 2.1 *Suppose that the quasi-ordered metric space* $(X,d,\preceq)$ *is mixed-monotonically complete. Consider the functions* $\mathbf{F} : (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ *and* $\mathbf{f} : (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ *satisfying* $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ *for some* $p \in \mathbb{N}$. *Let* $\mathbf{x}_0$ *be a mixed* $\preceq$*-monotone seed element in* X^m. *Assume that the functions* $\mathbf{F}$ *and* $\mathbf{f}$ *satisfy the following conditions*:

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative*;
- $\mathbf{f}$ *has the sequentially mixed* $\preceq$*-monotone property*;
- $\mathbf{F}^p$ *is continuous with respect to* $\mathbf{f}$ *on* X^m;
- *each* f_k *is continuous on* X^m *for* $k = 1,\ldots,m$.

Suppose that there exist a function $\rho : X \times X \to \mathbb{R}_+$ *and a function of the contractive factor* $\varphi : [0,\infty) \to [0,1)$ *such that, for any two* $\preceq$*-mixed comparable elements* $\mathbf{x}$ *and* $\mathbf{y}$ *in* X^m, *the* *inequalities*

$$\rho\big(x^{(k)}, y^{(k)}\big) \le d\big(x^{(k)}, y^{(k)}\big) \tag{9}$$

and

$$d\big(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\big) \le \varphi\big(\rho\big(f_k(\mathbf{x}), f_k(\mathbf{y})\big)\big) \cdot \rho\big(f_k(\mathbf{x}), f_k(\mathbf{y})\big) \tag{10}$$

are satisfied for all $k = 1, \ldots, m$. Then $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}}$ such that each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed in (3) *for all $k = 1, \ldots, m$.*

Proof We consider the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (3). Since $\mathbf{x}_0$ is a mixed $\preceq$-monotone seed element in X^m, *i.e.*, $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone sequence, from observation (d) of Remark 2.2, it follows that, for each $n \in \mathbb{N}$, $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ are $\preceq$-mixed comparable. According to the inequalities (10), we obtain

$$\begin{aligned} d\big(f_k(\mathbf{x}_{n+1}), f_k(\mathbf{x}_n)\big) &= d\big(F_k^p(\mathbf{x}_n), F_k^p(\mathbf{x}_{n-1})\big) \\ &\le \varphi\big(\rho\big(f_k(\mathbf{x}_n), f_k(\mathbf{x}_{n-1})\big)\big) \cdot \rho\big(f_k(\mathbf{x}_n), f_k(\mathbf{x}_{n-1})\big). \end{aligned} \tag{11}$$

Since $\mathbf{f}$ has the sequentially mixed $\preceq$-monotone property, we see that $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone sequence. From observation (d) of Remark 2.2, it follows that, for each $n \in \mathbb{N}$, $\mathbf{f}(\mathbf{x}_n)$ and $\mathbf{f}(\mathbf{x}_{n+1})$ are $\preceq$-mixed comparable. Let

$$\xi_n = \rho\big(f_k(\mathbf{x}_n), f_k(\mathbf{x}_{n-1})\big).$$

Then, using (9) and (11), we obtain

$$\xi_{n+1} = \rho\big(f_k(\mathbf{x}_{n+1}), f_k(\mathbf{x}_n)\big)\big) \le d\big(f_k(\mathbf{x}_{n+1}), f_k(\mathbf{x}_n)\big) \le \varphi(\xi_n) \cdot \xi_n < \xi_n, \tag{12}$$

which also says that the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is strictly decreasing. Let $0 < \gamma = \sup_n \varphi(\xi_n) < 1$. From (12), it follows that

$$d\big(f_k(\mathbf{x}_{n+1}), f_k(\mathbf{x}_n)\big) \le \gamma \cdot \xi_n \quad \text{and} \quad \xi_{n+1} \le \gamma \cdot \xi_n,$$

which implies

$$d\big(f_k(\mathbf{x}_{n+1}), f_k(\mathbf{x}_n)\big) \le \gamma^n \cdot \xi_1. \tag{13}$$

For $n_1, n_2 \in \mathbb{N}$ with $n_1 > n_2$, since $0 < \gamma < 1$, from (13), it follows that

$$\begin{aligned} d\big(f_k(\mathbf{x}_{n_1}), f_k(\mathbf{x}_{n_2})\big) &\le \sum_{j=n_2}^{n_1-1} d\big(f_k(\mathbf{x}_{j+1}), f_k(\mathbf{x}_j)\big) \quad \text{(by the triangle inequality)} \\ &\le \xi_1 \cdot \sum_{j=n_2}^{n_1-1} \gamma^j \quad \big(\text{by (13)}\big) \\ &\le \frac{\xi_1 \cdot \gamma^{n_2} \cdot (1-\gamma^{n_1-n_2})}{1-\gamma} < \frac{\xi_1 \cdot \gamma^{n_2}}{1-\gamma} \to 0 \quad \text{as } n_2 \to \infty, \end{aligned}$$

which also says that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X for any fixed k. Since $\mathbf{f}$ has the sequentially mixed $\preceq$-monotone property, *i.e.*, $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone Cauchy sequence for $k = 1,\ldots,m$, by the mixed $\preceq$-monotone completeness of X, there exists $\widehat{x}^{(k)} \in X$ such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ as $n \to \infty$ for $k = 1,\ldots,m$. By Remark 2.3, it follows that $\mathbf{f}(\mathbf{x}_n) \to \widehat{\mathbf{x}}$ as $n \to \infty$. Since each f_k is continuous on X^m, we also have

$$f_k\big(\mathbf{f}(\mathbf{x}_n)\big) \to f_k(\widehat{\mathbf{x}}) \quad \text{as } n \to \infty.$$

Since $\mathbf{F}^p$ is continuous with respect to $\mathbf{f}$ on X^m, by Proposition 2.1, given any $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbf{x} \in X^m$ with $d(\widehat{x}^{(k)}, f_k(\mathbf{x})) < \delta$ for all $k = 1,\ldots,m$ imply

$$d\big(F_k^p(\widehat{\mathbf{x}}), F_k^p(\mathbf{x})\big) < \frac{\epsilon}{2} \quad \text{for all } k = 1,\ldots,m. \tag{14}$$

Since $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ as $n \to \infty$ for all $k = 1,\ldots,m$, given $\zeta = \min\{\epsilon/2, \delta\} > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d\big(f_k(\mathbf{x}_n), \widehat{x}^{(k)}\big) < \zeta \leq \delta \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq n_0 \text{ and for all } k = 1,\ldots,m. \tag{15}$$

For each $n \geq n_0$, by (14) and (15), it follows that

$$d\big(F_k^p(\widehat{\mathbf{x}}), F_k^p(\mathbf{x}_n)\big) < \frac{\epsilon}{2} \quad \text{for all } k = 1,\ldots,m. \tag{16}$$

Therefore, we obtain

$$\begin{aligned} d\big(F_k^p(\widehat{\mathbf{x}}), \widehat{x}^{(k)}\big) &\leq d\big(F_k^p(\widehat{\mathbf{x}}), f_k(\mathbf{x}_{n_0+1})\big) + d\big(f_k(\mathbf{x}_{n_0+1}), \widehat{x}^{(k)}\big) \\ &= d\big(F_k^p(\widehat{\mathbf{x}}), F_k^p(\mathbf{x}_{n_0})\big) + d\big(f_k(\mathbf{x}_{n_0+1}), \widehat{x}^{(k)}\big) \\ &< \frac{\epsilon}{2} + \zeta \quad \big(\text{by (15) and (16)}\big) \\ &\leq \epsilon \quad \text{for all } k = 1,\ldots,m. \end{aligned}$$

Since ϵ is any positive number, we conclude that $d(F_k^p(\widehat{\mathbf{x}}), \widehat{x}^{(k)}) = 0$ for all $k = 1,\ldots,m$, which also says that $F_k^p(\widehat{\mathbf{x}}) = \widehat{x}^{(k)}$ for all $k = 1,\ldots,m$, *i.e.*, $\mathbf{F}^p(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$. This completes the proof. □

Remark 2.4 We have the following observations.

- In Theorem 2.1, if we assume that the quasi-ordered metric space $(X, d, \preceq)$ is complete (not mixed-monotonically complete), then the assumption for $\mathbf{f}$ having the sequentially mixed $\preceq$-monotone property can be dropped, since the proof is still valid in this case.
- The assumptions for the inequalities (9) and (10) are weak, since we just assume that it is satisfied for $\preceq$-mixed comparable elements. In other words, if $\mathbf{x}$ and $\mathbf{y}$ are not $\preceq$-mixed comparable, we do not need to check the inequalities (9) and (10).

In Theorem 2.1, we can consider a different function ρ that is defined on $X^m \times X^m$ instead of $X \times X$. Then we can have the following result.

Theorem 2.2 *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is mixed-monotonically complete. Consider the functions $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ and $\mathbf{f} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preceq$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions*:

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative*;
- $\mathbf{f}$ *has the sequentially mixed $\preceq$-monotone property*;
- $\mathbf{F}^p$ *is continuous with respect to* $\mathbf{f}$ *on X^m*;
- *each f_k is continuous on X^m for $k = 1, \ldots, m$.*

Suppose that there exist a function $\rho : X^m \times X^m \to \mathbb{R}_+$ and a function of the contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any two $\preceq$-mixed comparable elements $\mathbf{x}$ and $\mathbf{y}$ in X^m, the inequalities

$$\rho(\mathbf{x}, \mathbf{y}) \le d\left(x^{(k)}, y^{(k)}\right) \tag{17}$$

and

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \le \varphi\left(\rho\left(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})\right)\right) \cdot \rho\left(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})\right) \tag{18}$$

are satisfied for all $k = 1, \ldots, m$. Then $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}}$ such that each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed in (3) *for all $k = 1, \ldots, m$.*

Proof Using a similar argument to the proof of Theorem 2.1, we can obtain the desired results. □

By considering the mixed $\preccurlyeq_I$-monotone seed element instead of mixed $\preceq$-monotone seed element, the assumptions for the inequalities (9) and (10) can be weaken, which is shown below.

Theorem 2.3 *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is mixed-monotonically complete. Consider the functions $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ and $\mathbf{f} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preccurlyeq_I$-monotone seed element in X^m, and let $(X^m, \preccurlyeq_{(\mathbf{f},\mathbf{F},\mathbf{x}_0)}) \equiv (X^m, \preccurlyeq_I)$ be a quasi-ordered set induced by $(\mathbf{f}, \mathbf{F}, \mathbf{x}_0)$. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions*:

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative*;
- $\mathbf{f}$ *has the sequentially mixed $\preccurlyeq_I$-monotone property or the sequentially mixed $\preceq$-monotone property*;
- $\mathbf{F}^p$ *is continuous with respect to* $\mathbf{f}$ *on X^m*;
- *each f_k is continuous on X^m for $k = 1, \ldots, m$.*

Suppose that there exist a function $\rho : X \times X \to \mathbb{R}_+$ and a function of the contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any $\mathbf{x}, \mathbf{y} \in X^m$ with $\mathbf{y} \preccurlyeq_I \mathbf{x}$ or $\mathbf{x} \preccurlyeq_I \mathbf{y}$, the inequalities

$$\rho\left(x^{(k)}, y^{(k)}\right) \le d\left(x^{(k)}, y^{(k)}\right) \tag{19}$$

and

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \le \varphi\left(\rho\left(f_k(\mathbf{x}), f_k(\mathbf{y})\right)\right) \cdot \rho\left(f_k(\mathbf{x}), f_k(\mathbf{y})\right) \tag{20}$$

are satisfied for all $k = 1,\dots,m$. Then $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}}$ such that each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed in (3) *for all $k = 1,\dots,m$.*

Proof We consider the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (3). Since $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element in X^m, it follows that $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a mixed $\preccurlyeq_I$-monotone sequence, *i.e.*, for each $n \in \mathbb{N}$, $\mathbf{x}_{n-1} \preccurlyeq_I \mathbf{x}_n$ or $\mathbf{x}_n \preccurlyeq_I \mathbf{x}_{n-1}$. According to the inequalities (20), we obtain

$$\begin{aligned} d\bigl(f_k(\mathbf{x}_{n+1}),f_k(\mathbf{x}_n)\bigr) &= d\bigl(F_k^p(\mathbf{x}_n),F_k^p(\mathbf{x}_{n-1})\bigr) \\ &\leq \varphi\bigl(\rho\bigl(f_k(\mathbf{x}_n),f_k(\mathbf{x}_{n-1})\bigr)\bigr)\cdot\rho\bigl(f_k(\mathbf{x}_n),f_k(\mathbf{x}_{n-1})\bigr). \end{aligned}$$

Using the argument in the proof of Theorem 2.1, we can show that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X for any fixed k. Now, we consider the following cases.

- Suppose that $\mathbf{f}$ has the sequentially mixed $\preccurlyeq_I$-monotone property. We see that $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preccurlyeq_I$-monotone sequence; that is, for each $n \in \mathbb{N}$, $\mathbf{f}(\mathbf{x}_n) \preccurlyeq_I \mathbf{f}(\mathbf{x}_{n+1})$ or $\mathbf{f}(\mathbf{x}_{n+1}) \preccurlyeq_I \mathbf{f}(\mathbf{x}_n)$. Since $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X for any fixed k, from observation (b) of Remark 2.2, we also see that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone Cauchy sequence for $k = 1,\dots,m$.
- Suppose that $\mathbf{f}$ has the sequentially mixed $\preceq$-monotone property. Since $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a mixed $\preccurlyeq_I$-monotone sequence, by part (b) of Remark 2.2, it follows that $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ in X is a mixed $\preceq$-monotone sequence for all $k = 1,\dots,m$. Therefore, we see that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone Cauchy sequence for $k = 1,\dots,m$.

By the mixed $\preceq$-monotone completeness of X, there exists $\widehat{x}^{(k)} \in X$ such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ as $n \to \infty$ for $k = 1,\dots,m$. The remaining proof follows from the same argument in the proof of Theorem 2.1. This completes the proof. □

Remark 2.5 We have the following observations.

- In Theorem 2.3, if we assume that the quasi-ordered metric space $(X,d,\preceq)$ is complete (not mixed-monotonically complete), then the assumption for $\mathbf{f}$ having the sequentially mixed $\preccurlyeq_I$-monotone can be dropped, since the proof is still valid in this case.
- From the observation (a) of Remark 2.1, we see that the assumptions for the inequalities (19) and (20) are indeed weaken by comparing to the inequalities (9) and (10).
- We can also obtain a similar result when the inequalities (19) and (20) in Theorem 2.3 are replaced by the inequalities (17) and (18), respectively.

Next, we shall study the coincidence point without considering the continuity of $\mathbf{F}^p$. However, we need to introduce the concept of mixed-monotone convergence given below.

Definition 2.7 Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order '$\preceq$'. We say that $(X,d,\preceq)$ preserves the *mixed-monotone convergence* if and only if, for each mixed $\preceq$-monotone sequence $\{x_n\}_{n\in\mathbb{N}}$ that converges to $\widehat{x}$, we have $x_n \preceq \widehat{x}$ or $\widehat{x} \preceq x_n$ for each $n \in \mathbb{N}$.

Remark 2.6 Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order '$\preceq$' and preserve the mixed-monotone convergence. Suppose that $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a sequence in the product space X^m such that each sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone convergence sequence with limit point $\widehat{x}^{(k)}$ for $k = 1,\dots,m$. Then we have the following observations.

(a) For each $n \in \mathbb{N}$, $\mathbf{x}_n$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable.
(b) For each $n \in \mathbb{N}$, there exists a disjoint pair I_n and J_n (which depend on n) of $\{1,\ldots,m\}$ such that $\mathbf{x}_n \preccurlyeq_{I_n} \widehat{\mathbf{x}}$ or $\widehat{\mathbf{x}} \preccurlyeq_{I_n} \mathbf{x}_n$, where I_n or J_n is allowed to be empty set.

Definition 2.8 Let I and J be a disjoint pair of $\{1,2,\ldots,m\}$. Given a quasi-ordered set $(X,\preceq)$, we consider the quasi-ordered set $(X^m,\preccurlyeq_I)$ defined in (1), and the function $\mathbf{f}: X^m \to X^m$.

- The function $\mathbf{f}$ is said to have the *$\preceq$-comparable property* if and only if, given any two $\preceq$-comparable elements $\mathbf{x}$ and $\mathbf{y}$ in X^m, the function values $\mathbf{f}(\mathbf{x})$ and $\mathbf{f}(\mathbf{y})$ are $\preceq$-comparable.
- The function $\mathbf{f}$ is said to have the *$\preccurlyeq_I$-comparable property* if and only if, given any two $\preccurlyeq_I$-comparable elements $\mathbf{x}$ and $\mathbf{y}$ in X^m, the function values $\mathbf{f}(\mathbf{x})$ and $\mathbf{f}(\mathbf{y})$ are $\preccurlyeq_I$-comparable.

Theorem 2.4 *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is mixed-monotonically complete and preserves the mixed-monotone convergence. Consider the functions $\mathbf{F}: (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ and $\mathbf{f}: (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preceq$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions*:

- *$\mathbf{F}$ and $\mathbf{f}$ are commutative*;
- *$\mathbf{f}$ has the $\preceq$-comparable property and the sequentially mixed $\preceq$-monotone property*;
- *each f_k is continuous on X^m for $k = 1,\ldots,m$.*

Suppose that there exist a function $\rho: X \times X \to \mathbb{R}_+$ and a function of the contractive factor $\varphi: [0,\infty) \to [0,1)$ such that, for any two $\preceq$-mixed comparable elements $\mathbf{x}$ and $\mathbf{y}$ in X^m, the inequalities

$$\rho\big(x^{(k)},y^{(k)}\big) \leq d\big(x^{(k)},y^{(k)}\big) \tag{21}$$

and

$$d\big(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\big) \leq \varphi\big(\rho\big(f_k(\mathbf{x}),f_k(\mathbf{y})\big)\big) \cdot \rho\big(f_k(\mathbf{x}),f_k(\mathbf{y})\big) \tag{22}$$

are satisfied for all $k = 1,\ldots,m$. Then the following statements hold true.

(i) *There exists $\widehat{\mathbf{x}} \in X^m$ of $\mathbf{F}$ such that $\mathbf{F}^p(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}})$. If $p = 1$, then $\widehat{\mathbf{x}}$ is a coincidence point of $\mathbf{F}$ and $\mathbf{f}$.*
(ii) *If there exists another $\widehat{\mathbf{y}} \in X^m$ such that $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ are $\preceq$-mixed comparable satisfying $\mathbf{F}^p(\widehat{\mathbf{y}}) = \mathbf{f}(\widehat{\mathbf{y}})$, then $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$.*
(iii) *Suppose that $\widehat{\mathbf{x}}$ is obtained from part* (i). *If $\widehat{\mathbf{x}}$ and $\mathbf{F}(\widehat{\mathbf{x}})$ are $\preceq$-mixed comparable, then $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$ for any $q \in \mathbb{N}$.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed in (3) *for all $k = 1,\ldots,m$.*

Proof From the proof of Theorem 2.1, we can construct a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ and $f_k(\mathbf{f}(\mathbf{x}_n)) \to f_k(\widehat{\mathbf{x}})$ as $n \to \infty$, where $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone sequence for all $k = 1,\ldots,n$. Since $f_k(\mathbf{f}(\mathbf{x}_n)) \to f_k(\widehat{\mathbf{x}})$ as $n \to \infty$, given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d\big(f_k\big(\mathbf{f}(\mathbf{x}_n)\big),f_k(\widehat{\mathbf{x}})\big) < \frac{\epsilon}{2} \tag{23}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$ and for all $k = 1, \ldots, m$. Since $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone convergence sequence for all $k = 1, \ldots, n$, from observation (a) of Remark 2.6, we see that $\mathbf{f}(\mathbf{x}_n)$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable for each $n \in \mathbb{N}$. Since $\mathbf{f}$ has the $\preceq$-comparable property, it follows that $\mathbf{f}(\mathbf{f}(\mathbf{x}_n))$ and $\mathbf{f}(\widehat{\mathbf{x}})$ are $\preceq$-mixed comparable. For each $n \geq n_0$, we have

$$\begin{aligned} d\big(F_k^p(\widehat{\mathbf{x}}), F_k^p\big(\mathbf{f}(\mathbf{x}_n)\big)\big) &\leq \varphi\big(\rho\big(f_k(\widehat{\mathbf{x}}), f_k\big(\mathbf{f}(\mathbf{x}_n)\big)\big)\big) \cdot \rho\big(f_k(\widehat{\mathbf{x}}), f_k\big(\mathbf{f}(\mathbf{x}_n)\big)\big) \quad \big(\text{by (22)}\big) \\ &< \rho\big(f_k(\widehat{\mathbf{x}}), f_k\big(\mathbf{f}(\mathbf{x}_n)\big)\big) \leq d\big(f_k(\widehat{\mathbf{x}}), f_k\big(\mathbf{f}(\mathbf{x}_n)\big)\big) \quad \big(\text{by (21)}\big) \\ &< \frac{\epsilon}{2} \quad \big(\text{by (23)}\big), \end{aligned} \tag{24}$$

Since $\mathbf{F}$ and $\mathbf{f}$ are commutative, we have $\mathbf{f}(\mathbf{F}^p(\mathbf{x})) = \mathbf{F}^p(\mathbf{f}(\mathbf{x}))$ for all $\mathbf{x} \in X^m$, which also implies

$$f_k\big(\mathbf{f}(\mathbf{x}_n)\big) = f_k\big(\mathbf{F}^p(\mathbf{x}_{n-1})\big) = F_k^p\big(\mathbf{f}(\mathbf{x}_{n-1})\big).$$

Now, we obtain

$$\begin{aligned} d\big(F_k^p(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{x}})\big) &\leq d\big(F_k^p(\widehat{\mathbf{x}}), F_k^p\big(\mathbf{f}(\mathbf{x}_{n_0})\big)\big) + d\big(F_k^p\big(\mathbf{f}(\mathbf{x}_{n_0})\big), f_k(\widehat{\mathbf{x}})\big) \\ &= d\big(F_k^p(\widehat{\mathbf{x}}), F_k^p\big(\mathbf{f}(\mathbf{x}_{n_0})\big)\big) + d\big(f_k\big(\mathbf{f}(\mathbf{x}_{n_0+1})\big), f_k(\widehat{\mathbf{x}})\big) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \big(\text{by (23) and (24)}\big) \\ &= \epsilon. \end{aligned}$$

Since ϵ is any positive number, we conclude that $d(F_k^p(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{x}})) = 0$, which says that $F_k^p(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{x}})$ for all $k = 1, \ldots, m$, *i.e.*, $\mathbf{F}^p(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}})$. This proves part (i).

To prove part (ii), since $\mathbf{f}$ has the $\preceq$-comparable property, it follows that $\mathbf{f}(\widehat{\mathbf{x}})$ and $\mathbf{f}(\widehat{\mathbf{y}})$ are $\preceq$-mixed comparable. If $f_k(\widehat{\mathbf{x}}) \neq f_k(\widehat{\mathbf{y}})$, *i.e.*, $d(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})) \neq 0$, then we obtain

$$\begin{aligned} d\big(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\big) = d\big(F_k^p(\widehat{\mathbf{x}}), F_k^p(\widehat{\mathbf{y}})\big) &\leq \varphi\big(\rho\big(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\big)\big) \cdot \rho\big(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\big) \quad \big(\text{by (22)}\big) \\ &< \rho\big(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\big) \leq d\big(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\big) \quad \big(\text{by (21)}\big). \end{aligned}$$

This contradiction says that $f_k(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{y}})$ for all $k = 1, \ldots, m$, *i.e.*, $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$.

To prove part (iii), using the commutativity of $\mathbf{F}$ and $\mathbf{f}$, we have

$$\mathbf{f}\big(\mathbf{F}(\widehat{\mathbf{x}})\big) = \mathbf{F}\big(\mathbf{f}(\widehat{\mathbf{x}})\big) = \mathbf{F}\big(\mathbf{F}^p(\widehat{\mathbf{x}})\big) = \mathbf{F}^p\big(\mathbf{F}(\widehat{\mathbf{x}})\big). \tag{25}$$

By taking $\widehat{\mathbf{y}} = \mathbf{F}(\widehat{\mathbf{x}})$, the equalities (25) says that $\mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}^p(\widehat{\mathbf{y}})$. Since $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}} = \mathbf{F}(\widehat{\mathbf{x}})$ are $\preceq$-mixed comparable by the assumption, part (ii) says that

$$\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{f}\big(\mathbf{F}(\widehat{\mathbf{x}})\big) = \mathbf{F}\big(\mathbf{f}(\widehat{\mathbf{x}})\big),$$

which says that $\mathbf{f}(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$. Given any $q \in \mathbb{N}$, we have

$$\begin{aligned} \mathbf{F}\big(\mathbf{f}^q(\widehat{\mathbf{x}})\big) &= \mathbf{f}^{q-1}\big(\mathbf{F}\big(\mathbf{f}(\widehat{\mathbf{x}})\big)\big) \quad (\text{by the commutativity of } \mathbf{F} \text{ and } \mathbf{f}) \\ &= \mathbf{f}^{q-1}\big(\mathbf{f}(\widehat{\mathbf{x}})\big) = \mathbf{f}^q(\widehat{\mathbf{x}}), \end{aligned}$$

which says that $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$. This completes the proof. $\square$

Remark 2.7 We have the following observations.

- In Theorem 2.4, if we assume that the quasi-ordered metric space $(X, d, \preceq)$ is complete (not mixed-monotonically complete), then the assumption for $\mathbf{f}$ having the sequentially mixed $\preceq$-monotone property can be dropped, since the proof is still valid in this case.
- We can also obtain a similar result when the inequalities (21) and (22) in Theorem 2.4 are replaced by the inequalities (17) and (18), respectively.

Theorem 2.5 *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is mixed-monotonically complete and preserves the mixed-monotone convergence. Consider the functions $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ and $\mathbf{f} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preccurlyeq_I$-monotone seed element in X^m, and let $(X^m, \preccurlyeq_{(\mathbf{f},\mathbf{F},\mathbf{x}_0)}) \equiv (X^m, \preccurlyeq_I)$ be a quasi-ordered set induced by $(\mathbf{f}, \mathbf{F}, \mathbf{x}_0)$. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions*:

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative*;
- $\mathbf{f}$ *has the sequentially mixed $\preccurlyeq_I$-monotone property or the sequentially mixed $\preceq$-monotone property*;
- *each f_k is continuous on X^m for $k = 1, \ldots, m$*;
- $\mathbf{f}$ *has the $\preccurlyeq_{I^\circ}$-comparable property for any disjoint pair I° and J° of $\{1, \ldots, m\}$*.

Suppose that there exist a function $\rho : X \times X \to \mathbb{R}_+$ and a function of the contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any $\mathbf{x}, \mathbf{y} \in X^m$ and any disjoint pair I° and J° of $\{1, \ldots, m\}$ with $\mathbf{y} \preccurlyeq_{I^\circ} \mathbf{x}$ or $\mathbf{x} \preccurlyeq_{I^\circ} \mathbf{y}$, the inequalities

$$\rho\big(x^{(k)}, y^{(k)}\big) \le d\big(x^{(k)}, y^{(k)}\big) \tag{26}$$

and

$$d\big(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\big) \le \varphi\big(\rho\big(f_k(\mathbf{x}), f_k(\mathbf{y})\big)\big) \cdot \rho\big(f_k(\mathbf{x}), f_k(\mathbf{y})\big) \tag{27}$$

are satisfied for all $k = 1, \ldots, m$. Then the following statements hold true.

(i) *There exists $\widehat{\mathbf{x}} \in X^m$ of $\mathbf{F}$ such that $\mathbf{F}^p(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}})$. If $p = 1$, then $\widehat{\mathbf{x}}$ is a coincidence point of $\mathbf{F}$ and $\mathbf{f}$.*

(ii) *If there exist a disjoint pair I° and J° of $\{1, \ldots, m\}$ and another $\widehat{\mathbf{y}} \in X^m$ such that $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ are comparable with respect to the quasi-order '$\preccurlyeq_{I^\circ}$' satisfying $\mathbf{F}^p(\widehat{\mathbf{y}}) = \mathbf{f}(\widehat{\mathbf{y}})$, then $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$.*

(iii) *Suppose that $\widehat{\mathbf{x}}$ is obtained from part* (i). *If there exists a disjoint pair I° and J° of $\{1, \ldots, m\}$ such that $\widehat{\mathbf{x}}$ and $\mathbf{F}(\widehat{\mathbf{x}})$ are comparable with respect to the quasi-order '$\preccurlyeq_{I^\circ}$', then $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$ for any $q \in \mathbb{N}$.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed in (3) *for all $k = 1, \ldots, m$.*

Proof From the proof of Theorem 2.3, we can construct a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ and $f_k(\mathbf{f}(\mathbf{x}_n)) \to f_k(\widehat{\mathbf{x}})$ as $n \to \infty$, where $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone sequence for all $k = 1, \ldots, m$. Since $f_k(\mathbf{f}(\mathbf{x}_n)) \to f_k(\widehat{\mathbf{x}})$ as $n \to \infty$, given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d\big(f_k\big(\mathbf{f}(\mathbf{x}_n)\big), f_k(\widehat{\mathbf{x}})\big) < \frac{\epsilon}{2} \tag{28}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$ and for all $k = 1, \ldots, m$. Since $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone convergent sequence for all $k = 1, \ldots, m$, from observation (b) of Remark 2.6, we see that, for each $n \in \mathbb{N}$, there exists a subset I_n of $\{1, \ldots, m\}$ such that

$$\mathbf{f}(\mathbf{x}_n) \preccurlyeq_{I_n} \widehat{\mathbf{x}} \quad \text{or} \quad \widehat{\mathbf{x}} \preccurlyeq_{I_n} \mathbf{f}(\mathbf{x}_n). \tag{29}$$

Since $\mathbf{f}$ has the $\preccurlyeq_{I^\circ}$-comparable property for any subset I° of $\{1, \ldots, m\}$, it follows that

$$\mathbf{f}\bigl(\mathbf{f}(\mathbf{x}_n)\bigr) \preccurlyeq_{I_n} \mathbf{f}(\widehat{\mathbf{x}}) \quad \text{or} \quad \mathbf{f}(\widehat{\mathbf{x}}) \preccurlyeq_{I_n} \mathbf{f}\bigl(\mathbf{f}(\mathbf{x}_n)\bigr). \tag{30}$$

For each $n \geq n_0$, we obtain

$$\begin{aligned} d\bigl(F_k^p(\widehat{\mathbf{x}}), F_k^p\bigl(\mathbf{f}(\mathbf{x}_n)\bigr)\bigr) &\leq \varphi\bigl(\rho\bigl(f_k(\widehat{\mathbf{x}}), f_k\bigl(\mathbf{f}(\mathbf{x}_n)\bigr)\bigr)\bigr) \cdot \rho\bigl(f_k(\widehat{\mathbf{x}}), f_k\bigl(\mathbf{f}(\mathbf{x}_n)\bigr)\bigr) \quad \bigl(\text{by (29) and (27)}\bigr) \\ &< \rho\bigl(f_k(\widehat{\mathbf{x}}), f_k\bigl(\mathbf{f}(\mathbf{x}_n)\bigr)\bigr) \leq d\bigl(f_k(\widehat{\mathbf{x}}), f_k\bigl(\mathbf{f}(\mathbf{x}_n)\bigr)\bigr) \quad \bigl(\text{by (30) and (26)}\bigr) \\ &< \frac{\epsilon}{2} \quad \bigl(\text{by (28)}\bigr). \end{aligned}$$

Using the same argument in the proof of Theorem 2.4 immediately, we complete the proof. □

Remark 2.8 We have the following observations.

- Suppose that the inequalities (21) and (22) in Theorem 2.4, and that the inequalities (26) and (27) in Theorem 2.5 are satisfied for any $\mathbf{x}, \mathbf{y} \in X^m$. Then, from the proofs of Theorems 2.4 and 2.5, we can see that parts (ii) and (iii) can be changed as follows.

  (ii)′ If there exists another $\widehat{\mathbf{y}} \in X^m$ satisfying $\mathbf{F}^p(\widehat{\mathbf{y}}) = \mathbf{f}(\widehat{\mathbf{y}})$, then $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$.

  (iii)′ Suppose that $\widehat{\mathbf{x}}$ is obtained from part (i). Then $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$ for any $q \in \mathbb{N}$.

- We can also obtain a similar result when the inequalities (26) and (27) in Theorem 2.5 are replaced by the inequalities (17) and (18), respectively.

Next, we shall consider the uniqueness for a common fixed point in the $\preceq$-mixed comparable sense.

Definition 2.9 Let $(X, \preceq)$ be a quasi-order set. Consider the functions $\mathbf{F} : X^m \to X^m$ and $\mathbf{f} : X^m \to X^m$ defined on the product set X^m into itself. The common fixed point $\widehat{\mathbf{x}} \in X^m$ of $\mathbf{F}$ and $\mathbf{f}$ is *unique in the $\preceq$-mixed comparable sense* if and only if, for any other common fixed point $\mathbf{x}$ of $\mathbf{F}$ and $\mathbf{f}$, if $\mathbf{x}$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable, then $\mathbf{x} = \widehat{\mathbf{x}}$.

Theorem 2.6 *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is mixed-monotonically complete and preserves the mixed-monotone convergence. Consider the functions $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ and $\mathbf{f} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preceq$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions*:

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative*;
- $\mathbf{f}$ *has the $\preceq$-comparable property and the sequentially mixed $\preceq$-monotone property*;
- $\mathbf{F}^p$ *is continuous with respect to* $\mathbf{f}$ *on* X^m;
- *each f_k is continuous on X^m for $k = 1, \ldots, m$.*

Suppose that there exist a function $\rho : X \times X \to \mathbb{R}_+$ *and a function of the contractive factor* $\varphi : [0,\infty) \to [0,1)$ *such that, for any two* $\preceq$*-mixed comparable elements* $\mathbf{x}$ *and* $\mathbf{y}$ *in* X^m*, the inequalities*

$$\rho\bigl(x^{(k)}, y^{(k)}\bigr) \le d\bigl(x^{(k)}, y^{(k)}\bigr) \tag{31}$$

and

$$d\bigl(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\bigr) \le \varphi\bigl(\rho\bigl(f_k(\mathbf{x}), f_k(\mathbf{y})\bigr)\bigr) \cdot \rho\bigl(f_k(\mathbf{x}), f_k(\mathbf{y})\bigr) \tag{32}$$

are satisfied for all $k = 1, \dots, m$*. Then the following statements hold true.*

(i) $\mathbf{F}^p$ *and* $\mathbf{f}$ *have a unique common fixed point* $\widehat{\mathbf{x}}$ *in the* $\preceq$*-mixed comparable sense. Equivalently, if* $\widehat{\mathbf{y}}$ *is another common fixed point of* $\mathbf{F}^p$ *and* $\mathbf{f}$*, and is* $\preceq$*-mixed comparable with* $\widehat{\mathbf{x}}$*, then* $\widehat{\mathbf{y}} = \widehat{\mathbf{x}}$*.*

(ii) *For* $p \neq 1$*, suppose that* $\mathbf{F}(\widehat{\mathbf{x}})$ *and* $\widehat{\mathbf{x}}$ *obtained in* (i) *are* $\preceq$*-mixed comparable. Then* $\mathbf{F}$ *and* $\mathbf{f}$ *have a unique common fixed point* $\widehat{\mathbf{x}}$ *in the* $\preceq$*-mixed comparable sense.*

Moreover, each component $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ *constructed in* (3) *for all* $k = 1, \dots, m$*.*

Proof To prove part (i), from Remark 2.3 and part (i) of Theorem 2.4, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}^p(\widehat{\mathbf{x}})$. From Theorem 2.1, we also have $\mathbf{F}^p(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$. Therefore, we obtain

$$\widehat{\mathbf{x}} = \mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}^p(\widehat{\mathbf{x}}).$$

This shows that $\widehat{\mathbf{x}}$ is a common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$. For the uniqueness in the $\preceq$-mixed comparable sense, let $\widehat{\mathbf{y}}$ be another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$ such that $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable, *i.e.*, $\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}^p(\widehat{\mathbf{y}})$. By part (ii) of Theorem 2.4, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. Therefore, by the triangle inequality, we have

$$d(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \le d\bigl(\widehat{\mathbf{x}}, \mathbf{f}(\widehat{\mathbf{x}})\bigr) + d\bigl(\mathbf{f}(\widehat{\mathbf{x}}), \mathbf{f}(\widehat{\mathbf{y}})\bigr) + d\bigl(\mathbf{f}(\widehat{\mathbf{y}}), \widehat{\mathbf{y}}\bigr) = 0, \tag{33}$$

which says that $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. This proves part (i).

To prove part (ii), since $\mathbf{F}(\widehat{\mathbf{x}})$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable, part (iii) of Theorem 2.4 says that $\mathbf{f}(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$, *i.e.*, $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}(\mathbf{f}(\widehat{\mathbf{x}}))$, which implies $\widehat{\mathbf{x}} = \mathbf{F}(\widehat{\mathbf{x}})$, since $\widehat{\mathbf{x}} = \mathbf{f}(\widehat{\mathbf{x}})$. This shows that $\widehat{\mathbf{x}}$ is a common fixed point of $\mathbf{F}$ and $\mathbf{f}$. For the uniqueness in the $\preceq$-mixed comparable sense, let $\widehat{\mathbf{y}}$ be another common fixed point of $\mathbf{F}$ and $\mathbf{f}$ such that $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable, *i.e.*, $\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}(\widehat{\mathbf{y}})$. Then we have

$$\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}(\widehat{\mathbf{y}}) = \mathbf{F}\bigl(\mathbf{f}(\widehat{\mathbf{y}})\bigr) = \mathbf{F}^2(\widehat{\mathbf{y}}) = \cdots = \mathbf{F}^p(\widehat{\mathbf{y}}).$$

By part (ii) of Theorem 2.4, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. From (33), we can similarly obtain $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. This completes the proof. □

Remark 2.9 We can also obtain a similar result when the inequalities (31) and (32) in Theorem 2.6 are replaced by the inequalities (17) and (18), respectively.

Since we consider a metric space $(X,d,\preceq)$ endowed with a quasi-order '$\preceq$', given any disjoint pair I and J of $\{1,\ldots,p\}$, we can define a quasi-order '$\preccurlyeq_I$' on X^m as given in (1). Now, given any $\mathbf{x}\in X^m$, we define the chain $\mathfrak{C}(\preccurlyeq_I,\mathbf{x})$ containing $\mathbf{x}$ as follows:

$$\begin{aligned}\mathfrak{C}(\preccurlyeq_I,\mathbf{x}) &= \{\mathbf{y}\in X^m : \mathbf{y}\preccurlyeq_I \mathbf{x} \text{ or } \mathbf{x}\preccurlyeq_I \mathbf{y}\}\\ &= \{\mathbf{y}\in X^m : \mathbf{x} \text{ and } \mathbf{y} \text{ are comparable with respect to ` } \preccurlyeq_I \text{ '}\}.\end{aligned}$$

Next, we shall introduce the concept of chain-uniqueness for a common fixed point.

Definition 2.10 Let $(X,\preceq)$ be a quasi-order set. Consider the functions $\mathbf{F}:X^m\to X^m$ and $\mathbf{f}:X^m\to X^m$ defined on the product set X^m into itself. The common fixed point $\widehat{\mathbf{x}}\in X^m$ of $\mathbf{F}$ and $\mathbf{f}$ is called *chain-unique* if and only if, given any other common fixed point $\mathbf{x}$ of $\mathbf{F}$ and $\mathbf{f}$, if $\mathbf{x}\in\mathfrak{C}(\preccurlyeq_{I^\circ},\widehat{\mathbf{x}})$ for some disjoint pair I° and J° of $\{1,\ldots,m\}$, then $\mathbf{x}=\widehat{\mathbf{x}}$.

Theorem 2.7 *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is mixed-monotonically complete and preserves the mixed-monotone convergence. Consider the functions $\mathbf{F}:(X^m,\mathfrak{d})\to(X^m,\mathfrak{d})$ and $\mathbf{f}:(X^m,\mathfrak{d})\to(X^m,\mathfrak{d})$ satisfying $\mathbf{F}^p(X^m)\subseteq\mathbf{f}(X^m)$ for some $p\in\mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preccurlyeq_I$-monotone seed element in X^m, and let $(X^m,\preccurlyeq_{(\mathbf{f},\mathbf{F},\mathbf{x}_0)})\equiv(X^m,\preccurlyeq_I)$ be a quasi-ordered set induced by $(\mathbf{f},\mathbf{F},\mathbf{x}_0)$. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions*:

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative*;
- $\mathbf{f}$ *has the sequentially mixed $\preccurlyeq_I$-monotone property or the $\preceq$-monotone property*;
- $\mathbf{F}^p$ *is continuous with respect to* $\mathbf{f}$ *on X^m*;
- *each f_k is continuous on X^m for $k=1,\ldots,m$.*

Suppose that there exist a function $\rho:X\times X\to\mathbb{R}_+$ and a function of the contractive factor $\varphi:[0,\infty)\to[0,1)$ such that, for any $\mathbf{x},\mathbf{y}\in X^m$ and any disjoint pair I° and J° of $\{1,\ldots,m\}$ with $\mathbf{y}\preccurlyeq_{I^\circ}\mathbf{x}$ or $\mathbf{x}\preccurlyeq_{I^\circ}\mathbf{y}$, the inequalities

$$\rho\bigl(x^{(k)},y^{(k)}\bigr)\le d\bigl(x^{(k)},y^{(k)}\bigr) \tag{34}$$

and

$$d\bigl(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\bigr)\le\varphi\bigl(\rho\bigl(f_k(\mathbf{x}),f_k(\mathbf{y})\bigr)\bigr)\cdot\rho\bigl(f_k(\mathbf{x}),f_k(\mathbf{y})\bigr) \tag{35}$$

are satisfied for all $k=1,\ldots,m$. Then the following statements hold true.

(i) $\mathbf{F}^p$ *and* $\mathbf{f}$ *have a chain-unique common fixed point $\widehat{\mathbf{x}}$. Equivalently, if $\widehat{\mathbf{y}}\in\mathfrak{C}(\preccurlyeq_{I^\circ},\widehat{\mathbf{x}})$ is another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$ for some disjoint pair I° and J° of $\{1,\ldots,m\}$, then $\widehat{\mathbf{y}}=\widehat{\mathbf{x}}$.*

(ii) *For $p\neq 1$, suppose that $\mathbf{F}(\widehat{\mathbf{x}})$ and $\widehat{\mathbf{x}}$ obtained in* (i) *are comparable with respect to the quasi-order '$\preccurlyeq_{I^\circ}$' for some disjoint pair I° and J° of $\{1,\ldots,m\}$. Then $\mathbf{F}$ and $\mathbf{f}$ have a chain-unique common fixed point $\widehat{\mathbf{x}}$.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed in (3) *for all $k=1,\ldots,m$.*

Proof To prove part (i), from Remark 2.3 and part (i) of Theorem 2.5, we can show that $\widehat{\mathbf{x}}$ is a common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$. For the chain-uniqueness, let $\widehat{\mathbf{y}}$ be another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$ with $\widehat{\mathbf{y}} \preccurlyeq_{I^\circ} \widehat{\mathbf{x}}$ or $\widehat{\mathbf{x}} \preccurlyeq_{I^\circ} \widehat{\mathbf{y}}$ for some disjoint pair I° and J° of $\{1,\ldots,m\}$, *i.e.*, $\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}^p(\widehat{\mathbf{y}})$. By part (ii) of Theorem 2.5, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. Therefore, by the triangle inequality, we have

$$d(\widehat{\mathbf{x}},\widehat{\mathbf{y}}) \leq d\bigl(\widehat{\mathbf{x}},\mathbf{f}(\widehat{\mathbf{x}})\bigr) + d\bigl(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}(\widehat{\mathbf{y}})\bigr) + d\bigl(\mathbf{f}(\widehat{\mathbf{y}}),\widehat{\mathbf{y}}\bigr) = 0,$$

which says that $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. This proves part (i). Part (ii) can be similarly obtained by applying Theorem 2.5 to the argument in the proof of part (ii) of Theorem 2.6. This completes the proof. □

Remark 2.10 We have the following observations.

- Suppose that the inequalities (31) and (32) in Theorem 2.6, and that the inequalities (34) and (35) in Theorem 2.7 are satisfied for any $\mathbf{x},\mathbf{y} \in X^m$. Then, from Remark 2.8 and the proofs of Theorems 2.6 and 2.7, we can see that parts (i) and (ii) can be combined together to conclude that $\mathbf{F}$ and $\mathbf{f}$ have a unique common fixed point $\widehat{\mathbf{x}}$.
- We can also obtain a similar result when the inequalities (34) and (35) in Theorem 2.7 are replaced by the inequalities (17) and (18), respectively.

Now, we are going to weaken the concept of mixed-monotone completeness for the quasi-ordered metric space. Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order '$\preceq$'. We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X,\preceq)$ is $\preceq$-increasing if and only if $x_k \preceq x_{k+1}$ for all $k \in \mathbb{N}$. The concept of $\preceq$-decreasing sequence can be similarly defined. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X,\preceq)$ is called *$\preceq$-monotone* if and only if $\{x_n\}_{n\in\mathbb{N}}$ is either $\preceq$-increasing or $\preceq$-decreasing.

Let I and J be a disjoint pair of $\{1,2,\ldots,m\}$. We say that the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in $(X^m,\preccurlyeq_I)$ is $\preccurlyeq_I$-increasing if and only if $\mathbf{x}_n \preccurlyeq_I \mathbf{x}_{n+1}$ for all $n \in \mathbb{N}$. The concept of $\preccurlyeq_I$-decreasing sequence can be similarly defined. The sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in $(X^m,\preccurlyeq_I)$ is called *$\preccurlyeq_I$-monotone* if and only if $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is either $\preccurlyeq_I$-increasing or $\preccurlyeq_I$-decreasing.

Given a disjoint pair I and J of $\{1,2,\ldots,m\}$, let $\mathbf{f} : (X^m,\preccurlyeq_I) \to (X^m,\preccurlyeq_I)$ be a function defined on $(X^m,\preccurlyeq_I)$ into itself. We say that $\mathbf{f}$ is $\preccurlyeq_I$-increasing if and only if $\mathbf{x} \preccurlyeq_I \mathbf{y}$ implies $\mathbf{f}(\mathbf{x}) \preccurlyeq_I \mathbf{f}(\mathbf{y})$. The concept of $\preccurlyeq_I$-decreasing function can be similarly defined. The function $\mathbf{f}$ is called *$\preccurlyeq_I$-monotone* if and only if $\mathbf{f}$ is either $\preccurlyeq_I$-increasing or $\preccurlyeq_I$-decreasing.

In the previous section, we consider the mixed $\preccurlyeq_I$-monotone seed element. Now, we shall consider another concept of seed element. Given a disjoint pair I and J of $\{1,2,\ldots,m\}$, we say that the initial element $\mathbf{x}_0$ is a *$\preccurlyeq_I$-monotone seed element* of X^m if and only if the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (3) is a $\preccurlyeq_I$-monotone sequence. It is obvious that if $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element, then it is also a mixed $\preccurlyeq_I$-monotone seed element.

Definition 2.11 Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order '$\preceq$'. We say that $(X,d,\preceq)$ is *monotonically complete* if and only if each $\preceq$-monotone Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is convergent.

It is obvious that if $(X,d,\preceq)$ is a mixed-monotonically complete quasi-ordered metric space, then it is also a monotonically complete quasi-ordered metric space. However, the converse is not true. In other words, the concept of monotone completeness is weaker than that of mixed-monotone completeness.

Theorem 2.8 *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is monotonically complete. Consider the functions $\mathbf{F}:(X^m,\mathfrak{d})\to(X^m,\mathfrak{d})$ and $\mathbf{f}:(X^m,\mathfrak{d})\to(X^m,\mathfrak{d})$ satisfying $\mathbf{F}^p(X^m)\subseteq\mathbf{f}(X^m)$ for some $p\in\mathbb{N}$. Let $\mathbf{x}_0$ be a $\preccurlyeq_I$-monotone seed element in X^m, and let $(X^m,\preccurlyeq_{(\mathbf{f},\mathbf{F},\mathbf{x}_0)})\equiv(X^m,\preccurlyeq_I)$ be a quasi-ordered set induced by $(\mathbf{f},\mathbf{F},\mathbf{x}_0)$. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions*:

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative*;
- $\mathbf{f}$ *is $\preccurlyeq_I$-monotone*;
- $\mathbf{F}^p$ *is continuous with respect to* $\mathbf{f}$ *on* X^m;
- *each f_k is continuous on X^m for $k=1,\ldots,m$.*

Suppose that there exist a function $\rho:X^2\to\mathbb{R}_+$ and a function of the contractive factor $\varphi:[0,\infty)\to[0,1)$ such that, for any $\mathbf{x},\mathbf{y}\in X^m$ with $\mathbf{y}\preccurlyeq_I\mathbf{x}$ or $\mathbf{x}\preccurlyeq_I\mathbf{y}$, the inequalities

$$\rho\bigl(x^{(k)},y^{(k)}\bigr)\le d\bigl(x^{(k)},y^{(k)}\bigr)\tag{36}$$

and

$$d\bigl(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\bigr)\le\varphi\bigl(\rho\bigl(f_k(\mathbf{x}),f_k(\mathbf{y})\bigr)\bigr)\cdot\rho\bigl(f_k(\mathbf{x}),f_k(\mathbf{y})\bigr)\tag{37}$$

are satisfied for all $k=1,\ldots,m$. Then $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}}$ such that each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed in (3) *for all $k=1,\ldots,m$.*

Proof We consider the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (3). Since $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element in X^m, *i.e.*, $\mathbf{x}_n\preccurlyeq_I\mathbf{x}_{n+1}$ for all $n\in\mathbb{N}$ or $\mathbf{x}_{n+1}\preccurlyeq_I\mathbf{x}_n$ for all $n\in\mathbb{N}$, according to the inequalities (37), we obtain

$$\begin{aligned}d\bigl(f_k(\mathbf{x}_{n+1}),f_k(\mathbf{x}_n)\bigr)&=d\bigl(F_k^p(\mathbf{x}_n),F_k^p(\mathbf{x}_{n-1})\bigr)\\&\le\varphi\bigl(\rho\bigl(f_k(\mathbf{x}_n),f_k(\mathbf{x}_{n-1})\bigr)\bigr)\cdot\rho\bigl(f_k(\mathbf{x}_n),f_k(\mathbf{x}_{n-1})\bigr).\end{aligned}\tag{38}$$

Since $\mathbf{f}$ is $\preccurlyeq_I$-monotone, it follows that $\mathbf{f}(\mathbf{x}_n)\preccurlyeq_I\mathbf{f}(\mathbf{x}_{n+1})$ for all $n\in\mathbb{N}$ or $\mathbf{f}(\mathbf{x}_{n+1})\preccurlyeq_I\mathbf{f}(\mathbf{x}_n)$ for all $n\in\mathbb{N}$. Let

$$\xi_n=\rho\bigl(f_k(\mathbf{x}_n),f_k(\mathbf{x}_{n-1})\bigr).$$

Then, using (36) and (38), we obtain

$$\xi_{n+1}=\rho\bigl(f_k(\mathbf{x}_{n+1}),f_k(\mathbf{x}_n)\bigr)\le d\bigl(f_k(\mathbf{x}_{n+1}),f_k(\mathbf{x}_n)\bigr)\le\varphi(\xi_n)\cdot\xi_n.\tag{39}$$

Let $0<\gamma=\sup_n\varphi(\xi_n)<1$. From (39), it follows that

$$d\bigl(f_k(\mathbf{x}_{n+1}),f_k(\mathbf{x}_n)\bigr)\le\gamma\cdot\xi_n\quad\text{and}\quad\xi_{n+1}\le\gamma\cdot\xi_n,$$

which implies

$$d\bigl(f_k(\mathbf{x}_{n+1}),f_k(\mathbf{x}_n)\bigr)\le\gamma^n\cdot\xi_1.$$

Using the argument in the proof of Theorem 2.1, we can show that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X for any fixed $k=1,\ldots,n$. Since $\mathbf{f}$ is $\preccurlyeq_I$-monotone and $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-monotone sequence, it follows that $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-monotone sequence.

- If $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-increasing sequence, then $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preceq$-increasing Cauchy sequence for $k \in I$, and is a $\preceq$-decreasing Cauchy sequence for $k \in J$.
- If $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-decreasing sequence, then $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preceq$-decreasing Cauchy sequence for $k \in I$, and is a $\preceq$-increasing Cauchy sequence for $k \in J$.

By the monotone completeness of X, there exists $\widehat{x}^{(k)} \in X$ such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ as $n \to \infty$ for $k = 1,\ldots,m$. The remaining proof follows from the same argument in the proof of Theorem 2.1. This completes the proof. □

Remark 2.11 We can also obtain a similar result when the inequalities (36) and (37) in Theorem 2.8 are replaced by the inequalities (17) and (18), respectively.

Next, we shall study the coincidence point without considering the continuity of $\mathbf{F}^p$. However, we need to introduce the concept of monotone convergence given below.

Definition 2.12 Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order '$\preceq$'. We say that $(X,d,\preceq)$ preserves the *monotone convergence* if and only if, for each $\preceq$-monotone sequence $\{x_n\}_{n\in\mathbb{N}}$ that converges to $\widehat{x}$, either one of the following conditions is satisfied:

- if $\{x_n\}_{n\in\mathbb{N}}$ is a $\preceq$-increasing sequence, then $x_n \preceq \widehat{x}$ for each $n \in \mathbb{N}$;
- if $\{x_n\}_{n\in\mathbb{N}}$ is a $\preceq$-decreasing sequence, then $\widehat{x} \preceq x_n$ for each $n \in \mathbb{N}$.

Remark 2.12 Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order '$\preceq$' and preserve the monotone convergence. Given a disjoint pair I and J of $\{1,\ldots,m\}$, suppose that $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-monotone sequence such that each sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ converges to $\widehat{x}^{(k)}$ for $k = 1,\ldots,m$. We consider the following situation.

- If $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-increasing sequence, then $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ is a $\preceq$-increasing sequence for $k \in I$, and is a $\preceq$-decreasing sequence for $k \in J$. By the monotone convergence, we see that, for each $n \in \mathbb{N}$, $x_n^{(k)} \preceq \widehat{x}^{(k)}$ for $k \in I$ and $x_n^{(k)} \succeq \widehat{x}^{(k)}$ for $k \in J$, which shows that $\mathbf{x}_n \preccurlyeq_I \widehat{\mathbf{x}}$ for all $n \in \mathbb{N}$.
- If $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-decreasing sequence, then $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ is a $\preceq$-decreasing sequence for $k \in I$, and is a $\preceq$-increasing sequence for $k \in J$. By the monotone convergence, we see that, for each $n \in \mathbb{N}$, $x_n^{(k)} \succeq \widehat{x}^{(k)}$ for $k \in I$ and $x_n^{(k)} \preceq \widehat{x}^{(k)}$ for $k \in J$, which shows that $\mathbf{x}_n \succcurlyeq_I \widehat{\mathbf{x}}$ for all $n \in \mathbb{N}$.

Therefore, we conclude that $\mathbf{x}_n$ and $\widehat{\mathbf{x}}$ are comparable with respect to '$\preccurlyeq_I$' for all $n \in \mathbb{N}$.

Theorem 2.9 *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is monotonically complete and preserves the monotone convergence. Consider the functions $\mathbf{F} : (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ and $\mathbf{f} : (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a $\preccurlyeq_I$-monotone seed element in X^m, and let $(X^m, \preccurlyeq_{(\mathbf{f},\mathbf{F},\mathbf{x}_0)}) \equiv (X^m, \preccurlyeq_I)$ be a quasi-ordered set induced by $(\mathbf{f},\mathbf{F},\mathbf{x}_0)$. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions*:

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative*;
- $\mathbf{f}$ *is $\preccurlyeq_I$-monotone*;
- *each f_k is continuous on X^m for $k = 1,\ldots,m$.*

Suppose that there exist a function $\rho : X \times X \to \mathbb{R}_+$ and a function of the contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any $\mathbf{x},\mathbf{y} \in X^m$ with $\mathbf{y} \preccurlyeq_I \mathbf{x}$ or $\mathbf{x} \preccurlyeq_I \mathbf{y}$, the inequalities

$$\rho\big(x^{(k)},y^{(k)}\big) \le d\big(x^{(k)},y^{(k)}\big) \tag{40}$$

and

$$d\big(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\big) \leq \varphi\big(\rho\big(f_k(\mathbf{x}),f_k(\mathbf{y})\big)\big) \cdot \rho\big(f_k(\mathbf{x}),f_k(\mathbf{y})\big) \tag{41}$$

are satisfied for all $k = 1,\ldots,m$. *Then the following statements hold true.*

(i) *There exists* $\widehat{\mathbf{x}} \in X^m$ *of* $\mathbf{F}$ *such that* $\mathbf{F}^p(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}})$. *If* $p = 1$, *then* $\widehat{\mathbf{x}}$ *is a coincidence point of* $\mathbf{F}$ *and* $\mathbf{f}$.

(ii) *If there exists* $\widehat{\mathbf{y}} \in X^m$ *such that* $\mathbf{F}^p(\widehat{\mathbf{y}}) = \mathbf{f}(\widehat{\mathbf{y}})$ *with* $\widehat{\mathbf{x}} \preccurlyeq_I \widehat{\mathbf{y}}$ *or* $\widehat{\mathbf{y}} \preccurlyeq_I \widehat{\mathbf{x}}$, *then* $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$.

(iii) *Suppose that* $\widehat{\mathbf{x}}$ *is obtained from part* (i). *If* $\widehat{\mathbf{x}}$ *and* $\mathbf{F}(\widehat{\mathbf{x}})$ *are comparable with respect to '*$\preccurlyeq_I$*', then* $\mathbf{f}^q(\widehat{\mathbf{x}})$ *is a fixed point of* $\mathbf{F}$ *for any* $q \in \mathbb{N}$.

Moreover, each component $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ *constructed in* (3) *for all* $k = 1,\ldots,m$.

Proof From the proof of Theorem 2.8, we can construct a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ and $f_k(\mathbf{f}(\mathbf{x}_n)) \to f_k(\widehat{\mathbf{x}})$ as $n \to \infty$ for all $k = 1,\ldots,m$, where $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-monotone sequence. From Remark 2.12, it follows that, for each $n \in \mathbb{N}$, $\mathbf{f}(\mathbf{x}_n) \preccurlyeq_I \widehat{\mathbf{x}}$ or $\mathbf{f}(\mathbf{x}_n) \succcurlyeq_I \widehat{\mathbf{x}}$. Since $f_k(\mathbf{f}(\mathbf{x}_n)) \to f_k(\widehat{\mathbf{x}})$ as $n \to \infty$, given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d\big(f_k\big(\mathbf{f}(\mathbf{x}_n)\big),f_k(\widehat{\mathbf{x}})\big) < \frac{\epsilon}{2} \tag{42}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$ and for all $k = 1,\ldots,m$. Since $\mathbf{f}$ is $\preccurlyeq_I$-monotone, it follows that $\mathbf{f}(\mathbf{f}(\mathbf{x}_n)) \preccurlyeq_I \mathbf{f}(\widehat{\mathbf{x}})$ or $\mathbf{f}(\mathbf{f}(\mathbf{x}_n)) \succcurlyeq_I \mathbf{f}(\widehat{\mathbf{x}})$. For each $n \geq n_0$, it follows that

$$\begin{aligned} d\big(F_k^p(\widehat{\mathbf{x}}),F_k^p\big(\mathbf{f}(\mathbf{x}_n)\big)\big) &\leq \varphi\big(\rho\big(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}\big(\mathbf{f}(\mathbf{x}_n)\big)\big)\big) \cdot \rho\big(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}\big(\mathbf{f}(\mathbf{x}_n)\big)\big) \quad \big(\text{by (41)}\big) \\ &< \rho\big(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}\big(\mathbf{f}(\mathbf{x}_n)\big)\big) \leq d\big(f_k(\widehat{\mathbf{x}}),f_k\big(\mathbf{f}(\mathbf{x}_n)\big)\big) \quad \big(\text{by (40)}\big) \\ &< \frac{\epsilon}{2} \quad \big(\text{by (42)}\big). \end{aligned}$$

Using the same argument in the proof of part (i) of Theorem 2.4, part (i) of this theorem follows immediately.

To prove part (ii), since $\mathbf{f}$ is $\preccurlyeq_I$-monotone, we immediately have $\mathbf{f}(\mathbf{y}) \preccurlyeq_I \mathbf{f}(\mathbf{x})$ or $\mathbf{f}(\mathbf{x}) \preccurlyeq_I \mathbf{f}(\mathbf{y})$. If $f_k(\widehat{\mathbf{x}}) \neq f_k(\widehat{\mathbf{y}})$, *i.e.*, $d(f_k(\widehat{\mathbf{x}}),f_k(\widehat{\mathbf{y}})) \neq 0$, then we obtain

$$\begin{aligned} d\big(f_k(\widehat{\mathbf{x}}),f_k(\widehat{\mathbf{y}})\big) &= d\big(F_k^p(\widehat{\mathbf{x}}),F_k^p(\widehat{\mathbf{y}})\big) \leq \varphi\big(\rho\big(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}(\widehat{\mathbf{y}})\big)\big) \cdot \rho\big(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}(\widehat{\mathbf{y}})\big) \quad \big(\text{by (41)}\big) \\ &< \rho\big(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}(\widehat{\mathbf{y}})\big) \leq d\big(f_k(\widehat{\mathbf{x}}),f_k(\widehat{\mathbf{y}})\big) \quad \big(\text{by (40)}\big). \end{aligned}$$

This contradiction says that $f_k(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{y}})$ for all $k = 1,\ldots,m$, *i.e.*, $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. Finally, part (iii) follows from the same argument as in the proof of part (iii) of Theorem 2.4 immediately. This completes the proof. □

Remark 2.13 We have the following observations.

- Suppose that the inequalities (40) and (41) in Theorem 2.9 are assumed to be satisfied for any $\mathbf{x},\mathbf{y} \in X^m$. Then, from the proof of Theorem 2.9, we can see that parts (ii) and (iii) can be changed as follows.

  (ii)′ If there exists $\widehat{\mathbf{y}} \in X^m$ satisfying $\mathbf{F}^p(\widehat{\mathbf{y}}) = \mathbf{f}(\widehat{\mathbf{y}})$, then $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$.

(iii)′ Suppose that $\widehat{\mathbf{x}}$ is obtained from part (i). Then $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$ for any $q \in \mathbb{N}$.

- We can also obtain a similar result when the inequalities (40) and (41) in Theorem 2.8 are replaced by the inequalities (17) and (18), respectively.

Next, we shall study the $\preccurlyeq_I$-chain-uniqueness for the common fixed point, which is the different concept from Definition 2.10.

Definition 2.13 Let $(X, \preceq)$ be a quasi-order set. Consider the functions $\mathbf{F} : X^m \to X^m$ and $\mathbf{f} : X^m \to X^m$ defined on the product set X^m into itself. Given a disjoint pair I and J of $\{1, \ldots, m\}$, we recall that the chain $\mathfrak{C}(\preccurlyeq_I, \mathbf{x})$ containing $\mathbf{x}$ is given by

$$\mathfrak{C}(\preccurlyeq_I, \mathbf{x}) = \{\mathbf{y} \in X^m : \mathbf{y} \preccurlyeq_I \mathbf{x} \text{ or } \mathbf{x} \preccurlyeq_I \mathbf{y}\}.$$

The common fixed point $\widehat{\mathbf{x}} \in X^m$ of $\mathbf{F}$ and $\mathbf{f}$ is called *$\preccurlyeq_I$-chain-unique* if and only if, for any other common fixed point $\mathbf{x}$ of $\mathbf{F}$ and $\mathbf{f}$, if $\mathbf{x} \in \mathfrak{C}(\preccurlyeq_I, \widehat{\mathbf{x}})$, then $\mathbf{x} = \widehat{\mathbf{x}}$.

Theorem 2.10 *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is monotonically complete and preserves the monotone convergence. Consider the functions $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ and $\mathbf{f} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a $\preccurlyeq_I$-monotone seed element in X^m, and let $(X^m, \preccurlyeq_{(\mathbf{f},\mathbf{F},\mathbf{x}_0)}) \equiv (X^m, \preccurlyeq_I)$ be a quasi-ordered set induced by $(\mathbf{f}, \mathbf{F}, \mathbf{x}_0)$. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions*:

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative*;
- $\mathbf{f}$ *is $\preccurlyeq_I$-monotone*;
- $\mathbf{F}^p$ *is continuous with respect to* $\mathbf{f}$ *on* X^m;
- *each f_k is continuous on X^m for $k = 1, \ldots, m$.*

Suppose that there exist a function $\rho : X \times X \to \mathbb{R}_+$ and a function of the contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any $\mathbf{x}, \mathbf{y} \in X^m$ with $\mathbf{y} \preccurlyeq_I \mathbf{x}$ or $\mathbf{x} \preccurlyeq_I \mathbf{y}$, the inequalities

$$\rho(x^{(k)}, y^{(k)}) \le d(x^{(k)}, y^{(k)}) \tag{43}$$

and

$$d(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})) \le \varphi(\rho(f_k(\mathbf{x}), f_k(\mathbf{y}))) \cdot \rho(f_k(\mathbf{x}), f_k(\mathbf{y})) \tag{44}$$

are satisfied for all $k = 1, \ldots, m$. Then the following statements hold true.

(i) $\mathbf{F}^p$ *and* $\mathbf{f}$ *have a $\preccurlyeq_I$-chain-unique common fixed point* $\widehat{\mathbf{x}}$.

(ii) *For $p \neq 1$, suppose that $\mathbf{F}(\widehat{\mathbf{x}})$ and $\widehat{\mathbf{x}}$ obtained in* (i) *are comparable with respect to '$\preccurlyeq_I$'. Then* $\mathbf{F}$ *and* $\mathbf{f}$ *have a $\preccurlyeq_I$-chain-unique common fixed point* $\widehat{\mathbf{x}}$.

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n \in \mathbb{N}}$ constructed in (3) *for all $k = 1, \ldots, m$.*

Proof To prove part (i), from Proposition 2.3 and part (i) of Theorem 2.9, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}^p(\widehat{\mathbf{x}})$. From Theorem 2.8, we also have $\mathbf{F}^p(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$. Therefore, we obtain

$$\widehat{\mathbf{x}} = \mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}^p(\widehat{\mathbf{x}}).$$

This shows that $\widehat{\mathbf{x}}$ is a common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$. For the $\preccurlyeq_I$-chain-uniqueness, let $\widehat{\mathbf{y}}$ be another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$ such that $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{x}}$ are comparable with respect to '$\preccurlyeq_I$', *i.e.*, $\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}^p(\widehat{\mathbf{y}})$. By part (ii) of Theorem 2.9, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. Therefore, by the triangle inequality, we have

$$d(\widehat{\mathbf{x}},\widehat{\mathbf{y}}) \le d\bigl(\widehat{\mathbf{x}},\mathbf{f}(\widehat{\mathbf{x}})\bigr) + d\bigl(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}(\widehat{\mathbf{y}})\bigr) + d\bigl(\mathbf{f}(\widehat{\mathbf{y}}),\widehat{\mathbf{y}}\bigr) = 0,$$

which says that $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. This proves part (i). Part (ii) can be obtained by applying part (iii) of Theorem 2.9 to the similar argument in the proof of Theorem 2.6. This completes the proof. □

Remark 2.14 We can also obtain a similar result when the inequalities (43) and (44) in Theorem 2.10 are replaced by the inequalities (17) and (18), respectively.

3 Fixed points of functions having monotone property

We shall study the fixed points of functions having monotone property in the product space. Considering the function $\mathbf{F} : X^m \to X^m$, we shall define many monotonic concepts of $\mathbf{F}$ as follows.

Definition 3.1 Let $(X, \preceq)$ be a quasi-order set, and let I and J be the disjoint pair of $\{1,2,\ldots,m\}$. Consider the quasi-order set $(X^m, \preccurlyeq_I)$ and the function $\mathbf{F} : (X^m, \preccurlyeq_I) \to (X^m, \preccurlyeq_I)$.

- We say that $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_I)$-*increasing* if and only if $\mathbf{x} \preccurlyeq_I \mathbf{y}$ implies $\mathbf{F}(\mathbf{x}) \preccurlyeq_I \mathbf{F}(\mathbf{y})$.
- We say that $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_J)$-*increasing* if and only if $\mathbf{x} \preccurlyeq_I \mathbf{y}$ implies $\mathbf{F}(\mathbf{x}) \preccurlyeq_J \mathbf{F}(\mathbf{y})$.
- We say that $\mathbf{F}$ is $(\preccurlyeq_J, \preccurlyeq_I)$-*increasing* if and only if $\mathbf{x} \preccurlyeq_J \mathbf{y}$ implies $\mathbf{F}(\mathbf{x}) \preccurlyeq_I \mathbf{F}(\mathbf{y})$.
- We say that $\mathbf{F}$ is $(\preccurlyeq_J, \preccurlyeq_J)$-*increasing* if and only if $\mathbf{x} \preccurlyeq_J \mathbf{y}$ implies $\mathbf{F}(\mathbf{x}) \preccurlyeq_J \mathbf{F}(\mathbf{y})$.
- We say that $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_I)$-*decreasing* if and only if $\mathbf{x} \preccurlyeq_I \mathbf{y}$ implies $\mathbf{F}(\mathbf{x}) \succcurlyeq_I \mathbf{F}(\mathbf{y})$.
- We say that $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_J)$-*decreasing* if and only if $\mathbf{x} \preccurlyeq_I \mathbf{y}$ implies $\mathbf{F}(\mathbf{x}) \succcurlyeq_J \mathbf{F}(\mathbf{y})$.
- We say that $\mathbf{F}$ is $(\preccurlyeq_J, \preccurlyeq_I)$-*decreasing* if and only if $\mathbf{x} \preccurlyeq_J \mathbf{y}$ implies $\mathbf{F}(\mathbf{x}) \succcurlyeq_I \mathbf{F}(\mathbf{y})$.
- We say that $\mathbf{F}$ is $(\preccurlyeq_J, \preccurlyeq_J)$-*decreasing* if and only if $\mathbf{x} \preccurlyeq_J \mathbf{y}$ implies $\mathbf{F}(\mathbf{x}) \succcurlyeq_J \mathbf{F}(\mathbf{y})$.

Remark 3.1 From (2), we see that it suffices to consider the increasing cases. On the other hand, we also see that $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_I)$-increasing if and only if it is $(\preccurlyeq_J, \preccurlyeq_J)$-increasing, and $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_J)$-increasing if and only if it is $(\preccurlyeq_J, \preccurlyeq_I)$-increasing. Therefore, the cases in Definition 3.1 can be reduced to only consider the $(\preccurlyeq_I, \preccurlyeq_I)$-increasing and $(\preccurlyeq_I, \preccurlyeq_J)$-increasing cases. Since the $(\preccurlyeq_I, \preccurlyeq_J)$-increasing case is equivalent to the $(\preccurlyeq_I, \preccurlyeq_I)$-decreasing case, it follows that the cases in Definition 3.1 can be reduced to only consider the $(\preccurlyeq_I, \preccurlyeq_I)$-increasing and $(\preccurlyeq_I, \preccurlyeq_I)$-decreasing cases.

Theorem 3.1 *Suppose that the quasi-ordered metric space* $(X, d, \preceq)$ *is monotonically complete. Let* I *and* J *be a disjoint pair of* $\{1,2,\ldots,m\}$. *Assume that the function* $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ *is continuous on* X^m *and satisfies any one of the following conditions*:

(a) $\mathbf{F}$ *is* $(\preccurlyeq_I, \preccurlyeq_I)$-*increasing*;

(b) p *is an even integer and* $\mathbf{F}$ *is* $(\preccurlyeq_I, \preccurlyeq_I)$-*decreasing*.

Assume that there exist a function $\rho : X \times X \to \mathbb{R}_+$ *and a function of the contractive factor* $\varphi : [0,\infty) \to [0,1)$ *such that, for any* $\mathbf{x}, \mathbf{y} \in X^m$ *with* $\mathbf{y} \preccurlyeq_I \mathbf{x}$ *or* $\mathbf{x} \preccurlyeq_I \mathbf{y}$, *the inequalities*

$$\rho\bigl(x^{(k)}, y^{(k)}\bigr) \le d\bigl(x^{(k)}, y^{(k)}\bigr) \tag{45}$$

and

$$d\big(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\big) \le \varphi\big(\rho\big(f_k(\mathbf{x}),f_k(\mathbf{y})\big)\big) \cdot \rho\big(f_k(\mathbf{x}),f_k(\mathbf{y})\big) \tag{46}$$

are satisfied for all $k = 1,\ldots,m$ *and for some* $p \in \mathbb{N}$. *If there exists* $\mathbf{x}_0 \in X^m$ *such that* $\mathbf{x}_0 \preccurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$ *or* $\mathbf{x}_0 \succcurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$, *then the function* $\mathbf{F}^p$ *has a fixed point* $\widehat{\mathbf{x}} \in X^m$, *where each component* $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed from* (51) *for all* $k = 1,\ldots,m$.

Proof We consider the following cases.

- If $\mathbf{F}$ is $(\preccurlyeq_I,\preccurlyeq_I)$-increasing, then it follows that $\mathbf{F}^p$ is $(\preccurlyeq_I,\preccurlyeq_I)$-increasing.
- If $\mathbf{F}$ is $(\preccurlyeq_I,\preccurlyeq_I)$-decreasing and p is an even integer, then $\mathbf{F}^p$ is also $(\preccurlyeq_I,\preccurlyeq_I)$-increasing.

According to (51), we have $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$ or $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$. Since $\mathbf{x}_1 = \mathbf{F}^p(\mathbf{x}_0)$ and $\mathbf{x}_2 = \mathbf{F}^p(\mathbf{x}_1)$, it follows that $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$ implies $\mathbf{x}_1 \preccurlyeq_I \mathbf{x}_2$, and $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$ implies $\mathbf{x}_1 \succcurlyeq_I \mathbf{x}_2$. Therefore, if $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$, then we can generate a $\preccurlyeq_I$-increasing sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, and if $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$, then we can generate a $\preccurlyeq_I$-decreasing sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, which also says that the initial element $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.8 by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Remark 3.2 We can also obtain a similar result when the inequalities (45) and (46) in Theorem 3.1 are replaced by the inequalities (17) and (18), respectively.

Next, we can consider the chain-uniqueness and drop the assumption of continuity of $\mathbf{F}$ by assuming that $(X,d,\preceq)$ preserves the monotone convergence.

Theorem 3.2 *Suppose that the quasi-ordered metric space* $(X,d,\preceq)$ *is monotonically complete and preserves the monotone convergence. Let I and J be a disjoint pair of* $\{1,2,\ldots,m\}$. *Assume that the function* $\mathbf{F}: X^m \to X^m$ *satisfies any one of the following conditions*:

(a) $\mathbf{F}$ *is* $(\preccurlyeq_I,\preccurlyeq_I)$-*increasing*;
(b) p *is an even integer and* $\mathbf{F}$ *is* $(\preccurlyeq_I,\preccurlyeq_I)$-*decreasing*.

Assume that there exist a function $\rho: X\times X \to \mathbb{R}_+$ *and a function of the contractive factor* $\varphi:[0,\infty)\to[0,1)$ *such that, for any* $\mathbf{x},\mathbf{y}\in X^m$ *with* $\mathbf{y} \preccurlyeq_I \mathbf{x}$ *or* $\mathbf{x} \preccurlyeq_I \mathbf{y}$, *the inequalities*

$$\rho\big(x^{(k)},y^{(k)}\big) \le d\big(x^{(k)},y^{(k)}\big) \tag{47}$$

and

$$d\big(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\big) \le \varphi\big(\rho\big(f_k(\mathbf{x}),f_k(\mathbf{y})\big)\big) \cdot \rho\big(f_k(\mathbf{x}),f_k(\mathbf{y})\big) \tag{48}$$

are satisfied for all $k = 1,\ldots,m$ *and for some* $p \in \mathbb{N}$. *If there exists* $\mathbf{x}_0 \in X^m$ *such that* $\mathbf{x}_0 \preccurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$ *or* $\mathbf{x}_0 \succcurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$, *then the function* $\mathbf{F}^p$ *has a* $\preccurlyeq_I$-*chain-unique fixed point* $\widehat{\mathbf{x}} \in X^m$, *where each component* $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed from* (51) *for all* $k = 1,\ldots,m$.

Proof From the proof of Theorem 3.1, we see that the initial element $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.9 by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Remark 3.3 We can also obtain a similar result when the inequalities (47) and (48) in Theorem 3.2 are replaced by the inequalities (17) and (18), respectively.

Theorem 3.3 *Suppose that the quasi-ordered metric space* $(X, d, \preceq)$ *is mixed-monotonically complete. Let* I *and* J *be a disjoint pair of* $\{1, 2, \ldots, m\}$. *Assume that the function* $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ *is continuous on* X^m *and* $(\preccurlyeq_I, \preccurlyeq_I)$*-decreasing, and that there exist a function* $\rho : X \times X \to \mathbb{R}_+$ *and a function of the contractive factor* $\varphi : [0, \infty) \to [0, 1)$ *such that, for any* $\mathbf{x}, \mathbf{y} \in X^m$ *with* $\mathbf{y} \preccurlyeq_I \mathbf{x}$ *or* $\mathbf{x} \preccurlyeq_I \mathbf{y}$, *the inequalities*

$$\rho\left(x^{(k)}, y^{(k)}\right) \le d\left(x^{(k)}, y^{(k)}\right) \tag{49}$$

and

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \le \varphi\left(\rho\left(x^{(k)}, y^{(k)}\right)\right) \cdot \rho\left(x^{(k)}, y^{(k)}\right), \tag{50}$$

are satisfied for all and for some odd integer $p \in \mathbb{N}$. *If there exists* $\mathbf{x}_0 \in X^m$ *such that* $\mathbf{x}_0 \preccurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$ *or* $\mathbf{x}_0 \succcurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$, *then the function* $\mathbf{F}^p$ *has a fixed point* $\widehat{\mathbf{x}} \in X^m$, *where each component* $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed below*

$$\mathbf{x}_n = \mathbf{F}^p(\mathbf{x}_{n-1}) \tag{51}$$

for all $k = 1, \ldots, m$.

Proof Since $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_I)$-decreasing and p is an odd integer, it follow that $\mathbf{F}^p$ is $(\preccurlyeq_I, \preccurlyeq_I)$-decreasing. We see that $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$ implies $\mathbf{x}_1 \succcurlyeq_I \mathbf{x}_2$, and that $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$ implies $\mathbf{x}_1 \preccurlyeq_I \mathbf{x}_2$. Therefore, we can generate a $\preccurlyeq_I$-mixed-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, which also says that the initial element $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.3 by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Remark 3.4 We can also obtain a similar result when the inequalities (49) and (50) in Theorem 3.3 are replaced by the inequalities

$$\rho(\mathbf{x}, \mathbf{y}) \le d\left(x^{(k)}, y^{(k)}\right) \tag{52}$$

and

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \le \varphi\left(\rho(\mathbf{x}, \mathbf{y})\right) \cdot \rho(\mathbf{x}, \mathbf{y}), \tag{53}$$

respectively.

Next, we can consider the chain-uniqueness and drop the assumption of continuity of $\mathbf{F}$ by assuming that $(X, d, \preceq)$ preserves the mixed-monotone convergence.

Theorem 3.4 *Suppose that the quasi-ordered metric space* $(X, d, \preceq)$ *is mixed-monotonically complete and preserves the mixed-monotone convergence. Suppose that there exists a disjoint pair* I *and* J *of* $\{1, 2, \ldots, m\}$ *and* $\mathbf{x}_0 \in X^m$ *such that the following conditions are satisfied*:

- *the function* $\mathbf{F} : X^m \to X^m$ *is* $(\preccurlyeq_I, \preccurlyeq_I)$*-decreasing.*
- $\mathbf{x}_0 \preccurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$ *or* $\mathbf{x}_0 \succcurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$.

Assume that there exists a function $\rho : X \times X \to \mathbb{R}_+$ *and a function of the contractive factor* $\varphi : [0, \infty) \to [0, 1)$ *such that, for any* $\mathbf{x}, \mathbf{y} \in X^m$ *and any disjoint pair* I° *and* J° *of* $\{1, \ldots, m\}$ *with* $\mathbf{y} \preccurlyeq_{I^\circ} \mathbf{x}$ *or* $\mathbf{x} \preccurlyeq_{I^\circ} \mathbf{y}$, *the inequalities*

$$\rho\bigl(x^{(k)}, y^{(k)}\bigr) \le d\bigl(x^{(k)}, y^{(k)}\bigr) \tag{54}$$

and

$$d\bigl(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\bigr) \le \varphi\bigl(\rho\bigl(x^{(k)}, y^{(k)}\bigr)\bigr) \cdot \rho\bigl(x^{(k)}, y^{(k)}\bigr), \tag{55}$$

are satisfied for all $k = 1, \ldots, m$ *and for some odd integer* $p \in \mathbb{N}$. *Then the function* $\mathbf{F}^p$ *has a chain-unique fixed point* $\widehat{\mathbf{x}} \in X^m$, *where each component* $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed from* (51) *for all* $k = 1, \ldots, m$.

Proof From the proof of Theorem 3.3, we see that the initial element $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.5 by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Remark 3.5 We can also obtain a similar result when the inequalities (54) and (55) in Theorem 3.4 are replaced by the inequalities (52) and (53), respectively.

4 Fixed points of functions having comparable property

We shall study the fixed points of functions having the comparable property in the product space.

Definition 4.1 Let I and J be a disjoint pair of $\{1, 2, \ldots, m\}$. Given a quasi-ordered set $(X, \preceq)$, we consider the corresponding quasi-ordered set $(X^m, \preccurlyeq_I)$.

- The function $\mathbf{F} : X^m \to X^m$ is said to have the $\preceq$*-mixed comparable property* if and only if, for any two $\preceq$-mixed comparable elements $\mathbf{x}$ and $\mathbf{y}$ in X^m, the function values $\mathbf{F}(\mathbf{x})$ and $\mathbf{F}(\mathbf{y})$ in X^m are $\preceq$-mixed comparable.
- The function $\mathbf{F} : (X^m, \preccurlyeq_I) \to (X^m, \preccurlyeq_I)$ is said to have the $\preccurlyeq_I$*-comparable property* if and only if, for any two elements $\mathbf{x}, \mathbf{y} \in X^m$ with $\mathbf{x} \preccurlyeq_I \mathbf{y}$ or $\mathbf{y} \preccurlyeq_I \mathbf{x}$ (*i.e.*, $\mathbf{x}$ and $\mathbf{y}$ are comparable with respect to '$\preccurlyeq_I$'), one has either $\mathbf{F}(\mathbf{x}) \preccurlyeq_I \mathbf{F}(\mathbf{y})$ or $\mathbf{F}(\mathbf{y}) \preccurlyeq_I \mathbf{F}(\mathbf{x})$ (*i.e.*, the function values $\mathbf{F}(\mathbf{x})$ and $\mathbf{F}(\mathbf{y})$ in X^m are comparable with respect to '$\preccurlyeq_I$').

It is obvious that if $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_I)$-increasing or $(\preccurlyeq_I, \preccurlyeq_I)$-decreasing, then it also has the $\preccurlyeq_I$-comparable property.

Theorem 4.1 *Suppose that the quasi-ordered metric space* $(X, d, \preceq)$ *is mixed-monotonically complete. Assume that the function* $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ *is continuous on* X^m *and has the* $\preceq$*-mixed comparable property, and that there exist a function* $\rho : X \times X \to \mathbb{R}_+$ *and a function of the contractive factor* $\varphi : [0, \infty) \to [0, 1)$ *such that, for any two* $\preceq$*-mixed comparable elements* $\mathbf{x}$ *and* $\mathbf{y}$ *in* X^m, *the inequalities*

$$\rho\bigl(x^{(k)}, y^{(k)}\bigr) \le d\bigl(x^{(k)}, y^{(k)}\bigr) \tag{56}$$

and

$$d\big(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\big) \le \varphi\big(\rho\big(x^{(k)},y^{(k)}\big)\big)\cdot\rho\big(x^{(k)},y^{(k)}\big), \tag{57}$$

are satisfied for all $k=1,\dots,m$ *and for some* $p\in\mathbb{N}$. *If there exists* $\mathbf{x}_0\in X^m$ *such that* $\mathbf{x}_0$ *and* $\mathbf{F}^p(\mathbf{x}_0)$ *are* $\preceq$*-mixed comparable, then* $\mathbf{F}^p$ *has a fixed point* $\widehat{\mathbf{x}}$ *such that each component* $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed from* (51) *for all* $k=1,\dots,m$.

Proof According to (51), it follows that $\mathbf{x}_0$ and $\mathbf{x}_1$ are $\preceq$-mixed comparable. Since $\mathbf{F}$ has the $\preceq$-mixed comparable property, we see that $\mathbf{F}^p$ has also the $\preceq$-mixed comparable property. It follows that $\mathbf{x}_1=\mathbf{F}^p(\mathbf{x}_0)$ and $\mathbf{x}_2=\mathbf{F}^p(\mathbf{x}_1)$ are also $\preceq$-mixed comparable. Therefore, we can generate a mixed $\preceq$-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ by observation (d) of Remark 2.2, which also says that the initial element $\mathbf{x}_0$ is a mixed $\preceq$-monotone seed element in X^m. Since $\mathbf{F}$ is continuous on X^m, it follows that $\mathbf{F}^p$ is also continuous on X^m. Therefore, the result follows from Theorem 2.1 immediately by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Remark 4.1 We can also obtain a similar result when the inequalities (56) and (57) in Theorem 4.1 are replaced by the inequalities (52) and (53), respectively.

Next, we can drop the assumption of continuity of $\mathbf{F}$ by assuming that $(X,d,\preceq)$ preserves the mixed-monotone convergence.

Theorem 4.2 *Suppose that the quasi-ordered metric space* $(X,d,\preceq)$ *is mixed-monotonically complete and preserves the mixed-monotone convergence. Assume that the function* $\mathbf{F}:X^m\to X^m$ *has the* $\preceq$*-mixed comparable property, and that there exist a function* $\rho:X\times X\to\mathbb{R}_+$ *and a function of the contractive factor* $\varphi:[0,\infty)\to[0,1)$ *such that, for any two* $\preceq$*-mixed comparable elements* $\mathbf{x}$ *and* $\mathbf{y}$ *in* X^m, *the inequalities*

$$\rho\big(x^{(k)},y^{(k)}\big)\le d\big(x^{(k)},y^{(k)}\big) \tag{58}$$

and

$$d\big(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\big) \le \varphi\big(\rho\big(x^{(k)},y^{(k)}\big)\big)\cdot\rho\big(x^{(k)},y^{(k)}\big), \tag{59}$$

are satisfied for all $k=1,\dots,m$ *and for some* $p\in\mathbb{N}$. *Suppose that there exists* $\mathbf{x}_0\in X^m$ *such that* $\mathbf{x}_0$ *and* $\mathbf{F}^p(\mathbf{x}_0)$ *are* $\preceq$*-mixed comparable. Then the following statements hold true.*

- (i) *There exists a unique fixed point* $\widehat{\mathbf{x}}$ *of* $\mathbf{F}^p$ *in the* $\preceq$*-mixed comparable sense.*
- (ii) *For* $p\neq 1$, *we further assume that the function* $\mathbf{F}^p$ *is continuous on* X^m, *and that* $\mathbf{F}(\widehat{\mathbf{x}})$ *and* $\widehat{\mathbf{x}}$ *obtained in* (i) *are* $\preceq$*-mixed comparable. Then* $\widehat{\mathbf{x}}$ *is a unique fixed point of* $\mathbf{F}$ *in the* $\preceq$*-mixed-monotone sense.*

Moreover, each component $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed from* (51) *for all* $k=1,\dots,m$ *for all* $k=1,\dots,m$.

Proof According to the argument in the proof of Theorem 4.1, we see that the initial element $\mathbf{x}_0$ is a mixed $\preceq$-monotone seed element in X^m. Therefore, part (i) follows from Theorem 2.4 immediately by taking $\mathbf{f}$ as the identity function. Also, part (ii) follows from Theorem 2.6 immediately by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Remark 4.2 We can also obtain a similar result when the inequalities (58) and (59) in Theorem 4.2 are replaced by the inequalities (52) and (53), respectively.

Theorem 4.3 *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is mixed-monotonically complete. Given a disjoint pair I and J of $\{1, \ldots, m\}$, assume that the following conditions are satisfied*:

- *the function $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ is continuous on X^m and has the $\preccurlyeq_I$-comparable property*;
- *there exists $\mathbf{x}_0 \in X^m$ such that $\mathbf{x}_0$ and $\mathbf{F}^p(\mathbf{x}_0)$ are comparable with respect to the quasi-order '$\preccurlyeq_I$' for some $p \in \mathbb{N}$.*

Assume that there exist a function $\rho : X \times X \to \mathbb{R}_+$ and a function of the contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any two $\preceq$-mixed comparable elements $\mathbf{x}$ and $\mathbf{y}$ in X^m, the inequalities

$$\rho\big(x^{(k)}, y^{(k)}\big) \le d\big(x^{(k)}, y^{(k)}\big) \tag{60}$$

and

$$d\big(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\big) \le \varphi\big(\rho\big(x^{(k)}, y^{(k)}\big)\big) \cdot \rho\big(x^{(k)}, y^{(k)}\big), \tag{61}$$

are satisfied for all $k = 1, \ldots, m$. Then $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}}$ such that each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ constructed from (51) *for all $k = 1, \ldots, m$.*

Proof According to (51), we see that $\mathbf{x}_0$ and $\mathbf{x}_1$ are comparable with respect to '$\preccurlyeq_I$'. Since $\mathbf{F}$ has the $\preccurlyeq_I$-comparable property, we see that $\mathbf{F}^p$ has also the $\preccurlyeq_I$-comparable property. It follows that $\mathbf{x}_1 = \mathbf{F}^p(\mathbf{x}_0)$ and $\mathbf{x}_2 = \mathbf{F}^p(\mathbf{x}_1)$ are also comparable with respect to '$\preccurlyeq_I$'. Therefore, we can generate a mixed $\preccurlyeq_I$-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, which also says that the initial element $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element in X^m. Since $\mathbf{F}$ is continuous on X^m, it follows that $\mathbf{F}^p$ is also continuous on X^m. Therefore, the result follows from Theorem 2.3 immediately by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Remark 4.3 We can also obtain a similar result when the inequalities (60) and (61) in Theorem 4.3 are replaced by the inequalities (52) and (53), respectively.

Next, we can drop the assumption of continuity of $\mathbf{F}$ by assuming that $(X, d, \preceq)$ preserves the mixed-monotone convergence.

Theorem 4.4 *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is mixed-monotonically complete and preserves the mixed-monotone convergence. Given a disjoint pair I and J of $\{1, \ldots, m\}$, assume that the following conditions are satisfied*:

- *the function $\mathbf{F} : X^m \to X^m$ has the $\preccurlyeq_I$-comparable property*;
- *there exists $\mathbf{x}_0 \in X^m$ such that $\mathbf{x}_0$ and $\mathbf{F}^p(\mathbf{x}_0)$ are comparable with respect to the quasi-order '$\preccurlyeq_I$' for some $p \in \mathbb{N}$.*

Suppose that there exists a function $\rho : X \times X \to \mathbb{R}_+$ and a function of the contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any $\mathbf{x}, \mathbf{y} \in X^m$ and any disjoint pair I° and J° of $\{1,\ldots,m\}$ with $\mathbf{y} \preccurlyeq_{I^\circ} \mathbf{x}$ or $\mathbf{x} \preccurlyeq_{I^\circ} \mathbf{y}$, the inequalities

$$\rho\left(x^{(k)}, y^{(k)}\right) \le d\left(x^{(k)}, y^{(k)}\right) \tag{62}$$

and

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \le \varphi\left(\rho\left(x^{(k)}, y^{(k)}\right)\right) \cdot \rho\left(x^{(k)}, y^{(k)}\right), \tag{63}$$

are satisfied for all $k = 1,\ldots,m$. Then the following statements hold true.

(i) *There exists a chain-unique fixed point $\widehat{\mathbf{x}}$ of $\mathbf{F}^p$.*

(ii) *For $p \neq 1$, we further assume that the function $\mathbf{F}^p$ is continuous on X^m, and that $\mathbf{F}(\widehat{\mathbf{x}})$ and $\widehat{\mathbf{x}}$ obtained in* (i) *are comparable with respect to '$\preccurlyeq_{I^\circ}$' for some disjoint pair I° and J° of $\{1,\ldots,m\}$. Then $\widehat{\mathbf{x}}$ is a chain-unique fixed point of $\mathbf{F}$.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ constructed from (51) *for all $k = 1,\ldots,m$.*

Proof According to the argument in the proof of Theorem 4.3, we see that the initial element $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element in X^m. Therefore, part (i) follows from Theorem 2.5 immediately by taking $\mathbf{f}$ as the identity function. Also, part (ii) follows from Theorem 2.7 immediately by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Remark 4.4 We can also obtain a similar result when the inequalities (62) and (63) in Theorem 4.4 are replaced by the inequalities (52) and (53), respectively.

5 Applications to the system of integral equations

Let $\mathcal{C}([0,T],\mathbb{R})$ be the space of all continuous functions from $[0,T]$ into $\mathbb{R}$. We also denote by $\mathcal{C}^m([0,T],\mathbb{R})$ the product space of $\mathcal{C}([0,T],\mathbb{R})$ for m times. In the sequel, we shall consider a metric d and a quasi-order '$\preceq$' on $\mathcal{C}([0,T],\mathbb{R})$ such that $(\mathcal{C}([0,T],\mathbb{R}), d, \preceq)$ is monotonically complete or mixed-monotonically complete and preserves the monotone convergence.

Given continuous functions $G : [0,T] \times [0,T] \to \mathbb{R}_+$ and $g^{(k)} : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ for $k = 1,\ldots,m$, we consider the following system of integral equations:

$$\int_0^T G(s,t)\left[g^{(k)}\left(s,\mathbf{w}(s)\right) + \lambda w^{(k)}(s)\right] ds = w^{(k)}(t) \tag{64}$$

for $k = 1,\ldots,m$, where $\lambda \ge 0$. We shall find $\mathbf{w}^* \in \mathcal{C}^m([0,T],\mathbb{R})$ such that the systems of integral equations (64) are all satisfied, where $w^{(k*)} \in \mathcal{C}([0,T],\mathbb{R})$ is the kth component of $\mathbf{w}^*$ for $k = 1,\ldots,m$. The solution $\mathbf{w}^*$ will be in the sense of chain-uniqueness.

For the vector-valued function $\mathbf{h} : [0,T] \to \mathbb{R}^m$ defined on $[0,T]$, the kth component function of $\mathbf{h}$ is denoted by $h^{(k)}$ for $k = 1,\ldots,m$. The integral of $\mathbf{h}$ on $[0,T]$ is defined as the following vector in $\mathbb{R}^m$:

$$\int_0^T \mathbf{h}(s)\,ds = \left(\int_0^T h^{(1)}(s)\,ds, \int_0^T h^{(2)}(s)\,ds, \ldots, \int_0^T h^{(m)}(s)\,ds\right) \in \mathbb{R}^m.$$

Now, we define a vector-valued functions $\mathbf{g} : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ by $\mathbf{g} = (g^{(1)}, g^{(2)}, \ldots, g^{(m)})$. Then the system of integral equations as shown in (64) can be written as the following vectorial form of integral equation:

$$\int_0^T G(s,t)\big[\mathbf{g}\big(s, \mathbf{w}(s)\big) + \lambda \mathbf{w}(s)\big]\, ds = \mathbf{w}(t), \tag{65}$$

where $\lambda \geq 0$. Equivalently, we shall find $\mathbf{w}^* \in \mathcal{C}^m([0,T], \mathbb{R})$ such that (65) is satisfied, which also says that $\mathbf{w}^*$ is a solution of (65).

Definition 5.1 Consider the quasi-ordered metric space $(\mathcal{C}([0,T], \mathbb{R}), d, \preceq)$.

(a) We say that $\mathbf{w}^*$ is a *unique solution* of the system of integral equations (65) in the $\preceq$*-mixed comparable sense* if and only if the following conditions are satisfied:
  - $\mathbf{w}^*$ is a solution of (65);
  - if $\bar{\mathbf{w}}$ is another solution of (65) such that $\mathbf{w}^*$ and $\bar{\mathbf{w}}$ are $\preceq$-mixed comparable, then $\mathbf{w}^* = \bar{\mathbf{w}}$.

Given a disjoint pair I and J of $\{1, \ldots, m\}$, consider the product space $(\mathcal{C}^m([0,T], \mathbb{R}), \mathfrak{d}, \preccurlyeq_I)$.

(b) We say that $\mathbf{w}^*$ is a $\preccurlyeq_I$*-chain-unique solution* of the system of integral equations (65) if and only if the following conditions are satisfied:
  - $\mathbf{w}^*$ is a solution of (65);
  - if $\bar{\mathbf{w}}$ is another solution of (65) satisfying $\mathbf{w}^* \preccurlyeq_I \bar{\mathbf{w}}$ or $\bar{\mathbf{w}} \preccurlyeq_I \mathbf{w}^*$ (*i.e.*, $\mathbf{w}^*$ and $\bar{\mathbf{w}}$ are comparable with respect to $\preccurlyeq$), then $\mathbf{w}^* = \bar{\mathbf{w}}$.

Theorem 5.1 *Suppose that the quasi-ordered metric space $(\mathcal{C}([0,T], \mathbb{R}), d, \preceq)$ is monotonically complete and preserves the monotone convergence. Let I and J be a disjoint pair of $\{1, 2, \ldots, m\}$. Define the function $\mathbf{F} : (\mathcal{C}^m([0,T], \mathbb{R}), \preccurlyeq_I) \to (\mathcal{C}^m([0,T], \mathbb{R}), \preccurlyeq_I)$ by*

$$\mathbf{F}(\mathbf{w})(t) = \int_0^T G(s,t)\big[\mathbf{g}\big(s, \mathbf{w}(s)\big) + \lambda \mathbf{w}(s)\big]\, ds.$$

Suppose that the following conditions are satisfied:
- $\mathbf{F}$ *is $(\preccurlyeq_I, \preccurlyeq_I)$-increasing*;
- *there exist a function $\rho : \mathcal{C}([0,T], \mathbb{R}) \times \mathcal{C}([0,T], \mathbb{R}) \to \mathbb{R}_+$ and a function of the contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any $\mathbf{w}, \bar{\mathbf{w}} \in \mathcal{C}^m([0,T], \mathbb{R})$ with $\mathbf{w} \preccurlyeq_I \bar{\mathbf{w}}$ or $\bar{\mathbf{w}} \preccurlyeq_I \mathbf{w}$, the inequalities*

  $$\rho\big(w^{(k)}, \bar{w}^{(k)}\big) \leq d\big(w^{(k)}, \bar{w}^{(k)}\big) \tag{66}$$

  and

  $$d\big(F_k(\mathbf{w}), F_k(\bar{\mathbf{w}})\big) \leq \varphi\big(\rho\big(w^{(k)}, \bar{w}^{(k)}\big)\big) \cdot \rho\big(w^{(k)}, \bar{w}^{(k)}\big) \tag{67}$$

  are satisfied for all $k = 1, \ldots, m$;
- *there exists $\mathbf{w}_0 \in \mathcal{C}^m([0,T], \mathbb{R})$ such that $\mathbf{w}_0 \preccurlyeq_I \mathbf{F}(\mathbf{w}_0)$ or $\mathbf{w}_0 \succcurlyeq_I \mathbf{F}(\mathbf{w}_0)$.*

Then there exists a $\preccurlyeq_I$-chain-unique solution of the system of integral equations (65).

Proof Since $\mathfrak{d}$ is defined in (5) or (6), we immediately have that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving convergence. Using condition (a) and considering

$p = 1$ in Theorem 3.2, we see that $\mathbf{F}$ has a $\preccurlyeq_I$-chain-unique fixed point $\mathbf{w}^*$ in $\mathcal{C}^m([0,T],\mathbb{R})$. In other words, we have

$$\int_0^T G(s,t)\big[\mathbf{g}\big(s,\mathbf{f}\big(\mathbf{w}^*(s)\big)\big) + \lambda\mathbf{f}\big(\mathbf{w}^*(s)\big)\big]\,ds = \mathbf{F}\big(\mathbf{w}^*\big) = \mathbf{w}^*,$$

which says that $\mathbf{w}^*$ is a $\preccurlyeq_I$-chain-unique solution of the vectorial form of the integral equation (65). This completes the proof. □

Remark 5.1 The assumption for the inequalities (66) and (67) are really weak, since we just assume that they are satisfied for $\preccurlyeq_I$-comparable elements. In other words, if $\mathbf{x}$ and $\mathbf{y}$ are not $\preccurlyeq_I$-comparable, we do not need to check the inequalities (66) and (67).

Theorem 5.2 *Suppose that the quasi-ordered metric space $(\mathcal{C}([0,T],\mathbb{R}),d,\preceq)$ is monotonically complete and preserves the monotone convergence. Let I and J be a disjoint pair of $\{1,2,\ldots,m\}$. Define the function $\mathbf{F} : (\mathcal{C}^m([0,T],\mathbb{R}),\preccurlyeq_I) \to (\mathcal{C}^m([0,T],\mathbb{R}),\preccurlyeq_I)$ by*

$$\mathbf{F}(\mathbf{w})(t) = \int_0^T G(s,t)\big[\mathbf{g}\big(s,\mathbf{w}(s)\big) + \lambda\mathbf{w}(s)\big]\,ds.$$

Suppose that the following conditions are satisfied:
- $\mathbf{F}$ *is $(\preccurlyeq_I,\preccurlyeq_I)$-increasing;*
- *there exist a function $\rho : \mathcal{C}^m([0,T],\mathbb{R}) \times \mathcal{C}^m([0,T],\mathbb{R}) \to \mathbb{R}_+$ and a function of the contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any $\mathbf{w},\bar{\mathbf{w}} \in \mathcal{C}^m([0,T],\mathbb{R})$ with $\mathbf{w} \preccurlyeq_I \bar{\mathbf{w}}$ or $\bar{\mathbf{w}} \preccurlyeq_I \mathbf{w}$, the inequalities*

$$\rho(\mathbf{w},\bar{\mathbf{w}}) \le d\big(w^{(k)},\bar{w}^{(k)}\big) \tag{68}$$

  and

$$d\big(F_k(\mathbf{w}),F_k(\bar{\mathbf{w}})\big) \le \varphi\big(\rho(\mathbf{w},\bar{\mathbf{w}})\big)\cdot\rho(\mathbf{w},\bar{\mathbf{w}}) \tag{69}$$

  are satisfied for all $k = 1,\ldots,m$;
- *there exists $\mathbf{w}_0 \in \mathcal{C}^m([0,T],\mathbb{R})$ such that $\mathbf{w}_0 \preccurlyeq_I \mathbf{F}(\mathbf{w}_0)$ or $\mathbf{w}_0 \succcurlyeq_I \mathbf{F}(\mathbf{w}_0)$.*

Then there exists a $\preccurlyeq_I$-chain-unique solution of the system of integral equations (65).

Proof By applying Remark 3.3 to the argument in the proof of Theorem 5.1, we can obtain the desired result. □

Corollary 5.1 *Suppose that the quasi-ordered metric space $(\mathcal{C}([0,T],\mathbb{R}),d,\preceq)$ is monotonically complete and preserves the monotone convergence. Let I and J be a disjoint pair of $\{1,2,\ldots,m\}$. Define the function $\mathbf{F} : (\mathcal{C}^m([0,T],\mathbb{R}),\preccurlyeq_I) \to (\mathcal{C}^m([0,T],\mathbb{R}),\preccurlyeq_I)$ by*

$$\mathbf{F}(\mathbf{w})(t) = \int_0^T G(s,t)\big[\mathbf{g}\big(s,\mathbf{w}(s)\big) + \lambda\mathbf{w}(s)\big]\,ds.$$

Suppose that the following conditions are satisfied:

- $\mathbf{F}$ *is* $(\preccurlyeq_I, \preccurlyeq_I)$*-increasing*;
- *there exists a function of the contractive factor* $\varphi : [0,\infty) \to [0,1)$ *such that, for any* $\mathbf{w}, \bar{\mathbf{w}} \in \mathcal{C}^m([0,T],\mathbb{R})$ *with* $\mathbf{w} \preccurlyeq_I \bar{\mathbf{w}}$ *or* $\bar{\mathbf{w}} \preccurlyeq_I \mathbf{w}$, *the inequalities*

$$d\big(F_k(\mathbf{w}), F_k(\bar{\mathbf{w}})\big) \le \varphi\Big(\min_{k=1,\ldots,m} d\big(w^{(k)}, \bar{w}^{(k)}\big)\Big) \cdot \Big(\min_{k=1,\ldots,m} d\big(w^{(k)}, \bar{w}^{(k)}\big)\Big)$$

  are satisfied for all $k = 1, \ldots, m$;
- *there exists* $\mathbf{w}_0 \in \mathcal{C}^m([0,T],\mathbb{R})$ *such that* $\mathbf{w}_0 \preccurlyeq_I \mathbf{F}(\mathbf{w}_0)$ *or* $\mathbf{w}_0 \succcurlyeq_I \mathbf{F}(\mathbf{w}_0)$.

Then there exists a $\preccurlyeq_I$*-chain-unique solution of the system of integral equations* (65).

Proof By taking

$$\rho(\mathbf{w}, \bar{\mathbf{w}}) = \min_{k=1,\ldots,m} d\big(w^{(k)}, \bar{w}^{(k)}\big),$$

the desired result follows from Theorem 5.2 immediately. □

Lemma 5.1 *For any* $a, b \in \mathcal{C}([0,T],\mathbb{R})$, *we define*

$$a \preceq^* b \quad \text{if and only if} \quad a(s) \le b(s) \quad \text{for all } s \in [0,T]. \tag{70}$$

Then the quasi-ordered metric space $(\mathcal{C}([0,T],\mathbb{R}), d, \preceq^*)$ *preserves the monotone convergence*.

Proof Let $\{a_n\}_{n\in\mathbb{N}}$ be an $\preceq$-increasing sequence in $(\mathcal{C}([0,T],\mathbb{R}), d, \preceq)$, and let $\widehat{a}$ be the d-limit of $\{a_n\}_{n\in\mathbb{N}}$. Suppose that there exists $n_1 \in \mathbb{N}$ such that $a_{n_1} \not\preceq \widehat{a}$; that is, there exists $s_0 \in [0,T]$ such that $a_{n_1}(s_0) > \widehat{a}(s_0)$. Since $a_n(s) \le a_{n+1}(s)$ for all $s \in [0,T]$ and $n \in \mathbb{N}$, it follows that $a_{n+1}(s_0) \ge a_n(s_0) > \widehat{a}(s_0)$ for all $n \ge n_1$, which contradicts the convergence $a_n(s_0) \to \widehat{a}(s_0)$. Therefore, we must have $a_n(s) \le \widehat{a}(s)$ for all $s \in [0,T]$, *i.e.*, $a_n \preceq \widehat{a}$. If $\{a_n\}_{n\in\mathbb{N}}$ is a $\preceq$-decreasing sequence in $(\mathcal{C}([0,T],\mathbb{R}), d, \preceq)$ and converges to $\widehat{a}$, then we can similarly show that $a_n \succeq \widehat{a}$ for all $n \in \mathbb{N}$. This completes the proof. □

The following result is well known.

Lemma 5.2 *For any* $a, b \in \mathcal{C}([0,T],\mathbb{R})$, *we define*

$$d^*(a,b) = \sup_{s\in[0,T]} \big|a(s) - b(s)\big|. \tag{71}$$

Then the quasi-ordered metric space $(\mathcal{C}([0,T],\mathbb{R}), d^*, \preceq)$ *is complete*.

Given a disjoint pair I and J of $\{1,2,\ldots,m\}$, we can consider a quasi-ordered set $(\mathbb{R}^m, \preccurlyeq_I^{(m)})$ that depends on I, where, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$,

$$\mathbf{x} \preccurlyeq_I^{(m)} \mathbf{y} \quad \text{if and only if} \quad x^{(k)} \le y^{(k)} \quad \text{for } k \in I \quad \text{and} \quad y^{(k)} \le x^{(k)} \quad \text{for } k \in J.$$

Then we have the following interesting existence.

Theorem 5.3 *Let $(\mathcal{C}([0,T],\mathbb{R}), d^*, \preceq^*)$ be a quasi-ordered metric space with the metric d^* and the quasi-order $\preceq^*$ defined in* (71) *and* (70), *respectively. Let I and J be a disjoint pair of $\{1,2,\ldots,m\}$. Define the function $\mathbf{F} : (\mathcal{C}^m([0,T],\mathbb{R}), \preccurlyeq_I^*) \to (\mathcal{C}^m([0,T],\mathbb{R}), \preccurlyeq_I^*)$ by*

$$\mathbf{F}(\mathbf{w})(t) = \int_0^T G(s,t)\big[\mathbf{g}\big(s,\mathbf{w}(s)\big) + \lambda\mathbf{w}(s)\big]\,ds,$$

where $\preccurlyeq_I^*$ is defined in (1) *according to $\preceq^*$. Suppose that the following conditions are satisfied*:

- $\mathbf{F}$ *is $(\preccurlyeq_I^*, \preccurlyeq_I^*)$-increasing*;
- *there exists a function $\rho : \mathcal{C}([0,T],\mathbb{R}) \times \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}_+$ such that, for any $\mathbf{a}, \mathbf{b} \in \mathcal{C}^m([0,T],\mathbb{R})$ with $\mathbf{a} \preccurlyeq_I^* \mathbf{b}$ or $\mathbf{b} \preccurlyeq_I^* \mathbf{a}$, the inequalities $\rho(a^{(k)}, b^{(k)}) \le d^*(a^{(k)}, b^{(k)})$ are satisfied for all $k = 1,\ldots,m$*;
- *there exists a function of the contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ with $\mathbf{x} \preccurlyeq_I^{(m)} \mathbf{y}$ or $\mathbf{y} \preccurlyeq_I^{(m)} \mathbf{x}$, the inequalities*

$$\big|g^{(k)}(s,\mathbf{x}) + \lambda x^{(k)} - g^{(k)}(s,\mathbf{y}) - \lambda y^{(k)}\big| \le \bar{\phi}^*\big(x^{(k)}, y^{(k)}\big) \cdot \widehat{\phi}^*\big(x^{(k)}, y^{(k)}\big) \tag{72}$$

  are satisfied for $k = 1,\ldots,m$, where $\lambda \ge 0$, and the functions $\bar{\phi}^* : \mathbb{R}^2 \to \mathbb{R}_+$ and $\widehat{\phi}^* : \mathbb{R}^2 \to \mathbb{R}_+$ satisfy the following inequalities: for $\mathbf{a}, \mathbf{b} \in \mathcal{C}^m([0,T],\mathbb{R})$

$$\widehat{\phi}^*\big(a^{(k)}(s), b^{(k)}(s)\big) \le \rho\big(a^{(k)}, b^{(k)}\big) \quad \textit{for } s \in [0,T] \tag{73}$$

  and

$$\sup_{t\in[0,T]} \int_0^T G(s,t) \cdot \bar{\phi}^*\big(a^{(k)}(s), b^{(k)}(s)\big)\,ds \le \varphi\big(\rho\big(a^{(k)}, b^{(k)}\big)\big) \tag{74}$$

  for $k = 1,\ldots,m$;
- *there exists $\mathbf{w}_0 \in \mathcal{C}^m([0,T],\mathbb{R})$ such that $\mathbf{w}_0 \preccurlyeq_I^* \mathbf{F}(\mathbf{w}_0)$ or $\mathbf{F}(\mathbf{w}_0) \preccurlyeq_I^* \mathbf{w}_0$.*

Then there exists a $\preccurlyeq_I^*$-chain-unique solution of the system of integral equations (65).

Proof Lemmas 5.1 and 5.2 say that the quasi-ordered metric space $(\mathcal{C}([0,T],\mathbb{R}), d^*, \preceq^*)$ is complete and preserves the monotone convergence. For $\mathbf{w} \preccurlyeq_I^* \bar{\mathbf{w}}$ or $\bar{\mathbf{w}} \preccurlyeq_I^* \mathbf{w}$, it means, for each $s \in [0,T]$,

$$w^{(k)}(s) \le \bar{w}^{(k)}(s) \quad \text{for } k \in I \quad \text{and} \quad \bar{w}^{(k)}(s) \le w^{(k)}(s) \quad \text{for } k \in J$$

or

$$w^{(k)}(s) \le \bar{w}^{(k)}(s) \quad \text{for } k \in I \quad \text{and} \quad \bar{w}^{(k)}(s) \le w^{(k)}(s) \quad \text{for } k \in J,$$

which also says that

$$\mathbf{w}(s) \preccurlyeq_I^{(m)} \bar{\mathbf{w}}(s) \quad \text{or} \quad \bar{\mathbf{w}}(s) \preccurlyeq_I^{(m)} \mathbf{w}(s) \quad \text{for each } s \in [0,T]. \tag{75}$$

Then we have

$$\begin{aligned}
&d\bigl(F_k(\mathbf{w}),F_k(\bar{\mathbf{w}})\bigr)\\
&\quad=\sup_{t\in[0,T]}\bigl|F_k(\mathbf{w})(t)-F_k(\bar{\mathbf{w}})(t)\bigr|\\
&\quad=\sup_{t\in[0,T]}\int_0^T G(s,t)\cdot\bigl|g^{(k)}\bigl(s,\mathbf{w}(s)\bigr)+\lambda w^{(k)}(s)-g^{(k)}\bigl(s,\bar{\mathbf{w}}(s)\bigr)-\lambda\bar{w}^{(k)}(s)\bigr|\,ds\\
&\quad\le\sup_{t\in[0,T]}\int_0^T G(s,t)\cdot\bar{\phi}^*\bigl(w^{(k)}(s),\bar{w}^{(k)}(s)\bigr)\cdot\widehat{\phi}^*\bigl(w^{(k)}(s),\bar{w}^{(k)}(s)\bigr)\,ds\quad\bigl(\text{by (75) and (72)}\bigr)\\
&\quad\le\sup_{t\in[0,T]}\int_0^T G(s,t)\cdot\bar{\phi}^*\bigl(w^{(k)}(s),\bar{w}^{(k)}(s)\bigr)\cdot\rho\bigl(w^{(k)},\bar{w}^{(k)}\bigr)\,ds\quad\bigl(\text{by (73)}\bigr)\\
&\quad\le\varphi\bigl(\rho\bigl(w^{(k)},\bar{w}^{(k)}\bigr)\bigr)\cdot\rho\bigl(w^{(k)},\bar{w}^{(k)}\bigr)\quad\bigl(\text{by (74)}\bigr).
\end{aligned}$$

Using Theorem 5.1, we complete the proof. □

Remark 5.2 The assumption for the inequalities (72) is really weak, since we just assume that it is satisfied for $\preccurlyeq_I^{(m)}$-comparable elements. In other words, if $\mathbf{x}$ and $\mathbf{y}$ are not $\preccurlyeq_I^{(m)}$-comparable, we do not need to check the inequalities (72).

Corollary 5.2 *Let $(\mathcal{C}([0,T],\mathbb{R}),d^*,\preceq^*)$ be a quasi-ordered metric space with the metric d^* and the quasi-order $\preceq^*$ defined in* (71) *and* (70), *respectively. Let I and J be a disjoint pair of $\{1,2,\dots,m\}$. Define the function $\mathbf{F}:(\mathcal{C}^m([0,T],\mathbb{R}),\preccurlyeq_I^*)\to(\mathcal{C}^m([0,T],\mathbb{R}),\preccurlyeq_I^*)$ by*

$$\mathbf{F}(\mathbf{w})(t)=\int_0^T G(s,t)\bigl[\mathbf{g}\bigl(s,\mathbf{w}(s)\bigr)+\lambda\mathbf{w}(s)\bigr]\,ds,$$

where $\preccurlyeq_I^*$ is defined in (1) *according to $\preceq^*$. Suppose that the following conditions are satisfied*:

- $\mathbf{F}$ *is $(\preccurlyeq_I^*,\preccurlyeq_I^*)$-increasing*;
- *there exists a function of the contractive factor $\varphi:[0,\infty)\to[0,1)$ such that, for any $\mathbf{x},\mathbf{y}\in\mathbb{R}^m$ with $\mathbf{x}\preccurlyeq_I^{(m)}\mathbf{y}$ or $\mathbf{y}\preccurlyeq_I^{(m)}\mathbf{x}$, the inequalities*

  $$\bigl|g^{(k)}(s,\mathbf{x})+\lambda x^{(k)}-g^{(k)}(s,\mathbf{y})-\lambda y^{(k)}\bigr|\le\bigl|x^{(k)}-y^{(k)}\bigr|\cdot\bar{\phi}^*\bigl(x^{(k)},y^{(k)}\bigr)$$

  are satisfied for $k=1,\dots,m$, where $\lambda\ge 0$, and the function $\bar{\phi}^*:\mathbb{R}^2\to\mathbb{R}_+$ satisfies the inequalities: *for $a,b\in\mathcal{C}([0,T],\mathbb{R})$,*

  $$\sup_{t\in[0,T]}\int_0^T G(s,t)\cdot\bar{\phi}^*\bigl(a^{(k)}(s),b^{(k)}(s)\bigr)\,ds\le\varphi\bigl(d^*\bigl(a^{(k)},b^{(k)}\bigr)\bigr)$$

  for $k=1,\dots,m$;
- *there exists $\mathbf{w}_0\in\mathcal{C}^m([0,T],\mathbb{R})$ such that $\mathbf{w}_0\preccurlyeq_I^*\mathbf{F}(\mathbf{w}_0)$ or $\mathbf{F}(\mathbf{w}_0)\preccurlyeq_I^*\mathbf{w}_0$.*

Then there exists a $\preccurlyeq_I^*$-chain-unique solution of the system of integral equations (65).

Proof We take $\rho=d^*$ and define the function $\widehat{\phi}^*:\mathbb{R}^2\to\mathbb{R}_+$ by

$$\widehat{\phi}^*\bigl(x^{(k)},y^{(k)}\bigr)=\bigl|x^{(k)}-y^{(k)}\bigr|.$$

Since

$$\left|a^{(k)}(s) - b^{(k)}(s)\right| \leq d^*\left(a^{(k)}, b^{(k)}\right) \quad \text{for all } s \in [0, T],$$

the desired result follows from Theorem 5.3 immediately, and the proof is complete. □

Theorem 5.4 *Let* $(\mathcal{C}([0,T],\mathbb{R}), d^*, \preceq^*)$ *be a quasi-ordered metric space with the metric* d^* *and the quasi-order* $\preceq^*$ *defined in* (71) *and* (70), *respectively. Let* I *and* J *be a disjoint pair of* $\{1,2,\ldots,m\}$. *Define the function* $\mathbf{F} : (\mathcal{C}^m([0,T],\mathbb{R}), \preccurlyeq_I^*) \to (\mathcal{C}^m([0,T],\mathbb{R}), \preccurlyeq_I^*)$ *by*

$$\mathbf{F}(\mathbf{w})(t) = \int_0^T G(s,t)\left[\mathbf{g}\left(s, \mathbf{w}(s)\right) + \lambda \mathbf{w}(s)\right] ds,$$

where $\preccurlyeq_I^*$ *is defined in* (1) *according to* $\preceq^*$. *Suppose that the following conditions are satisfied*:

- $\mathbf{F}$ *is* $(\preccurlyeq_I^*, \preccurlyeq_I^*)$*-increasing*;
- *there exists a function* $\rho : \mathcal{C}^m([0,T],\mathbb{R}) \times \mathcal{C}^m([0,T],\mathbb{R}) \to \mathbb{R}_+$ *such that, for any* $\mathbf{a}, \mathbf{b} \in \mathcal{C}^m([0,T],\mathbb{R})$ *with* $\mathbf{a} \preccurlyeq_I^* \mathbf{b}$ *or* $\mathbf{b} \preccurlyeq_I^* \mathbf{a}$, *the inequalities* $\rho(\mathbf{a},\mathbf{b}) \leq d^*(a^{(k)}, b^{(k)})$ *are satisfied for all* $k = 1, \ldots, m$;
- *there exists a function of the contractive factor* $\varphi : [0,\infty) \to [0,1)$ *such that, for any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ *with* $\mathbf{x} \preccurlyeq_I^{(m)} \mathbf{y}$ *or* $\mathbf{y} \preccurlyeq_I^{(m)} \mathbf{x}$, *the inequalities*

$$\left|g^{(k)}(s,\mathbf{x}) + \lambda x^{(k)} - g^{(k)}(s,\mathbf{y}) - \lambda y^{(k)}\right| \leq \bar{\phi}(\mathbf{x},\mathbf{y}) \cdot \widehat{\phi}(\mathbf{x},\mathbf{y}), \tag{76}$$

  are satisfied for $k = 1, \ldots, m$, *where* $\lambda \geq 0$, *and the functions* $\bar{\phi} : \mathbb{R}^{2m} \to \mathbb{R}_+$ *and* $\widehat{\phi} : \mathbb{R}^{2m} \to \mathbb{R}_+$ *satisfy the following inequalities*: *for* $\mathbf{a}, \mathbf{b} \in \mathcal{C}^m([0,T],\mathbb{R})$,

$$\widehat{\phi}\left(\mathbf{a}(s), \mathbf{b}(s)\right) \leq \rho(\mathbf{a},\mathbf{b}) \quad \textit{for all } s \in [0,T] \tag{77}$$

  and

$$\sup_{t\in[0,T]} \int_0^T G(s,t) \cdot \bar{\phi}\left(\mathbf{a}(s), \mathbf{b}(s)\right) ds \leq \varphi\left(\rho(\mathbf{a},\mathbf{b})\right); \tag{78}$$

- *there exists* $\mathbf{w}_0 \in \mathcal{C}^m([0,T],\mathbb{R})$ *such that* $\mathbf{w}_0 \preccurlyeq_I^* \mathbf{F}(\mathbf{w}_0)$ *or* $\mathbf{F}(\mathbf{w}_0) \preccurlyeq_I^* \mathbf{w}_0$.

Then there exists a $\preccurlyeq_I^*$*-chain-unique solution of the system of integral equations* (65).

Proof We first have

$$\begin{aligned}
& d\left(F_k(\mathbf{w}), F_k(\bar{\mathbf{w}})\right) \\
&\quad = \sup_{t\in[0,T]} \left|F_k(\mathbf{w})(t) - F_k(\bar{\mathbf{w}})(t)\right| \\
&\quad = \sup_{t\in[0,T]} \int_0^T G(s,t) \cdot \left|g^{(k)}\left(s,\mathbf{w}(s)\right) + \lambda w^{(k)}(s) - g^{(k)}\left(s,\bar{\mathbf{w}}(s)\right) - \lambda \bar{w}^{(k)}(s)\right| ds \\
&\quad \leq \sup_{t\in[0,T]} \int_0^T G(s,t) \cdot \bar{\phi}\left(\mathbf{w}(s), \bar{\mathbf{w}}(s)\right) \cdot \widehat{\phi}\left(\mathbf{w}(s), \bar{\mathbf{w}}(s)\right) ds \quad \text{(by (76) and (75))}
\end{aligned}$$

$$\begin{aligned}&\le \sup_{t\in[0,T]}\int_0^T G(s,t)\cdot\bar{\phi}\big(\mathbf{w}(s),\bar{\mathbf{w}}(s)\big)\cdot\rho(\mathbf{w},\bar{\mathbf{w}})\,ds \quad \big(\text{by (77)}\big)\\ &\le \varphi\big(\rho(\mathbf{w},\bar{\mathbf{w}})\big)\cdot\rho(\mathbf{w},\bar{\mathbf{w}}) \quad \big(\text{by (78)}\big).\end{aligned}$$

By applying Theorem 5.2 to the argument in the proof of Theorem 5.3, the desired result can be obtained immediately. □

Corollary 5.3 *Let* $(\mathcal{C}([0,T],\mathbb{R}),d^*,\preceq^*)$ *be a quasi-ordered metric space with the metric* d^* *and the quasi-order* $\preceq^*$ *defined in* (71) *and* (70), *respectively. Let* I *and* J *be a disjoint pair of* $\{1,2,\ldots,m\}$. *Define the function* $\mathbf{F}:(\mathcal{C}^m([0,T],\mathbb{R}),\preccurlyeq_I^*)\to(\mathcal{C}^m([0,T],\mathbb{R}),\preccurlyeq_I^*)$ *by*

$$\mathbf{F}(\mathbf{w})(t)=\int_0^T G(s,t)\big[\mathbf{g}\big(s,\mathbf{w}(s)\big)+\lambda\mathbf{w}(s)\big]\,ds,$$

where $\preccurlyeq_I^*$ *is defined in* (1) *according to* $\preceq^*$. *Suppose that the following conditions are satisfied*:

- $\mathbf{F}$ *is* $(\preccurlyeq_I^*,\preccurlyeq_I^*)$-*increasing*;
- *there exists a function of the contractive factor* $\varphi:[0,\infty)\to[0,1)$ *such that, for any* $\mathbf{x},\mathbf{y}\in\mathbb{R}^m$ *with* $\mathbf{x}\preccurlyeq_I^{(m)}\mathbf{y}$ *or* $\mathbf{y}\preccurlyeq_I^{(m)}\mathbf{x}$, *the inequalities*

  $$\big|g^{(k)}(s,\mathbf{x})+\lambda x^{(k)}-g^{(k)}(s,\mathbf{y})-\lambda y^{(k)}\big|\le\Big(\min_{k=1,\ldots,m}\big|x^{(k)}-y^{(k)}\big|\Big)\cdot\bar{\phi}(\mathbf{x},\mathbf{y})$$

  are satisfied for $k=1,\ldots,m$, *where* $\lambda\ge 0$, *and the function* $\bar{\phi}:\mathbb{R}^{2m}\to\mathbb{R}_+$ *satisfies the following inequality*: *for* $\mathbf{a},\mathbf{b}\in\mathcal{C}^m([0,T],\mathbb{R})$,

  $$\sup_{t\in[0,T]}\int_0^T G(s,t)\cdot\bar{\phi}\big(\mathbf{a}(s),\mathbf{b}(s)\big)\,ds\le\varphi\Big(\min_{k=1,\ldots,m}d^*\big(a^{(k)},b^{(k)}\big)\Big);$$

- *there exists* $\mathbf{w}_0\in\mathcal{C}^m([0,T],\mathbb{R})$ *such that* $\mathbf{w}_0\preccurlyeq_I^*\mathbf{F}(\mathbf{w}_0)$ *or* $\mathbf{F}(\mathbf{w}_0)\preccurlyeq_I^*\mathbf{w}_0$.

Then there exists a $\preccurlyeq_I^*$-*chain-unique solution of the system of integral equations* (65).

Proof For $\mathbf{a},\mathbf{b}\in\mathcal{C}^m([0,T],\mathbb{R})$, we define the function $\rho:\mathcal{C}^m([0,T],\mathbb{R})\times\mathcal{C}^m([0,T],\mathbb{R})\to\mathbb{R}_+$ by

$$\rho(\mathbf{a},\mathbf{b})=\min_{k=1,\ldots,m}d^*\big(a^{(k)},b^{(k)}\big) \tag{79}$$

and the function $\widehat{\phi}:\mathbb{R}^{2m}\to\mathbb{R}_+$ by

$$\widehat{\phi}(\mathbf{x},\mathbf{y})=\min_{k=1,\ldots,m}\big|x^{(k)}-y^{(k)}\big|.$$

Since

$$\big|a^{(k)}(s)-b^{(k)}(s)\big|\le d^*\big(a^{(k)},b^{(k)}\big)\quad\text{for all } s\in[0,T],$$

we have

$$\widehat{\phi}\big(\mathbf{a}(s),\mathbf{b}(s)\big)=\min_{k=1,\ldots,m}\big|a^{(k)}(s)-b^{(k)}(s)\big|\le\min_{k=1,\ldots,m}d^*\big(a^{(k)},b^{(k)}\big)=\rho(\mathbf{a},\mathbf{b}) \tag{80}$$

for all $s \in [0, T]$. The desired result follows from Theorem 5.4 immediately, and the proof is complete. □

Compared to Corollary 5.3, we consider the different type of inequalities below.

Theorem 5.5 *Let $(\mathcal{C}([0,T],\mathbb{R}), d^*, \preceq^*)$ be a quasi-ordered metric space with the metric d^* and the quasi-order $\preceq^*$ defined in* (71) *and* (70), *respectively. Let I and J be a disjoint pair of $\{1,2,\ldots,m\}$. Define the function $\mathbf{F} : (\mathcal{C}^m([0,T],\mathbb{R}), \preccurlyeq_I^*) \to (\mathcal{C}^m([0,T],\mathbb{R}), \preccurlyeq_I^*)$ by*

$$\mathbf{F}(\mathbf{w})(t) = \int_0^T G(s,t)\big[\mathbf{g}\big(s, \mathbf{w}(s)\big) + \lambda \mathbf{w}(s)\big]\,ds,$$

where $\preccurlyeq_I^*$ is defined in (1) *according to $\preceq^*$. Suppose that the following conditions are satisfied*:

- $\mathbf{F}$ *is $(\preccurlyeq_I^*, \preccurlyeq_I^*)$-increasing*;
- *there exists a function of the contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ with $\mathbf{x} \preccurlyeq_I^{(m)} \mathbf{y}$ or $\mathbf{y} \preccurlyeq_I^{(m)} \mathbf{x}$, the inequalities*

$$\big|g^{(k)}(s,\mathbf{x}) + \lambda x^{(k)} - g^{(k)}(s,\mathbf{y}) - \lambda y^{(k)}\big| \le \Big(\min_{k=1,\ldots,m} \big|x^{(k)} - y^{(k)}\big|\Big) \cdot \bar{\phi}^*\big(x^{(k)}, y^{(k)}\big) \tag{81}$$

  are satisfied for $k = 1,\ldots,m$, where $\lambda \ge 0$, and the function $\bar{\phi}^* : \mathbb{R}^2 \to \mathbb{R}_+$ satisfies the inequalities: *for $a, b \in \mathcal{C}([0,T],\mathbb{R})$,*

$$\sup_{t\in[0,T]} \int_0^T G(s,t) \cdot \bar{\phi}^*\big(a^{(k)}(s), b^{(k)}(s)\big)\,ds \le \varphi\Big(\Big(\min_{k=1,\ldots,m} d^*\big(a^{(k)}, b^{(k)}\big)\Big)\Big) \tag{82}$$

  for $k = 1,\ldots,m$;
- *there exists $\mathbf{w}_0 \in \mathcal{C}^m([0,T],\mathbb{R})$ such that $\mathbf{w}_0 \preccurlyeq_I^* \mathbf{F}(\mathbf{w}_0)$ or $\mathbf{F}(\mathbf{w}_0) \preccurlyeq_I^* \mathbf{w}_0$.*

Then there exists a $\preccurlyeq_I^*$-chain-unique solution of the system of integral equations (65).

Proof For $\mathbf{a}, \mathbf{b} \in \mathcal{C}^m([0,T],\mathbb{R})$, we define a function $\rho : \mathcal{C}^m([0,T],\mathbb{R}) \times \mathcal{C}^m([0,T],\mathbb{R}) \to \mathbb{R}_+$ by (79). Now, we have

$$\begin{aligned}
& d\big(F_k(\mathbf{w}), F_k(\bar{\mathbf{w}})\big) \\
&\quad = \sup_{t\in[0,T]} \big|F_k(\mathbf{w})(t) - F_k(\bar{\mathbf{w}})(t)\big| \\
&\quad = \sup_{t\in[0,T]} \int_0^T G(s,t) \cdot \big|g^{(k)}\big(s,\mathbf{w}(s)\big) + \lambda w^{(k)}(s) - g^{(k)}\big(s,\bar{\mathbf{w}}(s)\big) - \lambda \bar{w}^{(k)}(s)\big|\,ds \\
&\quad \le \sup_{t\in[0,T]} \int_0^T G(s,t) \cdot \bar{\phi}^*\big(w^{(k)}(s), \bar{w}^{(k)}(s)\big) \cdot \Big(\min_{k=1,\ldots,m} \big|w^{(k)}(s) - \bar{w}^{(k)}(s)\big|\Big)\,ds \\
&\qquad \big(\text{by (75) and (81)}\big) \\
&\quad \le \sup_{t\in[0,T]} \int_0^T G(s,t) \cdot \bar{\phi}^*\big(w^{(k)}(s), \bar{w}^{(k)}(s)\big) \cdot \rho(\mathbf{w}, \bar{\mathbf{w}})\,ds \quad \big(\text{by (80)}\big) \\
&\quad \le \varphi\big(\rho(\mathbf{w}, \bar{\mathbf{w}})\big) \cdot \rho(\mathbf{w}, \bar{\mathbf{w}}) \quad \big(\text{by (82) and (79)}\big).
\end{aligned}$$

Using Theorem 5.2, we complete the proof. □

Competing interests
The author declares that he has no competing interests.

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References

1. Choudhurya, BS, Kundub, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. **73**, 2524-2531 (2010)
2. Latif, A, Tweddle, I: Some results on coincidence points. Bull. Aust. Math. Soc. **59**, 111-117 (1999)
3. Tarafdar, E, Watson, PJ: A coincidence point theorem and related results. Appl. Math. Lett. **11**, 37-40 (1998)
4. Väth, M: Coincidence points of function pairs based on compactness properties. Glasg. Math. J. **44**, 209-230 (2002)
5. Abbas, M, Khan, AR, Nazir, T: Coupled common fixed point results in two generalized metric spaces. Appl. Math. Comput. **217**, 6328-6336 (2011)
6. Ahmed, MA: Common fixed points of hybrid maps and an application. Comput. Math. Appl. **60**, 1888-1894 (2010)
7. Bhatt, A, Chandraa, H, Sahu, DR: Common fixed point theorems for occasionally weakly compatible mappings under relaxed conditions. Nonlinear Anal. **73**, 176-182 (2010)
8. Chang, S-S, Joseph Lee, HW, Chan, CK: A new hybrid method for solving a generalized equilibrium problem, solving a variational inequality problem and obtaining common fixed points in Banach spaces, with applications. Nonlinear Anal. **73**, 2260-2270 (2010)
9. Ćirić, L: Common fixed point theorems for a family of non-self mappings in convex metric spaces. Nonlinear Anal. **71**, 1662-1669 (2009)
10. Ćirić, L, Samet, B, Vetro, C: Common fixed point theorems for families of occasionally weakly compatible mappings. Math. Comput. Model. **53**, 631-636 (2011)
11. Ćirić, L, Ume, JS: Some common fixed point theorems for weakly compatible mappings. J. Math. Anal. Appl. **314**, 488-499 (2006)
12. Colao, V, Marino, G: Strong convergence for a minimization problem on points of equilibrium and common fixed points of an infinite family of nonexpansive mappings. Nonlinear Anal. **73**, 3513-3524 (2010)
13. Liu, Q-Y, Liu, Z-B, Huang, N-J: Approximating the common fixed points of two sequences of uniformly quasi-Lipschitzian mappings in convex metric spaces. Appl. Math. Comput. **216**, 883-889 (2010)
14. Moradi, S, Fathi, Z, Analouee, E: The common fixed point of single-valued generalized φ_f-weakly contractive mappings. Appl. Math. Lett. **24**, 771-776 (2011)
15. Qin, X, Cho, SY, Zhou, H: Common fixed points of a pair of non-expansive mappings with applications to convex feasibility problems. Glasg. Math. J. **52**, 241-252 (2010)
16. Mizoguchi, N, Takahashi, W: Fixed point theorems for multivalued mappings on complete metric spaces. J. Math. Anal. Appl. **141**, 177-188 (1989)
17. Takahashi, W: Fixed point theorems for new nonlinear mappings in a Hilbert space. J. Nonlinear Convex Anal. **11**, 79-88 (2010)
18. Nadler, SB Jr.: Multi-valued contractive mappings. Pac. J. Math. **30**, 475-488 (1969)
19. Suzuki, T: Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's. J. Math. Anal. Appl. **340**, 752-755 (2008)

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