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Research article

The viscosity solutions of a nonlinear equation related to the *p-Laplacian*

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Abstract: The viscosity solutions of a nonlinear equation related to the *p-Laplacian* are considered. Besides there is a damping term in the equation, a nonlocal function is added. By considering the regularized problem and using Moser iteration technique, we get the uniformly local bounded properties of the solutions and the L^p-norm for the gradients. By the compactness theorem, we prove the existence of the viscosity solution of the equation.

Keywords: Nonlinear equation; *p-Laplacian*; moser iteration; viscosity solution
Mathematics Subject Classification: 35K55, 35K65, 35B40.

1. Introduction

The objective of the paper is to study the nonnegative weak solutions of nonlinear parabolic equation with the type

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) - a(x)u^{mq_1}|\nabla u^m|^{p_1} + f_0(u^m)\int_\Omega K(y)|u^m(y,t)|^\beta dy + g(x), \quad \text{in } S = \Omega\times(0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with smooth boundary $\partial\Omega$, $\int_\Omega K(y)|u(y,t)|^\beta dy$ represents a nonlocal function dependent on spatial domain Ω, $a(x) \geq 0$ is a bounded function, $K(x)$ and $g(x)$ are bounded functions too, and ∇ is the spatial gradient operator. We assume that $p > 1,\ m > 1,\ p_1 \leq 2$, $p > 2p_1,\ N \geq 1$,

$$0 \leq u_0^m(x) \in L^{q-1+\frac{1}{m}}(\Omega), q > 1,\ |f_0(s)| \leq c|s|^{\frac{1}{m}},\ s \in R^1 = (-\infty,\infty). \tag{1.4}$$

As usual, the here and after, the constants c may be different from one to another. The equation with the type of (1.1) has been suggested as the mathematical model for a variety of problems in mechanics, physics and biology, which can be found in [10, 11, 15, 17] et al. Equation (1.1) has been widely researched, whether it is linear or nonlinear, is uniformly parabolic or degenerate parabolic. In what follows, we only give a very roughly review.

If $a(x) = g(x) = f_0 \equiv 0$, the existence of nonnegative solution of the problem (1.1)-(1.3), defined in weak sense, is well established (see [10], [6] et al.).

If $g(x) = f \equiv 0$, some special cases of equation (1.1) had been researched by Bertsh [3], Zhou [36] and Zhang [34] et al. For examples, the existence and the properties of the viscosity solution to the following equation are obtained in [3, 36]

$$u_t = u\triangle u - \gamma|\nabla u|^2, \tag{1.5}$$

where γ is a positive constant. The existence and the properties of the viscosity solution to the following equation are obtained in [34]

$$u_t = \triangle u - b(x)|u|^{q-1}|\nabla u|^2, \tag{1.6}$$

where $b(x)$ is a known function. The most important characteristic of the equation (1.5) or (1.6) lies in that, generally, the uniqueness of the solutions is not true, one can refer to [4,9,29,34,36] for the details. Thus, for the equation with the type of (1.1), one mainly concerns with the existence of the viscosity solution and the related properties such as the large time behavior, one can refer to [8, 20, 33, 35] et al. for some progresses in the direction.

But if $p_1 = 0$, it is well-known the uniqueness of the solutions is true. Aassila [1] studied equation (1.1) when $p = 2, m = 1$ and proved the existence of solution by Schauder fixed point theorem, studied the convergence of the solution towards a steady state by using the point of view in dynamical systems. Cholewa and Dlotko [7], Teman [28] considered the following problem

$$u_t - \operatorname{div}(|\ \nabla u\ |^{p-2}\ \nabla u) + |u|^\alpha u = f_0(u) + g(x), \tag{1.7}$$

and proved the existence of global attractor in L^2 which is in fact a bounded set in $W_0^{1,p} \bigcap L^{\alpha+2}$. Chen [20] studied the long time behavior of solutions for following equation

$$u_t - \operatorname{div}(|\ \nabla u\ |^{p-2}\ \nabla u) + a(x)|u|^\alpha u = f_0(u)\int_\Omega K(y)|u(y,t)|^\beta dy + g(x), \tag{1.8}$$

and obtained the existence and L^p estimate of the global attractor.

While the papers, first by Nakao-Chen [25] and later by Chen-Wang [6], had studies the global existence and the gradient estimate for the quasilinear parabolic equation of m-Laplacian type with a nonlinear convection term, the typical equations included in [6, 25] are with the form as

$$u_t = \operatorname{div}(u^r|\nabla u|^{p-2}\nabla u) + \nabla A(u). \tag{1.9}$$

In our paper, we will study the global solution of equation (1.1) with the initial value (1.2) and homogeneous boundary value (1.3) by the usual regularized method. The main techniques are inspired by [6, 25]. However, due to the local and the nonlocal nonlinearity of the equation we considered, even to prove the initial value condition, we have to put some restrictions in the exponents of m, p, p_1, q_1. In particular, as we have said, instead of the nonlinear convection term $\nabla A(u)$ in equation (1.9), equation (1.1) contains the damping term $-a(x)u^{mq_1}|\nabla u^{mp_1}|$, the uniqueness of the solutions generally is not true.

We can only prove the uniqueness of the solutions under the condition $p_1 = 0$. If $p_1 \neq 0$ we only can prove the uniqueness of the viscosity solutions. At the same time, comparing with [5], since equation (1.1) is more complicated, how to get the estimate in the gradient term of the solution, and how to prove the continuity of the solution etc, become more difficult. A clear promotion lies in that we put not any restrictions in the derivative $f_0'(s)$ of the function $f_0(s)$, while it must satisfy that $|f_0'(s)| \leq c|s|^{r-1}$ in [5]. Other related works on equation (1.1), one can refer to the references [2,14,16,18,19,22,24,27,30–32] et al.

Now we quote the following definition.

Definition 1.1. *A nonnegative function $u(x,t)$ is called a weak solution of (1.1)-(1.3) if u satisfies*

(i)

$$u \in L^{\infty}_{loc}(0,\infty; L^{\infty}(\Omega)), \tag{1.10}$$

$$u_t \in L^2_{loc}(0,\infty; L^2(\Omega)), \quad u^m \in L^{\infty}_{loc}(0,\infty; W_0^{1,p}(\Omega)), \tag{1.11}$$

(ii)

$$\iint_S \left[u\varphi_t - |\nabla u^m|^{p-2} \nabla u^m \cdot \nabla\varphi - a(x)u^{mq_1}|\nabla u^m|^{p_1}\varphi \right] dxdt$$

$$+ \iint_S \left[f_0(u^m) \int_\Omega K(y)|u^m(y,t)|^{\beta} dy + g(x) \right] \varphi dxdt = 0, \ \forall \varphi \in C_0^1(S); \tag{1.12}$$

(iii)

$$\lim_{t\to 0} \int_\Omega |u(x,t) - u_0(x)| dx = 0. \tag{1.13}$$

We are to get the solution of problem (1.1)-(1.3) by considering the regularized equation

$$u_t = \mathrm{div}((|\nabla u^m|^2 + \frac{1}{k})^{\frac{p-2}{2}} \nabla u^m) - a(x)u^{mq_1}|\nabla u^m|^{p_1} + f_0(u^m) \int_\Omega K(y)|u^m(y,t)|^{\beta} dy + g(x), \tag{1.14}$$

with the initial value (1.2) and the homogeneous boundary value (1.3). Here $0 \leq u_{0k}(x)$ is a suitable smooth function such that $u_{0k}(x) \in L^\infty(\Omega)$, $\lim_{k\to\infty} \|u_{0k}^m\|_{q-1+\frac{1}{m}} = \|u_0^m\|_{q-1+\frac{1}{m}}$.

Definition 1.2. *If u_k is the solution of the initial boundary value problem of (1.14)-(1.2)-(1.3), $\lim_{k\to\infty} u_k = u$, a.e in S, u is a weak solution of (1.1)-(1.3), then u is said to be a viscosity solution.*

We need some important lemmas in order to get our results.

Lemma 1.1. *If $1 \leq l < N$, $1+\beta \leq q$, $1 \leq r \leq q \leq (1+\beta)Nl/(N-l)$, $u^{1+\beta} \in W^{1,l}(\Omega)$, then*

$$\|u\|_q \leq c^{1/(1+\beta)} \|u\|_r^{1-\theta} \|u^{1+\beta}\|_{1,l}^{\theta/(1+\beta)}, \tag{1.15}$$

where $\theta = (\beta+1)(r^{-1} - q^{-1})/(N^{-1} - l^{-1} + (\beta+1)r^{-1})$.

This lemma is a general version of Gagliardo-Nirenberg inequality, it is first proved by M. Nakao [23].

Lemma 1.2. *Let $y(t)$ be a nonnegative function on $(0,T]$. If it satisfies*

$$y'(t) + At^{\lambda\theta-1}y^{1+\theta}(t) \leq Bt^{-k}y(t) + Ct^{-\delta}, 0 < t \leq T, \tag{1.16}$$

where $A, \theta > 0$, $\lambda\theta \geq 1$, $B, C \geq 0, k \leq 1$, then

$$y(t) \leq A^{-\frac{1}{\theta}}(2\lambda + 2BT^{1-k})^{\frac{1}{\theta}} t^{-\lambda} + 2C(\lambda + BT^{1-k})^{-1} t^{1-\delta}, 0 < t \leq T. \tag{1.17}$$

This lemma can be found in [26].

Lemma 1.3. *Suppose* $L_1 \geq 1$, $r, R, M > 0$, $\lambda_1 > 0$. *For* $n = 2, 3, \cdots$, *let*

$$L_n = RL_{n-1} - M,\ \theta_n = NR(1 - L_{n-1}L_n^{-1})(N(R-1) + r)^{-1},$$

$$\beta_n = (L_n + M)\theta_n^{-1} - L_n,\ \lambda_n = (1 + \lambda_{n-1}(\beta_n - M))\beta_n^{-1}.$$

Then

$$\lim_{n\to\infty} \lambda_n = \frac{L_1\lambda_1 r + N}{l_1 + MN}. \tag{1.18}$$

This lemma also was first proved in [25], then used in [6].

In our paper, we assume that $p > 1 + \frac{1}{m}$, so equation (1.1) is a doubly degenerate parabolic equation. By considering the solution u_k of the regularized problem (1.14)-(1.2)-(1.3) and using Moser iteration technique, we get u_k's local bounded properties and the local bounded properties of the L^p-norm of the gradient ∇u_k. By the compactness theorem, we get the existence of the viscosity solution of the diffusion equation itself. In details, we will prove the following theorems.

Theorem 1.1. *It is supposed that* K, g *are suitable smooth bounded functions,* $a(x) \in C(\overline{\Omega})$ *and exists* $a_0 > 0$, *such that* $a(x) \geq a_0$ *in* Ω, f_0 *satisfies (1.4). If* $p > 1 + \frac{1}{m}$, $u_0(x) \geq 0$,

$$u_0^m(x) \in L^{q-1+\frac{1}{m}}(\Omega), 3 > q > 2 - \frac{1}{m}, \tag{1.19}$$

$$p_1 \leq 2,\ 2p_1 < p,\ \beta < \max\{p - 1 - \frac{1}{m}, q - 1 + \frac{1}{m}\}, \tag{1.20}$$

$$\epsilon = \max\{\frac{mNq_1}{Nm(p-1) - N + mq} + \frac{p_1(m(p-1) + m - 2)}{m(p-1) - 1}, \frac{(\beta + m)N}{Nm(p-1) - N + mq}\} < 1, \tag{1.21}$$

then the problem (1.1)-(1.3) has a weak viscosity solution u, *satisfying*

$$u^m \in L^\infty_{\rm loc}(0, \infty; L^{q+1-\frac{1}{m}}(\Omega)) \bigcap L^\infty_{loc}(0, \infty; W_0^{1,p}(\Omega)), \tag{1.22}$$

and

$$\|u^m(t)\|_\infty \leq c(1 + t^{-\lambda})(1 + t)^{-1/(p-1-\frac{1}{m})}, t > 0, \tag{1.23}$$

where $\lambda = N(pq + (p - 1 - \frac{1}{m})N)^{-1}$. *Moreover, if* $p > 2$, *then*

$$\|\nabla u^m\|_p \leq c(1 + t^{-\delta_1})(1 + t)^{-\sigma}, t > 0, \tag{1.24}$$

where

$$\delta_1 = \max\{1 + \frac{m-1}{m(p-1) - 1}, \delta - 1\}, \delta = \max\{\frac{m+1}{m}, 2\beta\},$$

and

$$\sigma = \frac{p[m(2q_1 + 1) - 1] + mp_1}{[m(p-1) - 1](p - p_1)}.$$

Remark 1.1. *The condition (1.21) is only used to prove (1.13). We conjecture that this condition can be weaken.*

Theorem 1.2. *Let* u *be a nonnegative weak solution of problem (1.1)-(1.3). If* $g(x) \leq 0$, $f_0'(s) \geq 0$, *if* $p > 1 + \frac{1}{m}$, $p_1 + q_1 > (p - 1)$ *then*

$$\mathrm{supp}u(., s) \subset \mathrm{supp}u(., t), \tag{1.25}$$

for all s, t *with* $0 < s < t$.

2. The L^∞ estimation of the solution

Instead of considering the regularized problem (1.14)-(1.2)-(1.3) directly as one deals with the case $m = 1$, we have to consider the following approximate problem. For small $s > 0$, we consider

$$u_t = \operatorname{div}((|\nabla u^m|^2 + \frac{1}{k})^{\frac{p-2}{2}} \nabla u^m) - a(x)u^{mq_1}|\nabla u^m|^{p_1} + f_0(u^m)\int_\Omega K(y)|u^m(y,t)|^\beta dy + g(x), \tag{2.1}$$

$$u(x,0) = u_{0k}(x) + s, x \in \Omega, \tag{2.2}$$

$$u(x,t) = s, x \in \partial\Omega, t \geq 0, \tag{2.3}$$

where $0 \leq u_{0k}(x)$ is a suitable smooth function such that $u_{0k}(x) \in L^\infty(\Omega)$, $\lim_{k\to\infty} \|u_{0k}^m\|_{q-1+\frac{1}{m}} = \|u_0^m\|_{q-1+\frac{1}{m}}$.

Similar as the chapter 8 of [13], in which the existence of the initial boundary value problem of the quaslinear equation in the divergent form is obtained, by Leray-Schauder fixed point theory, using the condition $p_1 \leq 2$, we know that problem (2.1)-(2.3) has a nonnegative classical solution u_{ks}, we omit the details here.

Let $s \to 0$. In a similar way as [33], we are able to prove that

$$u_{ks} \to u_k, \text{ in } C(S),$$

$$\nabla u_{ks}^m \rightharpoonup \nabla u_k^m, \text{ in } L^p(S),$$

$$u_{kst} \rightharpoonup \nabla u_{kt}, \text{ in } L^2(S),$$

$$|\nabla u_{ks}^m|^{p-2}\nabla u_{ksx_i}^m \rightharpoonup * \, |\nabla u_k^m|^{p-2}\nabla u_{kx_i}^m, \text{weakly star in } L^\infty_{\text{loc}}(0,\infty; L^{\frac{p}{p-1}}(\Omega)),$$

and u_k is the solution of equation (2.1) with the following initial boundary values

$$u(x,0) = u_{0k}(x), x \in \Omega, \tag{2.4}$$

$$u(x,t) = 0, x \in \partial\Omega, t \geq 0. \tag{2.5}$$

Lemma 2.1. *Assume that*

(H_1) $a(x) \in C(\overline{\Omega})$ and exists $a_0 > 0$, such that $a(x) \geq a_0$ in Ω;

(H_2) $f_0(s) \in C(R^1)$, $|f_0(s)| \leq K_0|s|^{\frac{1}{m}}$, for some $K_0 > 0$.

(H_3) $g(x), K(x) \in L^\infty$.

In addition, $\beta + \frac{1}{m} < q_1$, $3 > q \geq 2 - \frac{1}{m}$, then $u_k^m \in L^\infty_{loc}(0,\infty; L^{q-1+\frac{1}{m}}(\Omega))$ and

$$\|u_k^m\|_{q-1+\frac{1}{m}} \leq c(1+t)^{-\frac{1}{p-1-\frac{1}{m}}}, t \geq 0. \tag{2.6}$$

Proof. In the proof what follows, we only denote u_k as u for simplicity. We only give the proof of the case $q > 2 - \frac{1}{m}$, if $q = 2 - \frac{1}{m}$, one can get the conclusion just a minor version. Let $A_n = (q-2)n^{3-q}$, $B_n = (3-q)n^{2-q}$, and

$$f_n(s) = \begin{cases} s^{q-1}, & \text{if } s \geq \frac{1}{n}, \\ A_n s^2 + B_n s, & \text{if } 0 \leq s < \frac{1}{n}. \end{cases}$$

Suppose that $n > k$, multiply (2.1) with $f_n(u^m)$ and integrate it on Ω. Since $f'(s) > 0$,then we have

$$\int_\Omega f_n(u^m)\mathrm{div}(|\nabla u^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)dx = -\int_\Omega (|\nabla u^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2 f_n'(u^m)dx$$

$$\leq -\int_\Omega |\nabla u^m|^p f_n'(u^m)dx = -\int_\Omega |\nabla \int_0^{u^m} (f_n'(s))^{\frac{1}{p}}ds|^p\, dx, \tag{2.7}$$

$$-\int_\Omega a(x)f_n(u^m)u^{mq_1}|\nabla u^m|^{p_1}dx \leq 0. \tag{2.8}$$

Suppose that $| f_0(s) | \leq K_0 s^r$. Then

$$|\int_{\Omega\bigcap\{u^m\leq\frac{1}{n}\}} f_0(u^m)f_n(u^m)\int_\Omega K(y)|u^m(y,t)|^\beta dydx|$$

$$\leq c(K)\int_{\Omega\bigcap\{u^m\leq\frac{1}{n}\}} u^{mr}(A_n u^{2m} + B_n u^m)dx\int_\Omega |u|^{m\beta}dy$$

$$\leq c(K)n^{1-q-r}\int_\Omega |u|^{m\beta}dy \leq c(K)n^{1-q-r}\|u^m\|^\beta_{q-1+\frac{1}{m}}.$$

If $r = \frac{1}{m}$,

$$|\int_{\Omega\bigcap\{u^m>\frac{1}{n}\}} f_0(u^m)f_n(u^m)\int_\Omega K(y)|u^m(y,t)|^\beta dydx|$$

$$\leq c(K)\int_\Omega u^{m(r+q-1)}dx\int_\Omega |u|^{m\beta}dy \leq c\|u^m\|^{q-1+\frac{1}{m}+\beta}_{q-1+\frac{1}{m}},$$

we have

$$|\int_\Omega f_0(u^m)f_n(u^m)\int_\Omega K(y)|u^m(y,t)|^\beta dydx|$$

$$\leq c\|u^m\|^\beta_{q-1+\frac{1}{m}}[n^{1-q-\frac{1}{m}} + \|u^m\|^{q-1+\frac{1}{m}}_{q-1+\frac{1}{m}}]. \tag{2.9}$$

$$|\int_{\Omega\bigcap\{u^m>\frac{1}{n}\}} f_n(u^m)g(x)dx| \leq c(g)\int_\Omega u^{m(q-1)}dx \leq c(g)\|u^m\|^{q-1}_{q-1+\frac{1}{m}}. \tag{2.10}$$

From the above calculations, we have

$$\int_\Omega f_n(u^m)u_t dx + \int_\Omega |\nabla \int_0^{u^m}(f_n'(s))^{\frac{1}{p}}ds|^p\, dx \leq c\|u^m\|^{q-1+\frac{1}{m}+\beta}_{q-1+\frac{1}{m}} + O(\frac{1}{n^{q-1}}), \tag{2.11}$$

by Poincare inequality, we have

$$\int_\Omega f_n(u^m)u_t dx + c\int_\Omega |\int_0^{u^m}(f_n'(s))^{\frac{1}{p}}ds|^p\, dx \leq c\|u^m\|^{q-1+\frac{1}{m}+\beta}_{q-1+\frac{1}{m}} + O(\frac{1}{n^{q-1}}). \tag{2.12}$$

Let $n \to \infty$ in (2.12). We can deduce that

$$\frac{d}{dt}\int_\Omega u^{m(q-1)+1}dx + c\int_\Omega u^{m[q-1+\frac{1}{m}+p-1-\frac{1}{m}]}dx \leq c\|u^m\|^{q-1+\frac{1}{m}+\beta}_{q-1+\frac{1}{m}}. \tag{2.13}$$

By Jessen inequality, from (2.13) we get

$$\frac{d}{dt}\|u^m\|_{q-1+\frac{1}{m}}^{q-1+\frac{1}{m}} + c\|u^m\|_{q-1+\frac{1}{m}}^{q-1+\frac{1}{m}+p-1-\frac{1}{m}} \leq c\|u^m\|_{q-1+\frac{1}{m}}^{q-1+\frac{1}{m}+\beta}.$$

If

$$\beta < p - 1 - \frac{1}{m}$$

by young inequality,

$$\frac{d}{dt}\|u^m\|_{q-1+\frac{1}{m}}^{q-1+\frac{1}{m}} + c\|u^m\|_{q-1+\frac{1}{m}}^{q-1+\frac{1}{m}+p-1-\frac{1}{m}} \leq c,$$

then

$$\|u^m\|_{q+1-\frac{1}{m}} \leq c(1+t)^{-\frac{1}{p-1-\frac{1}{m}}}.$$

We get the desired result. □

Lemma 2.2. *If $p > 1 + \frac{1}{m}$, u_k is the solution of problem (2.1)-(2.4)-(2.5), then*

$$\|u_k^m\|_\infty \leq ct^{-\lambda},\ 0 < t \leq 1, \tag{2.14}$$

$$\|u_k^m\|_\infty \leq c(1+t)^{-\frac{1}{p-1-\frac{1}{m}}}, t \geq 1, \tag{2.15}$$

where $\lambda = \frac{N}{(p-1-\frac{1}{m})N+qp}$.

Proof. Multiply (2.1) with $u^{m(l-1)}$, and integrate it on Ω, then

$$\begin{aligned}
\int_\Omega u^{m(l-1)}u_t dx &= \int_\Omega \mathrm{div}(|\nabla u^m| + \frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)u^{m(l-1)}dx - \int_\Omega a(x)u^{mq_1}|\nabla u^m|^{p_1}u^{m(l-1)}dx \\
&+ \int_\Omega f_0(u^m)u^{m(l-1)}\int_\Omega K(y)|u^m(y,t)|^\beta dydx + \int_\Omega g(x)u^{m(l-1)}dx \\
&= -(l-1)\int_\Omega (|\nabla u^m| + \frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2 u^{m(l-2)}dx - \int_\Omega a(x)u^{mq_1}|\nabla u^m|^{p_1}u^{m(l-1)}dx \\
&+ \int_\Omega K(y)|u^m(y,t)|^\beta dy\int_\Omega f_0(u^m)u^{m(l-1)}dx + \int_\Omega g(x)u^{m(l-1)}dx \\
&\leq -(l-1)\int_\Omega (|\nabla u^m| + \frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2 u^{m(l-2)}dx \\
&+c(K)\int_\Omega |u^m(y,t)|^\beta dy\int_\Omega u^{m(l-1)+1}dx + c(g)\int_\Omega u^{m(l-1)}dx,
\end{aligned}$$

which deduces that

$$\begin{aligned}
\frac{d}{dt}\|u^m\|_{l-1+\frac{1}{m}}^{l-1+\frac{1}{m}} + c(l-1+\frac{1}{m})^{2-p}\int_\Omega |\nabla u^{m\frac{p+l-1+\frac{1}{m}-1-\frac{1}{m}}{p}}|^p dx &\leq c\|u^m\|_{l-1+\frac{1}{m}}^{l-1+\frac{1}{m}}\|u^m\|_{q-1+\frac{1}{m}}^{q-1+\frac{1}{m}+\beta} + c\|u^m\|_{l-1+\frac{1}{m}}^{l-1} \\
&\leq \|u^m\|_{l-1+\frac{1}{m}}^{l-1+\frac{1}{m}} + c\|u^m\|_{l-1+\frac{1}{m}}^{l-1}, \text{(by (2.6))}.
\end{aligned}$$

Set $L = l - 1 + \frac{1}{m}$. Then

$$\frac{d}{dt}\|u^m\|_L^L + cL^{2-p}\int_\Omega |\nabla u^{m\frac{L+p-1-\frac{1}{m}}{p}}|^p\, dx \le c\|u^m\|_L^{L+\beta} + c\|u^m\|_L^{L-\frac{1}{m}}, \tag{2.16}$$

where c is a constant independent of l.

Now, if we choose $L_1 = q - 1 + \frac{1}{m}, L_n = rL_{n-1} - (p - 1 - \frac{1}{m}), \theta_n = rN(1 - L_{n-1}L_n^{-1})(p + N(r - 1))^{-1}$, $\mu_n = (L_n + p - 1 - \frac{1}{m})\theta_n^{-1} - L_n$, $r > 1 + (p - 1 - \frac{1}{m})q^{-1}$, $n = 2, 3, \cdots$, by Lemma 1.3, we have

$$\|u^m\|_{L_n} \le c^{p/(L_n+p-1-\frac{1}{m})}\|u^m\|_{L_{n-1}}^{1-\theta_n}\|\nabla u^{m(L_n+p-1-\frac{1}{m})/p}\|_p^{p\theta_n/(p-1-\frac{1}{m}+L_n)}. \tag{2.17}$$

If we choose $L = L_n$ in (2.16), by (2.17), we have

$$\frac{d}{dt}\|u^m\|_{L_n}^{L_n} + c^{-p/\theta_n}L_n^{2-p}\|u^m\|_{L_n}^{L_n+\mu_n}\|u^m\|_{L_{n-1}}^{p-1-\frac{1}{m}-\mu_n} \le c\|u^m\|_{L_n}^{L_n+\beta} + c\|u^m\|_{L_n}^{L_n-\frac{1}{m}}.\ 0 < t \le 1. \tag{2.18}$$

We will prove that there exist two bounded sequences $\{\xi_n\}, \{\lambda_n\}$ such that

$$\|u^m\|_{L_n} \le \xi_n t^{-\lambda_n},\ 0 < t \le 1. \tag{2.19}$$

Without loss of the generality, we may assume that $\|u^m\|_{L_n} \ge 1$. Otherwise, choosing $\xi_n \equiv 1$, (2.17) is true naturally. Thus, by (2.16), we have

$$\frac{d}{dt}\|u^m\|_{L_n}^{L_n} + c^{-p/\theta_n}L_n^{2-p}\|u^m\|_{L_n}^{L_n+\mu_n}\|u^m\|_{L_{n-1}}^{p-1-\frac{1}{m}-\mu_n} \le c\|u^m\|_{L_n}^{L_n+\beta}.\ 0 < t \le 1.$$

If $n = 1$, by Lemma 2.1, $\lambda_1 = 0, \xi_1 = \sup_{t\ge0}\|u^m(t)\|_{q-1+\frac{1}{m}}$ makes (2.19) sure. If (2.19) is true for $n - 1$, from (2.18),

$$\frac{d}{dt}\|u^m\|_{L_n}^{L_n} + c^{-p/\theta_n}L_n^{2-p}\|u^m\|_{L_n}^{L_n+\mu_n}\xi_{n-1}^{p-1-\frac{1}{m}-\mu_n}t^{-(p-1-\frac{1}{m}-\mu_n)\lambda_{n-1}} \le c\|u^m\|_{L_n}^{L_n+\beta}.\ 0 < t \le 1. \tag{2.20}$$

we can choose

$$\lambda_n = (\lambda_{n-1}(\mu_n - p + 1 + \frac{1}{m}) + 1)\mu_n^{-1},\ \xi_n = \xi_{n-1}(c^{p/\theta_n}L_n^{p-1}\lambda_n)^{1/\mu_n},\ n = 2, 3, \cdots,$$

$$\frac{d}{dt}\|u^m\|_{L_n}^{L_n} + c\|u^m\|_{L_n}^{L_n+\lambda_n} \le c\|u^m\|_{L_n}^{L_n+\beta}.\ 0 < t \le 1. \tag{2.21}$$

Suppose that

$$\beta < \frac{N}{(p - 1 - \frac{1}{m})N + qp}, \tag{2.22}$$

and notice that as $n \to \infty$, $\lambda_n \to \lambda = \frac{N}{(p-1-\frac{1}{m})N+pq}$.

$$\frac{d}{dt}\|u^m\|_{L_n}^{L_n} + c\|u^m\|_{L_n}^{L_n+\lambda_n} \le 0.\ 0 < t \le 1. \tag{2.23}$$

By Lemma 1.2 and (2.23), we know (2.19) is true.

Moreover, it is easy to see that $\{\xi_n\}$ is bounded. Thus, by Lemma 1.2, (2.14) is true.

To prove (2.15), we set $\tau = \log(1 + t), t \ge 1, w(\tau) = (1 + t)^{\frac{1}{p-1-\frac{1}{m}}}u^m(t)$. By (2.16), we have

$$\frac{d}{d\tau}\|w(\tau)\|_L^L + cL^{2-p}\|\nabla w^{\frac{L+p-1-\frac{1}{m}}{p}}\|_p^p \le \frac{L}{p - 1 - \frac{1}{m}}\|w(\tau)\|_L^L + c\|w(\tau)\|_L^{L+\beta},\ \tau \ge \log 2. \tag{2.21}$$

By the lemma 3.1 in [24], we can get (2.15), we omit details here. □

3. The L^∞ estimation of the gradient

Lemma 3.1. *If $p > \max\{2, 1+\frac{1}{m}\}$, u_k is the solution of problem (2.1)-(2.4)-(2.5), then*

$$\|\nabla u_k^m\|_p \le ct^{-(1+\frac{m-1}{m(p-1)-1})} + ct^{1-\delta},\ 0 < t \le 1, \tag{3.1}$$

$$\|\nabla u_k^m\|_p \le c(1+t)^{-\frac{p(m(2q_1+1)-1)+mp_1}{(m(p-1)-1)(p-p_1)}}, t \ge 1. \tag{3.2}$$

Here $\delta = \max\{\frac{m-1}{m}, 2\beta\}$.

Proof. Multiply (2.1) with u_t^m, and integrate it on Ω, then

$$m\int_\Omega u^{m-1}(u_t)^2dx = \int_\Omega \operatorname{div}((|\nabla u^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)u_t^m dx - \int_\Omega a(x)u^{mq_1}|\nabla u^m|^{p_1}u_t^m dx$$
$$+\int_\Omega f_0(u^m)u_t^m dx\int_\Omega K(y)|u^m(y,t)|^\beta dy + \int_\Omega g(x)u_t^m dx. \tag{3.3}$$

$$\int_\Omega \operatorname{div}((|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)u_t^m dx = -\int_\Omega(|\nabla u^m| + \frac{1}{k})^{\frac{p-2}{2}}\nabla u^m\nabla u_t^m dx$$
$$= -\frac{1}{2}\int_\Omega(|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|_t^2 dx$$
$$= -\frac{1}{2}\int_\Omega\frac{d}{dt}\int_0^{|\nabla u^m|^2}(s+\frac{1}{k})^{\frac{p-2}{2}}dsdx = -\frac{1}{2}\frac{d}{dt}\Gamma_k(|\nabla u^m|^2), \tag{3.4}$$

where we define that

$$\Gamma_k(|\nabla u^m|^2) = \int_\Omega\int_0^{|\nabla u^m|^2}(s+\frac{1}{k})^{\frac{p-2}{2}}dsdx.$$

At the same time,

$$|-a(x)u^{mq_1}|\nabla u^m|^{p_1}u_t^m dx| \le \frac{m}{2}\int_\Omega u^{m-1}(u_t)^2dx + c\int_\Omega |u^m|^{2q_1+\frac{m-1}{m}}|\nabla u^m|^{2p_1}dx. \tag{3.5}$$

By Lemma 2.1, using Young inequality and Hölder inequality,

$$|\int_\Omega f_0(u^m)u_t^m dx\int_\Omega K(y)|u^m(y,t)|^\beta dy|$$
$$\le c(\varepsilon\int_\Omega u^{m-1}(u_t)^2dx + c\int_\Omega u^{m+1}dx)\|u^m\|^\beta_{q-1+\frac{1}{m}}$$
$$\le c\varepsilon\int_\Omega u^{m-1}(u_t)^2dx + c\int_\Omega u^{m+1}dx$$

$$|\int_\Omega g(x)u_t^m dx| \le \varepsilon\int_\Omega u^{m-1}(u_t)^2dx + c\int_\Omega u^{m-1}dx.$$

By (3.3)-(3.5), we have

$$\int_\Omega u^{m-1}(u_t)^2dx + \frac{1}{m}\frac{d}{dt}\Gamma_k(|\nabla u^m|^2) \le c\int_\Omega |u^m|^{2q_1+\frac{m-1}{m}}|\nabla u^m|^{2p_1}dx + c\int_\Omega u^{m+1}dx + c\int_\Omega u^{m-1}dx. \tag{3.6}$$

Multiply (2.1) with u^m, and integrate it on Ω, then

$$\frac{1}{m+1}\int_\Omega \frac{d}{dt}u^{m+1}dx = \int_\Omega \mathrm{div}((|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}\nabla u^m)u^m dx - \int_\Omega a(x)u^{mq_1}|\nabla u^m|^{p_1}u^m dx$$
$$+\int_\Omega f_0(u^m)u^m\int_\Omega K(y)|u^m(y,t)|^\beta dydx + \int_\Omega g(x)u^m dx$$
$$= -\int_\Omega (|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2 dx - \int_\Omega a(x)u^{mq_1}|\nabla u^m|^{p_1}u^m dx$$
$$+\int_\Omega f_0(u^m)u^m\int_\Omega K(y)|u^m(y,t)|^\beta dydx + \int_\Omega g(x)u^m dx$$

and

$$\Gamma_k(|\nabla u^m|^2) \le \int_\Omega (|\nabla u^m|^2+\frac{1}{k})^{\frac{p-2}{2}}|\nabla u^m|^2 dx$$
$$= -\frac{1}{m+1}\int_\Omega \frac{d}{dt}u^{m+1}dx - \int_\Omega a(x)u^{mq_1}|\nabla u^m|^{p_1}u^m dx + \int_\Omega f_0(u^m)u^m\int_\Omega K(y)|u^m(y,t)|^\beta dydx + \int_\Omega g(x)u^m dx$$
$$\le \frac{1}{m+1}\|u^{\frac{m+1}{2}}\|_2\|u^{\frac{m-1}{2}}u_t\|_2 + c(K)\int_\Omega |u^m(y,t)|^\beta dy\int_\Omega u^{m+1}dx + c(g)\int_\Omega u^m dx,$$

so

$$\frac{1}{m}\frac{d}{dt}\Gamma_k(|\nabla u^m|^2) + (m+1)^2\|u^{\frac{m+1}{2}}\|_2^{-2}\Gamma_k^2(|\nabla u^m|^2)$$
$$\le c\int_\Omega |u^m|^{2q_1+\frac{m-1}{m}}|\nabla u^m|^{2p_1}dx + c\int_\Omega u^{m+1}dx + c\int_\Omega u^{m-1}dx$$
$$+c\|u^{\frac{m+1}{2}}\|_2^{-2}(\int_\Omega |u^m(y,t)|^\beta dy\int_\Omega u^{m+1}dx + \int_\Omega u^m dx)^2$$
$$\le c\int_\Omega |u^m|^{2q_1+\frac{m-1}{m}}|\nabla u^m|^{2p_1}dx + c\int_\Omega u^{m+1}dx + c\int_\Omega u^{m-1}dx$$
$$+c(\int_\Omega |u^m(y,t)|^\beta dy)^2\int_\Omega u^{m+1}dx + c\|u^{\frac{m+1}{2}}\|_2^{2\frac{m}{m+1}-2}. \tag{3.7}$$

Setting $2\gamma = 2q_1 + 1 - \frac{1}{m}$, for $\forall a \in [0, 2\gamma]$, if we notice that $p > 2p_1$, then we have

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^{2p_1}dx \le \|u^m(t)\|_\infty^a\left(\int_\Omega |u^m|^{\frac{(2\gamma-a)p}{p-2p_1}}dx\right)^{\frac{p-2p_1}{p}}\|\nabla u^m\|_p^{2p_1}. \tag{3.8}$$

If $2\gamma \ge (p-2p_1)(N+1)/N$, let $a = (2\gamma - (p-2p_1)(1+\frac{q}{N}))^+$. By Lemma 1.3,

$$\left(\int_\Omega |u^m|^{\frac{(2\gamma-a)p}{p-2p_1}}dx\right)^{\frac{p-2p_1}{p}} \le c\|u^m(t)\|_s^{(2\gamma-a)(1-\theta)}\|\nabla u^m\|_p^{p-2p_1}, \tag{3.9}$$

where $\theta = (s^{-1} - (1-\frac{2p_1}{p})(2\gamma-a)^{-1})/(N^{-1} - p^{-1} + s^{-1})$, and $s = (2\gamma - p + 2p_1 - a)N/(p-2p_1)$ when $2\gamma \ge (p-2p_1)(1+q/N)$, $s = q$ when $(p-2p_1)(1+N^{-1}) \le 2\gamma \le (p-2p_1)(1+q/N)$. By Lemma 2.1 and Lemma 2.2, from (3.8), we have

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^{2p_1}dx \le ct^{-\lambda a}\|\nabla u^m\|_p^p \le ct^{-\lambda a}\Gamma_k(|\nabla u^m|^2).\ 0 < t \le 1. \tag{3.10}$$

At the same time, if we choose $q = 2$ in Lemma 2.1, we have

$$\|u^m\|_{1+\frac{1}{m}} = \left(\int_\Omega u^{m+1}dx\right)^{\frac{m}{m+1}} \le c(1+t)^{-(p-1-\frac{m}{m+1})^{-1}} \le c,$$

and

$$\int_\Omega u^{m-1}dx \le ct^{\frac{m-1}{m}\lambda},\ \|u^{\frac{m+1}{2}}\|_2^2 = \int_\Omega u^{m+1}dx \le c. \tag{3.11}$$

By (3.7) and Lemma 2.2, we have

$$\Gamma_k'(t) + ct^{\frac{m+1}{m(p-1)-1}}\Gamma_k^2(t) \le ct^{-\lambda a}\Gamma_k(t) + c(t^{-\lambda\frac{m-1}{m}} + t^{-2\beta\lambda}),\ 0 < t \le 1. \tag{3.12}$$

If $2\gamma < (p-2p_1)(N+1)/N$ and $p - 2p_1 \le 2a \le 2\gamma$,

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^{2p_1}dx \le c\|\nabla u^m\|_1^{2a(1-\theta)}\|\nabla u^m\|_p^{2a\theta+2p_1} \le c\|\nabla u^m\|_p^p \le c\Gamma_k(|\nabla u^m|^2).\ 0 < t \le 1. \tag{3.13}$$

If $2\gamma < (p-2p_1)(N+1)/N$ and $p - 2p_1 \ge 2a \ge 0$,

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^2 dx \le c(1 + \|\nabla u^m\|_p^p) \le c(1 + \Gamma_k(|\nabla u^m|^2)).\ 0 < t \le 1. \tag{3.14}$$

(3.13) and (3.14) imply that (3.12) is still true when $2\gamma < (p-2p_1)(N+1)/N$. Using Lemma 1.2,

$$\Gamma_k(t) \le ct^{-(1+\frac{m-1}{m(p-1)-1})} + ct^{1-\delta},\ 0 < t \le 1,$$

where $\delta = \max\{\frac{m+1}{m}, 2\beta\}$. Then (3.1) is true. Now, we will prove (3.2). For $t \ge 1$, by (2.15)

$$\int_\Omega |u^m|^{2a}|\nabla u^m|^{2p_1}dx \le c\|\nabla u^m\|_p^2\|u^m(t)\|_{2\gamma p/p-2p_1}^{2\gamma} \le c(1+t)^{-2\gamma/(p-1-\frac{1}{m})}\|\nabla u^m\|_p^{2p_1}.\ t \ge 1. \tag{3.15}$$

$$\Gamma_k(|\nabla u^m|^2) = \int_0^{|\nabla u^m|^2}(s^2 + \frac{1}{k})^{\frac{p-2}{2}}ds \le c\|\nabla u^m\|_p^p = c(\|\nabla u^m\|_p^{2p_1})^{\frac{p}{2p_1}},\ t \ge 1. \tag{3.16}$$

$$\|u^{\frac{m+1}{2}}\|_2^2 = \left(\int_\Omega u^{m+1}dx\right)^2 \le c(1+t)^{-(p-1-\frac{1}{m})^{-1}},\ t \ge 1. \tag{3.17}$$

by (3.7), using (3.15)-(3.17)

$$\Gamma_k'(t) + c(1+t)^{-(p-1-\frac{1}{m})^{-1}}\Gamma_k^2(t) \le c(1+t)^{2\gamma/(p-1-\frac{1}{m})}(\Gamma_k(t))^{\frac{2p_1}{p}}$$

$$+c\int_\Omega u^{m+1}dx + c\int_\Omega u^{m-1}dx + c(\int_\Omega |u^m(y,t)|^\beta dy)^2\int_\Omega u^{m+1}dx + c\|u^{\frac{m+1}{2}}\|_2^{2(m-1)},$$

by Young inequality,

$$\Gamma_k'(t) + c(1+t)^{-(p-1-\frac{1}{m})^{-1}}\Gamma_k^2(t) \le c(1+t)^{\frac{-m(2\gamma p+p_1)}{(m(p-1)-1)(p-p_1)}} + c(1+t)^{-\frac{m(m+1)}{m(p-1)-1}}$$

$$= c(1+t)^{-\frac{p(m(2q_1+1)-1)+mp_1}{(m(p-1)-1)(p-p_1)}} + c(1+t)^{-\frac{m(m+1)}{m(p-1)-1}},$$

which means (3.2) is true. □

Lemma 3.2. *If $p > 1 + \frac{1}{m}$, u_k is the solution of problem (2.1) -(2.4)-(2.5), then*

$$\int_t^T \int_\Omega u_k^{m-1}(u_{kt})^2 dxds \leq ct^{-(1+\frac{m-1}{m(p-1)-1})} + ct^{-(\lambda\gamma+\frac{m-1}{m(p-1)-1})} + ct^{-\frac{1+m}{m}\lambda},\ 0 < t \leq T. \tag{3.18}$$

Proof. From (3.6), (3.10) and (2.14), we have

$$\int_\Omega u^{m-1}(u_t)^2 dx + \frac{1}{m}\frac{d}{dt}\Gamma_k(|\nabla u^m|^2) \leq c\int_\Omega |u^m|^{2q_1+\frac{m-1}{m}}|\nabla u^m|^{2p_1}dx + c\int_\Omega u^{m+1}dx + c\int_\Omega u^{m-1}dx$$

$$\begin{aligned}\int_t^T \int_\Omega u^{m-1}(u_t)^2 dxds &\leq \Gamma_k(t) + c\int_t^T \int_\Omega |u^m|^{2q_1+\frac{m-1}{m}}|\nabla u^m|^{2p_1}dxds + c\int_t^T \int_\Omega u^{m+1}dx\\ &\leq \Gamma_k(t) + c\int_t^T s^{-\lambda(2q_1+\frac{m-1}{m})}\Gamma_k(s)ds + c\int_t^T \int_\Omega u^{m+1}dx\\ &\leq ct^{-(1+\frac{m-1}{m(p-1)-1})} + ct^{-(\lambda\gamma+\frac{m-1}{m(p-1)-1})} + ct^{-\frac{1+m}{m}\lambda}.\end{aligned} \tag{3.19}$$

□

4. The proof of Theorem 1.1

The proof of Theorem 1.1 from Lemma 2.1, Lemma 2.2, Lemma 3.1 and Lemma 3.2, using the compactness theory (cf [21]), there is a sequence (still denoted it as $\{u_k\}$) of $\{u_k\}$ such that when $k \to \infty$, $u_k \to u$, *a.e.* in S and so

$$\lim_{k\to\infty} f_0(u_k^m)\int_\Omega K(y)|u_k^m(y,t)|^\beta dy = f_0(u^m)\int_\Omega K(y)|u^m(y,t)|^\beta dy.$$

Moreover, we have

$$u_k \rightharpoonup u, \text{weakly* star in } L^\infty_{\mathrm{loc}}(0,\infty; L^{m(q-1)+1}(\Omega)), \tag{4.1}$$

$$u_{kt} \rightharpoonup u_t,\ \text{weakly in } L^2(0,\infty; L^2(\Omega)), \nabla u_k^m \rightharpoonup \nabla u^m,\ \text{weakly in } L^p_{\mathrm{loc}}(0,\infty; L^p(\Omega)) \tag{4.2}$$

$$|\nabla u_k^m|^{p-2}u_{kx_i}^m \rightharpoonup \chi_i, \text{weakly* in } L^\infty_{\mathrm{loc}}(0,\infty; L^{\frac{p}{p-1}}(\Omega)), \tag{4.3}$$

$$a(x)u_k^{mq_1}|\nabla u_k^m|^{p_1} \rightharpoonup \nu, \text{weakly* in } L^\infty_{\mathrm{loc}}(0,\infty; L^{\frac{p}{p_1}}(\Omega)), \tag{4.4}$$

where $\chi = \{\chi_i : 1 \leq i \leq N\}$ and every χ_i is a function in $L^\infty_{loc}(0,T; L^{\frac{p}{p-1}}(\Omega))$, $\nu \in L^\infty_{loc}(0,\infty; L^{\frac{p}{p_1}}(\Omega))$. (4.1) and (4.2) are clearly true.

In what follows, we only need to prove that

$$\chi = |\nabla u^m|^{p-2}\nabla u^m,\ \text{ in } L^\infty_{\mathrm{loc}}(0,\infty; L^{\frac{p}{p-1}}(\Omega)). \tag{4.5}$$

and

$$\nu = a(x)u^{mq_1}|\nabla u^m|^{p_1},\ \text{ in } L^\infty_{\mathrm{loc}}(0,\infty; L^{\frac{p}{p_1}}(\Omega)). \tag{4.6}$$

It is easy to know that

$$\iint_S \left(u\varphi_t - \chi\cdot\nabla\varphi - \nu\varphi + f_0(u^m)\int_\Omega K(y)|u^m(y,t)|^\beta dy\varphi + g(x)\varphi\right)dxdt = 0,\ \forall\varphi \in C_0^\infty(S), \tag{4.7}$$

so, if we can prove that

$$\iint_S | \nabla u^m |^{p-2} \nabla u^m \cdot \nabla\varphi dxdt = \iint_S \chi \cdot \nabla\varphi dxdt, \ \forall\varphi \in C_0^1(S); \tag{4.8}$$

$$\iint_S a(x)u^{mq_1}|\nabla u^m|^{p_1}\varphi dxdt = \iint_S \nu\varphi dxdt, \ \ \forall\varphi \in C_0^1(S); \tag{4.9}$$

then (4.5),(4.6) and (1.12) are true.

First, for any $\psi \in C_0^\infty(S)$, $0 \leq \psi \leq 1$; $v^m \in L^p_{\mathrm{loc}}(0, T; W_0^{1,p}(\Omega))$, we have

$$\iint_S \psi(| \nabla u_k^m |^{p-2} \nabla u_k^m - | \nabla v^m |^{p-2} \nabla v^m) \cdot \nabla(u_k^m - v^m)dxdt \geq 0, \tag{4.10}$$

If we multiply with $u_k^m\psi$ on two sides of (2.1), then we have

$$\iint_S \psi\left(|\nabla u_k^m|^2 + \frac{1}{k}\right)^{\frac{p-2}{2}} |\nabla u_k^m|^2 dxdt = \frac{1}{m+1}\iint_S \psi_t u_k^{m+1}dxdt - \iint_S u_k^m\left(|\nabla u_k^m|^2 + \frac{1}{k}\right)^{\frac{p-2}{2}} \nabla u_k^m \cdot \nabla\psi dxdt$$

$$- \iint_S a(x)u_k^{m(q_1+1)}|\nabla u_k^m|^{p_1}\psi dxdt + \iint_S [f_0(u_k^m)\int_\Omega K(y)|u_k^m(y,t)|^\beta dy + g(x)]u_k^m\psi dxdt. \tag{4.11}$$

Noticing that when $1 < p < 2$,

$$|\nabla u_k^m|^2 \geq (|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p}{2}} - (\frac{1}{k})^{\frac{p}{2}},$$

$$(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}|\nabla u_k^m| \leq (|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-1}{2}},$$

and when $p \geq 2$,

$$(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}|\nabla u_k^m|^2 \geq |\nabla u_k^m|^p,$$

$$(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}|\nabla u_k^m| \leq (|\nabla u_k^m|^{p-1} + 1),$$

by (4.10), (4.11), we have

$$\begin{aligned}
&\frac{1}{m+1}\iint_S \psi_t u_k^{m+1}dxdt - \iint_S u_k^m\left(|\nabla u_k^m|^2 + \frac{1}{k}\right)^{\frac{p-2}{2}} \nabla u_k^m \cdot \nabla\psi dxdt \\
&- \iint_S a(x)u_k^{m(q_1+1)}|\nabla u_k^m|^{p_1}\psi dxdt + (\frac{1}{k})^{\frac{p-2}{2}}\mathrm{mes}\Omega \\
&+ \iint_S [f_0(u_k^m)\int_\Omega K(y)|u_k^m(y,t)|^\beta dy + g(x)]u_k^m\psi dxdt \\
&- \iint_S \psi|\nabla u_k^m|^{p-2}\nabla u_k^m \cdot \nabla v^m dxdt - \iint_S \psi|\nabla v^m|^{p-2}\nabla v^m \cdot \nabla(u_k^m - v^m)dxdt \geq 0.
\end{aligned} \tag{4.12}$$

Since

$$\left(|\nabla u_k^m|^2 + \frac{1}{k}\right)^{\frac{p-2}{2}} \nabla u_k^m = |\nabla u_k^m|^{p-2}\nabla u_k^m + \frac{p-2}{2k}\int_0^1 (|\nabla u_k^m|^2 + \frac{s}{k})^{\frac{p-4}{2}}ds\nabla u_k^m,$$

and

$$\lim_{k\to\infty}\iint_S \int_0^1 (|\nabla u_k^m|^2 + \frac{s}{k})^{\frac{p-4}{2}} ds \nabla u_k^m \cdot \nabla\psi u_k^m dxdt = 0,$$

if we let $k \to \infty$ in (4.12), we have

$$\begin{aligned}
&\frac{1}{m+1}\iint_S \psi_t u^{m+1} dxdt - \iint_S u^m v\psi dxdt - \iint_S u^m \chi \nabla\psi dxdt \\
&- \iint_S \psi\chi \cdot \nabla v^m dxdt - \iint_S \psi |\nabla v^m|^{p-2}\nabla v^m \cdot \nabla(u^m - v^m) dxdt \\
&+ \iint_S [f_0(u^m)\int_\Omega K(y)|u^m(y,t)|^\beta dy + g(x)]u^m\psi dxdt \geq 0. \qquad (4.13)
\end{aligned}$$

Now, we choose $\varphi = \psi u^m$ in (4.7),

$$\begin{aligned}
&\frac{1}{m+1}\iint_S \psi_t u^{m+1} dxdt - \iint_S u^m v\psi dxdt - \iint_S \chi \cdot \nabla\psi u^m dxdt \\
&+ \iint_S [f_0(u^m)\int_\Omega K(y)|u^m(y,t)|^\beta dy + g(x)]\psi u^m dxdt = \iint_S \psi\chi \cdot \nabla u^m dxdt.
\end{aligned}$$

From this formula and (4.13), we have

$$\iint_S \psi(\chi - |\nabla v^m|^{p-2}\nabla v^m) \cdot \nabla(u^m - v^m) dxdt \geq 0. \qquad (4.14)$$

Let $v^m = u^m - \lambda\varphi$, $\lambda \geq 0, \varphi \in C_0^\infty(S)$. Then

$$\iint_S \psi(\chi_i - |\nabla(u^m - \lambda\varphi)|^{p-2}(u^m - \lambda\varphi)_{x_i}) dxdt \geq 0.$$

Let $\lambda \to 0$. We obtain

$$\iint_S \psi(\chi_i - |\nabla u^m|^{p-2}u^m_{x_i}) dxdt \geq 0, \forall\varphi \in C_0^\infty(S).$$

Moreover, if we choose $\lambda \leq 0$, we are able to get

$$\iint_S \psi(\chi_i - |\nabla u^m|^{p-2}u^m_{x_i}) dxdt \leq 0, \forall\varphi \in C_0^\infty(S).$$

Now, if we choose ψ such that supp$\varphi \subset$ suppψ, and on suppφ, $\psi = 1$, then we can get (4.8).

By a process of limitation, we can choose the test function φ in (4.8) as u^m, then we have

$$\lim_{k\to 0}\iint_S |\nabla u_k^m|^p dxdt = \iint_S \chi \cdot \nabla u^m dxdt = \iint_S |\nabla u^m|^p dxdt. \qquad (4.15)$$

Due to (1.20), $2p_1 < p$, then by Hölder inequality, we have

$$\lim_{k\to 0}\iint_S |\nabla u_k^m|^{p_1} dxdt = \iint_S \chi \cdot \nabla u^m dxdt = \iint_S |\nabla u^m|^{p_1} dxdt. \qquad (4.16)$$

By a refinement of Fatou's lemma, the theorem 1.4.1 in [12], we are easy to prove (4.9), and so (1.12) is true.

Secondly, we are to prove (1.13).

For small $r > 0$, denote $\Omega_r = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \le r\}$. For any $\eta > 0$, let

$$\mathrm{sgn}_\eta(s) = \begin{cases} 1, & \text{if } s > \eta, \\ \frac{s}{\eta}, & \text{if } |s| \le \eta, \\ -1, & \text{if } s < -\eta. \end{cases}$$

For any given small $r > 0$, large enough k, l, we declare that

$$\int_{\Omega_{2r}} |u_k(x,t) - u_l(x,t)|dx \le \int_{\Omega_r} |u_k(x,0) - u_l(x,0)|dx + c_r(t), \tag{4.17}$$

where $c_r(t)$ is independent of k, l, and $\lim_{t\to 0} c_r(t) = 0$. By (2.1)

$$\begin{aligned} &\int_0^t \int_{\Omega_r} \varphi(u_{kt} - u_{lt})dxd\tau + \int_0^t \int_{\Omega_r} \nabla\varphi[(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u_k^m - (|\nabla u_l^m|^2 + \frac{1}{l})^{\frac{p-2}{2}}\nabla u_l^m]dxd\tau \\ &+ \int_0^t \int_{\Omega_r} a(x)(u_k^{mq_1}|\nabla u_k^m|^{p_1} - u_l^{mq_1}|\nabla u_l^m|^{p_1})\varphi dxd\tau \\ &+ \int_0^t \int_{\Omega_r} [f_0(u_k^m)\int_\Omega K(y)|u_k^m(y,t)|^\beta dy - f_0(u_l^m)\int_\Omega K(y)|u_l^m(y,t)|^\beta dy]\varphi dxd\tau = 0, \end{aligned} \tag{4.18}$$

for $\forall\varphi \in L^p(0,T; W_0^{1,p}(\Omega))$. Suppose that $\xi(x) \in C_0^1(\Omega_r)$ such that

$$0 \le \xi \le 1; \ \ \xi\,|_{\Omega_{2r}} = 1,$$

and choose $\varphi = \xi\mathrm{sgn}_\eta(u_k^m - u_l^m)$ in (4.18), then

$$\begin{aligned} &\int_0^t \int_{\Omega_r} \xi\mathrm{sgn}_\eta(u_k^m - u_l^m)(u_{kt} - u_{lt})dxd\tau \\ &+ \int_0^t \int_{\Omega_r} [(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u_k^m - (x|\nabla u_l^m|^2 + \frac{1}{l})^{\frac{p-2}{2}}\nabla u_l^m]\nabla\xi\mathrm{sgn}_\eta(u_k^m - u_l^m)dxd\tau \\ &+ \int_0^t \int_{\Omega_r} [(|\nabla u_k^m|^2 + \frac{1}{k})^{\frac{p-2}{2}}\nabla u_k^m - (x|\nabla u_l^m|^2 + \frac{1}{l})^{\frac{p-2}{2}}\nabla u_l^m]\nabla(u_k^m - u_l^m)\xi\mathrm{sgn}'_\eta(u_k^m - u_l^m)dxd\tau \\ &+ \int_0^t \int_{\Omega_r} a(x)(u_k^{mq_1}|\nabla u_k^m|^{p_1} - u_l^{mq_1}|\nabla u_l^m|^{p_1})\xi\mathrm{sgn}_\eta(u_k^m - u_l^m)dxd\tau \\ &+ \int_0^t \int_{\Omega_r} [f_0(u_k^m)\int_\Omega K(y)|u_k^m(y,t)|^\beta dy - f_0(u_l^m)\int_\Omega K(y)|u_l^m(y,t)|^\beta dy]\xi\mathrm{sgn}_\eta(u_k^m - u_l^m)dxd\tau \le 0. \end{aligned} \tag{4.19}$$

If we notice that the third term in the left hand side on (4.19) is nonnegative when $\eta \to 0$, then we have

$$\lim_{\eta\to 0}\int_0^t \int_{\Omega_r} \xi\mathrm{sgn}_\eta(u_k^m - u_l^m)(u_{kt} - u_{lt})dxd\tau$$

$$+\lim_{\eta\to 0}\int_0^t\int_{\Omega_r}[(|\nabla u_k^m|^2+\frac{1}{k})^{\frac{p-2}{2}}\nabla u_k^m-(|\nabla u_l^m|^2+\frac{1}{l})^{\frac{p-2}{2}}\nabla u_l^m]\nabla\xi \mathrm{sgn}_\eta(u_k^m-u_l^m)dxd\tau$$

$$+\lim_{\eta\to 0}\int_0^t\int_{\Omega_r}a(x)(u_k^{mq_1}|\nabla u_k^m|^{p_1}-u_l^{mq_1}|\nabla u_l^m|^{p_1})\xi \mathrm{sgn}_\eta(u_k^m-u_l^m)dxd\tau$$

$$+\lim_{\eta\to 0}\int_0^t\int_{\Omega_r}[f_0(u_k^m)\int_\Omega K(y)|u_k^m(y,t)|^\beta dy-f_0(u_l^m)\int_\Omega K(y)|u_l^m(y,t)|^\beta dy]\xi \mathrm{sgn}_\eta(u_k^m-u_l^m)dxd\tau=0. \quad (4.20)$$

At the same time,

$$\lim_{\eta\to 0}\int_0^t\int_{\Omega_r}\xi \mathrm{sgn}_\eta(u_k^m-u_l^m)(u_{kt}-u_{lt})dxd\tau=\int_0^t\int_{\Omega_r}\xi \mathrm{sgn}(u_k^m-u_l^m)(u_{kt}-u_{lt})dxd\tau$$

$$=\int_0^t\int_{\Omega_r}\xi \mathrm{sgn}(u_k-u_l)(u_{kt}-u_{lt})dxd\tau$$

$$\lim_{\eta\to 0}\int_0^t\int_{\Omega_r}\xi \mathrm{sgn}_\eta(u_k-u_l)(u_{kt}-u_{lt})dxd\tau=\lim_{\eta\to 0}\int_0^t\int_{\Omega_r}\xi(\int_0^{u_k-u_l}\mathrm{sgn}_\eta(s)ds)_\tau dxd\tau$$

$$=\lim_{\eta\to 0}\int_0^t\int_{\Omega_r}\xi\int_0^{u_k-u_l}\mathrm{sgn}_\eta(s)ds\,|_0^t\,dx=\int_{\Omega_r}\xi|u_k-u_l|dx-\int_{\Omega_r}\xi|u_{0k}-u_{0l}|dx. \quad (4.21)$$

By (4.20) (4.21), we have

$$\int_{\Omega_{2r}}\xi|u_k-u_l|dx\leq\int_{\Omega_r}|u_{0k}-u_{0l}|dx+c\int_0^t\int_{\Omega_r}[(|\nabla u_k^m|^2+\frac{1}{k})^{\frac{p-1}{2}}+(|\nabla u_l^m|^2+\frac{1}{l})^{\frac{p-1}{2}}]dxd\tau$$

$$+\int_0^t\int_{\Omega_r}a(x)|u_k^{mq_1}|\nabla u_k^m|^{p_1}-u_l^{mq_1}|\nabla u_l^m|^{p_1}|dxd\tau$$

$$\int_0^t\int_{\Omega_r}|f_0(u_k^m)\int_\Omega K(y)|u_k^m(y,t)|^\beta dy-f_0(u_l^m)\int_\Omega K(y)|u_l^m(y,t)|^\beta dy|dxd\tau. \quad (4.22)$$

By Lemma 2.2 and Lemma 3.1, if $0<t\leq 1$,

$$\int_0^t\int_{\Omega_r}a(x)|u_k^{mq_1}|\nabla u_k^m|^{p_1}-u_1^{mq_1}|\nabla u_1^m|^{p_1}|dxd\tau\leq c\int_0^t\int_{\Omega_r}t^{-\epsilon}dxd\tau,$$

which means (4.17) is true. Here

$$\epsilon=\max\{\frac{mNq_1}{Nm(p-1)-N+mq}+\frac{p_1(m(p-1)+m-2)}{m(p-1)-1},\frac{(\beta+m)N}{Nm(p-1)-N+mq}\}<1$$

Now, for any given small r, if k,l are large enough, by (4.17), we have

$$\int_{\Omega_{2r}}|u(x,t)-u_0(x)|dx\leq\int_{\Omega_r}|u(x,t)-u_k(x,t)|dx+\int_{\Omega_{2r}}|u_{0k}(x)-u_{0l}(x)|dx$$

$$+\int_{\Omega_{2r}}|u_l(x,t)-u_{0l}(x)|dx+\int_{\Omega_{2r}}|u_{0l}(x)-u_0(x)|dx$$

letting $t\to 0$, we get (1.13).

5. The uniqueness of the solutions and the proof of Theorem 1.2

As we have said in the introduction, the uniqueness of the solutions of problem (1.1)-(1.3) is not true generally. But it is not difficult to prove the following theorems.

Theorem 5.1. *Let u_1, u_2 be the two solutions of the problem (1.1)-(1.3) with the different initial values $u_{01}(x), u_{02}(x)$ respectively. If $(\frac{1}{m}+\beta-2)\setminus q_1 < 1$ and*

$$p_1 = 0, \tag{5.1}$$

then

$$\int_\Omega |u_1(x,t)-u_2(x,t)|dx \le \int_\Omega |u_{01}(x)-u_{02}(x)|dx,\ \forall t\ge 0. \tag{5.2}$$

Proof. Let $u_1(t), u_2(t)$ be two solutions of equation (1.1). Let $v_1 = u_1^m(t), v_2 = u_2^m(t)$. Denote $w(t) = v_1^{\frac{1}{m}}(t) - u_2^{\frac{1}{m}}(t)$, $v(t) = v_1(t)-v_2(t)$. Then $w(t), v_1(t), v_2(t)$ satisfy that

$$\begin{aligned} w'(t) - [\mathrm{div}(\|\nabla v_1\|^{p-2}\nabla v_1) - \mathrm{div}(\|\nabla v_2\|^{p-2}\nabla v_2) + a(x)(v_1^{q_1}-v_2^{q_1}) \\ = f_0(v_1)\int_\Omega K(y)|v_1|^\beta dy - f_0(v_2)\int_\Omega K(y)|v_2|^\beta dy. \end{aligned} \tag{5.3}$$

For any positive integer n, let $g_n(s)$ be an odd function and

$$g_n(s) = \begin{cases} 1, & \text{if } s > \frac{1}{n}, \\ n^2s^2e^{1-n^2s^2}, & \text{if } s \le \frac{1}{n}. \end{cases}$$

Clearly, when $|s|\ge n^{-1}$, $g_n'(s)=0$; when $|s|\le n^{-1}$, $0\le g_n'(s) = 6s^{-1}$.

Multiplying (5.3) with $g_n(v_1-v_2)$ and integrating on Ω, we have

$$\begin{aligned} &\int_\Omega g_n(v)w'(t)dx + \int_\Omega [|\nabla|v_1|^{p-2}|\nabla v_1 - |\nabla|v_2|^{p-2}|\nabla v_2]\nabla(v_1-v_2)g_n'(v)dx + \int_\Omega a(x)(v_1^{q_1}-v_2^{q_1})g_n(v)dx \\ &= \int_\Omega g_n(v)[f_0(v_1)\int_\Omega K(y)|v_1|^\beta dy - f_0(v_2)\int_\Omega K(y)|v_2|^\beta dy]dx. \end{aligned} \tag{5.4}$$

Moreover,

$$\lim_{n\to\infty}\int_\Omega g_n(v)w'(t)dx = \frac{d}{dt}\|w(t)\|_1,$$

$$\int_\Omega [|\nabla|v_1|^{p-2}|\nabla v_1 - |\nabla|v_2|^{p-2}|\nabla v_2]\nabla(v_1-v_2)g_n'(v)dx \ge 0,$$

$$\int_\Omega a(x)(v_1^{q_1}-v_2^{q_1})g_n(v)dx \ge 0,$$

$$\begin{aligned} &|\int_\Omega g_n(v)[f_0(v_1)\int_\Omega K(y)|v_1|^\beta dy - f_0(v_2)\int_\Omega K(y)|v_2|^\beta dy]dx| \\ &\le |\int_\Omega K(y)|v_1|^\beta dy\int_\Omega [f_0(v_1)-f_0(v_2)]dx| + c|\int_\Omega f_0(v_2)dx||\int_\Omega\int_{v_2}^{v_1} s^{\beta-1}dsdy| \end{aligned}$$

$$\leq c\|w(t)\|_1\|v_1\|_\beta^\beta + c\|v_2\|_1 \int_\Omega |\xi|^{\beta-1}|v(t)|dx,$$

where $\xi \in [v_1, v_2]$.

So

$$\frac{d}{dt}\|w(t)\|_1 \leq c\|w(t)\|_1\|v_1\|_\beta^\beta + c\|v_2\|_1(\|v_1\|_\beta^\beta + \|v_2\|_\beta^\beta), \tag{5.5}$$

By using (1.23)-(1.24) of Theorem 1.1 to (5.5), letting $n \to \infty$. By Gronwall's inequality, for any given $T > 0$, we can deduce that

$$\|w(t)\|_1 \equiv 0, 0 \leq t \leq T. \tag{5.6}$$

□

Another aim of the section is to prove the uniqueness of the viscosity solution of problem (1.1)-(1.3)

Theorem 5.2. *Suppose that $a(x)$ and $K(x)$ are bounded functions. If $u(x,t) \in L^\infty(S)$, $|\nabla u| \leq c$ in addition, $2 \geq p_1 \geq 1$, then the viscosity solution of (1.1)-(1.3) is unique.*

Proof. Let u, v be the two viscosity solutions of (1.1)-(1.3). Then there are two sequences $\{u_k\}$ and $\{v_l\}$, which are the solutions of problem (1.14)-(1.2)-(1.3), such that

$$\lim_{k\to\infty} u_k = u, \ \lim_{l\to\infty} v_l = v, \ \text{a.e.in } S. \tag{5.7}$$

Clearly, since $u(x,t), v(x,t) \in L^\infty(S)$, we may assume

$$\|u_k\|_\infty \leq c, \ \|v_l\|_\infty \leq c. \tag{5.8}$$

Let

$$w = u_k - v_l, \ \ w_1 = u_k^m - v_l^m.$$

Then

$$w_t = \left(a_{ij}(x,t)w_{1x_j}\right)_{x_i} + b(x,t,w,\nabla w), (x,t) \in \Omega\times(0,\infty) \tag{5.9}$$

$$w(x,0) = u_{0k}(x) - v_{0l}(x), \ x \in \Omega \tag{5.10}$$

$$w(x,t) = 0, \ (x,t) \in \partial\Omega\times(0,\infty), \tag{5.11}$$

where

$$a_{ij}(x,t) = \int_0^1 \left|s\nabla u_k^m + (1-s)\nabla v_l^m\right|^{p-2} ds\cdot\delta_{ij}$$

$$+\int_0^1 (p-2)\left|s\nabla u_k^m + (1-s)\nabla v_l^m\right|^{p-4}(su_{kx_i}^m + (1-s)v_{lx_i}^m)(su_{kx_j}^m + (1-s)v_{lx_j}^m)ds,$$

and since $p_1 \geq 1$, using the convexity of the function s^{p_1}, by (5.8), we have

$$b(x,t,w,\nabla w) = a(x)[u_k^{mq_1}|\nabla u_k^m|^{p_1} - v_l^{mq_1}|\nabla v_l^m|^{p_1}]$$

$$+f_0(u_k^m)\int_\Omega K(y)|u_k^m(y,t)|^\beta dy - f_0(v_l^m)\int_\Omega K(y)|v_l^m(y,t)|^\beta dy,$$

$$|b(x,t,w,\nabla w)| \leq c|\nabla(u_k^m - v_l^m)|^{p_1} \leq c|\nabla w|^{p_1} \leq c|\nabla w|^2 + c.$$

By the chapter 8 of [13], we know that

$$\|u_k(x,t) - v_l(x,t)\|_\infty \leq c\|u_{0k} - v_{0l}\|_\infty.$$

Let $k, l \to \infty$, we know that the uniqueness of the viscosity solution (1.1)-(1.3) is true. □

Suppose that the viscosity solution of problem (1.1)-(1.3) is unique in what follows. Then, by considering the regularized problem (1.14)-(1.2)-(1.3), we easily get the following Theorem 5.3, and Theorem 1.2 is a simple corollary of Theorem 5.3.

Theorem 5.3. *Let u be a weak solution of problem (1.1)-(1.3). If v satisfies*

$$v_t \geq div(|\nabla v^m|^{p-2}\nabla v^m) - a(x)v^{mq_1}|\nabla v^m|^{p_1}$$

$$+f_0(v^m)\int_\Omega K(y)|v^m(y,t)|^\beta dy + g(x) \quad in\ S = \Omega\times(0,\infty), \tag{5.12}$$

$$v(x,0) \geq u_0(x), \quad x \in \Omega, \tag{5.13}$$

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega\times(0,\infty), \tag{5.14}$$

then

$$u(x,t) \geq v(x,t), \ \forall (x,t) \in S. \tag{5.15}$$

Now, let

$$v(x,t) = u_{kr}(x,t) = ru_k(x, r^{m(p-1)-1}t), \quad r \in (0,1).$$

Then

$$v_t(x,t) = \text{div}(|Dv^m|^{p-2}Dv^m) - a(x)r^{m(p-1-q_1-p_1)}v^{mq_1}|Dv^m|^{p_1}$$

$$+r^{m[p-1-\beta]}f_0(r^{-m}v^m)\int_\Omega K(y)|v^m|^\beta dy + r^{m(p-1)}g(x), (x,t) \in \Omega\times(0,\infty) \tag{5.16}$$

$$v(x,0) = ru_k(x,0), x \in \Omega, \tag{5.17}$$

$$v(x,t) = 0, \ (x,t) \in \partial\Omega\times(0,\infty). \tag{5.18}$$

Noticing that $g(x) \leq 0$, $f_0(r^{-m}v^m) \geq f_0(v^m)$, and

$$p_1 + q_1 < p-1, p-1-\beta < 0, \ 0 < r < 1,$$

which implies that

$$r^{m(p-1-q_1-p_1)} < 1, \ r^{m[p-1-\beta]} > 1,$$

$$v_t(x,t) \geq \text{div}(|Dv^m|^{p-2}Dv^m) - a(x)v^{q_1 m}|Dv^m|^{p_1} + f_0(v^m)\int_\Omega K(y)|v^m|^\beta dy + g(x),$$

using the argument similar to that in the proof Lemma 3.5 of [35], we can prove

$$u_k \geq u_{kr}.$$

It follows that

$$\frac{u_k(x, r^{m(p-1)-1}t) - u_k(x,t)}{(r^{m(p-1)-1} - 1)t}$$

$$\geq \frac{r-1}{(1-r^{m(p-1)-1})t}u_k(x, r^{m(p-1)-1}t).$$

Letting $r \to 1$, we get

$$u_{kt} \geq -\frac{u_k}{(m(p-1)-1)t}. \tag{5.19}$$

By (5.19), we can easily get Theorem 1.2.

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

References

1. M. Aassila, *The influence of nonlocal nonlinearities on the long time behavior of solutions of diffusion problems,* J. of Diff. Equ., **192** (2003), 47-69.
2. A. V. Babin and M. I. Vishik, *Attractors of evolution equations,* North-Holland, Amsterdam, 1992.
3. M. Bertsch, R. Dal Passo and M. Ughi, *Discontinuous viscosity solutions of a degenerate parabolic equation,* Trans. Amer. Math. Soc., **320** (1990), 779-798.
4. M. Bertsch, R. Dal Passo and M. Ughi, *Non-uniqueness of solutions of a degenerate parabolic equation,* Ann. Math. Pura Appl., **161** (1992), 57-81.
5. C. Chen, *On global attractor for m-Laplacian parabolic equation with local and nonlocal nonlinearity,* J. Math. Anal. Appl., **337** (2008), 318-332.
6. C. Chen and R. Wang, *Global existence and L^∞ estimates of solution for doubly degenerate parabolic equation (in Chinese),* ACTA Math. Sinica, **44** (2001), 1089-1098.
7. J. W. Cholewa and T. Dlotko, *Global attractors in abstract parabolic problems,* London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, **278** (2000) .
8. A. Dall'Aglioa, D. Giachetti, I. Peral and S. *León, Global existence for some slightly super-linear parabolic equations with measure data,* J. Math. Anal. Appl., **345** (2008), 892-902.
9. R. Dal Passo and S. Luckhaus, *A degenerate diffusion problem not in divergence form,* J. Diff. Equa., **69** (1987), 1-14.
10. E. DiBenedetto, *Degenerate parabolic equations,* Universitext, Springer Verlag, 1993.
11. J. R. Esteban and J. L. Vazquez, *Homogeneous diffusion in R with power-like nonlinear diffusivity,* Arch. Rational Mech. Anal., **103** (1988), 39-88.
12. L. C. Evans, *Weak convergence methods for nonlinear partial differential equations, Conference Board of the Mathematical Sciences,* Regional Conferences Series in Mathematics Number **74**, 1998.
13. L. Gu, *Second order parabolic partial differential equations,* The Publishing Company of Xiamen University, China, 2002.
14. A. V. Ivanov, *Hölder estimates for quasilinear parabolic equations,* J.Soviet Mat., **56** (1991), 2320-2347.
15. A. S. Kalashnikkov, *Some problems of nonlinear parabolic equations of second order,* USSR. Math, Nauk, T., **42** (1987), 135-176.

16. S. Kamin and J. L. Vázquez, *Fundamental solutions and asymptotic behavior for the p-Laplacian equation,* Rev. Mat. Iberoamericana, **4** (1988), 339-354.

17. O. A. Ladyzenskaja, *New equations for the description of incompressible fluids and solvability in the large boundary value problem for them,* Proc. Steldov Inst. Math., **102** (1976), 95-118.

18. K. Lee, A. Petrosyan and J.L. Vázquez, *Large time geometric properties of solutions of the evolution p-Laplacian equation,* J. Diff. Equ., **229** (2006), 389-411.

19. K. Lee and J. L. Vázque, *Geometrical properties of solutions of the Porous Medium Equation for large times,* Indiana Univ. Math. J., **52** (2003), 991-1016.

20. P. Lei, Y. Li and P. Lin, *Null controllability for a semilinear parabolic equation with gradient quadratic growth,* Nonlinear Anal. T.M.A., **68** (2008), 73-82.

21. J. L. Lions, *Quelques méthodes de resolution des problè mes aux limites non linear,* Dound Gauthier-Villars, Paris, 1969.

22. J. Manfredi and V. Vespri, *Large time behavior of solutions to a class of doubly nonlinear parabolic equations,* Electronic J. Diff. Equ., **1994** (1994), 1-16.

23. M. Nakao, *Global solutions for some nonlinear parabolic equations with non-monotonic perturbations,* Non. Anal. TMA, **10** (1986), 299-314.

24. M. Nakao, *L^p estimates of solutions of some nonlinear degenerate diffusion equation,* J. Math. Soc. Japan, **37** (1985), 41-63.

25. M. Nakao and C. Chen, *Global existence and gradient estimate for the quasilinear parabolic equation of m-Laplacian type with a nonlinear convection term,* J. Diff. Equ., 162 (2000), 224-250.

26. Y. Ohara, *L^∞ estimates of solutions of some nonlinear degenerate parabolic equations,* Nonlinear Anal. TMA, **18** (1992), 413-426.

27. M. Pierre, *Uniqueness of solution of $u_t - \triangle\varphi(u) = 0$ with initial datum a measure,* Nonlinear Analysis TMA., **6** (1982), 175-187.

28. R. Temam, *Infinite-dimensional dynamical in mechanics and physics,* Springer-Verlag, New York, 1997.

29. M. Ughi, *A degenerate parabolic equation modelling the spread of an epidemic,* Ann. Math. Pura Appl., **143** (1986), 385-400.

30. M. Winkler, *Large time behavior of solutions to degenerate parabolic equations with absorption,* NoDEA. **8** (2001), 343-361.

31. Z. Wu, J. Zhao, J. Yin and H. Li, *Nonlinear diffusion equations,* Word Scientific Publishing, 2001.

32. J. Yuan, Z. Lian, L. Cao, J. Gao and J. Xu, *Extinction and positivity for a doubly nonlinear degenerate parabolic equation,* Acta Math. Sinica, Eng. Ser., **23** (2007), 1751-1756.

33. H. Zhan, *The Asymptotic Behavior of Solutions for a Class of Doubly Nonlinear Parabolic Equations,* J. Math. Anal. Appl., **370** (2010), 1-10.

34. Q. Zhang and P. Shi, *Global solutions and self-similar solutions of semilinear parabolic equations with nonlinear gradient terms,* Nonlinear Anal. T.M.A., **72** (2010), 2744-2752.

35. J. Zhao and H. Yuan, *The Cauchy problem of some doubly nonlinear degenerate parabolic equations (in Chinese),* Chinese Ann. Math., A, **16** (1995), 179-194.

36. W. Zhou and S. Cai, *The continuity of the viscosity of the Cauchy problem of a degenerate parabolic equation not in divergence form,* J. Jilin University (Natural Sci.), **42** (2004), 341-345.

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