Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 2(1): 161-194
DOI:10.3934/Math.2017.1.161
Received: 22 October 2016
Accepted: 16 January 2017
Published: 8 March 2017

Research article

Solvability for the non-isothermal Kobayashi–Warren–Carter system

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Abstract: In this paper, a system of parabolic type initial-boundary value problems are considered. The system $(\mathrm{S})_\nu$ is based on the non-isothermal model of grain boundary motion by [38], which was derived as an extending version of the "Kobayashi–Warren–Carter model" of grain boundary motion by [23]. Under suitable assumptions, the existence theorem of L^2-based solutions is concluded, as a versatile mathematical theory to analyze various Kobayashi–Warren–Carter type models.

Keywords: Non-isothermal grain boundary motion; Kobayashi–Warren–Carter type model; existence of L^2-based solution; weighted total variation; time-discretization

1. Introduction

Let $0 < T < \infty$ be a constant of time, and let $N \in \mathbb{N}$ be a constant of spatial dimension such that $1 \leq N \leq 3$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $\Gamma := \partial\Omega$ is smooth when $N > 1$. Besides, let us denote by $Q := (0, T) \times \Omega$ the product space of the time-interval $(0, T)$ and the spatial domain Ω, and similarly, let us set $\Sigma := (0, T) \times \Gamma$.

In this paper, we fix a constant $\nu \geq 0$, and consider the following system of initial-boundary value problems of parabolic types, denoted by $(\mathrm{S})_\nu$.

$(\mathrm{S})_\nu$:

$$\begin{cases} [u - \lambda(w)]_t - \mathit{\Delta} u = f \quad \text{in } Q, \\ Du \cdot \boldsymbol{n}_\Gamma + n_0(u - f_\Gamma) = 0 \quad \text{on } \Sigma, \\ u(0, x) = u_0(x), \quad x \in \Omega; \end{cases} \tag{1.1}$$

$$
\begin{cases}
w_t - \mathit{\Delta} w + \partial\gamma(w) + g_w(w,\eta) + \lambda'(w)u \\
\qquad\qquad +\alpha_w(w,\eta)|D\theta| + \nu^2\beta_w(w,\eta)|D\theta|^2 \ni 0 \ \text{ in } Q, \\
Dw \cdot \boldsymbol{n}_\Gamma = 0 \quad \text{on } \Sigma, \\
w(0,x) = w_0(x), \quad x \in \Omega;
\end{cases}
\tag{1.2}
$$

$$
\begin{cases}
\eta_t - \mathit{\Delta}\eta + g_\eta(w,\eta) + \alpha_\eta(w,\eta)|D\theta| + \nu^2\beta_\eta(w,\eta)|D\theta|^2 = 0 \quad \text{in } Q, \\
D\eta \cdot \boldsymbol{n}_\Gamma = 0 \quad \text{on } \Sigma, \\
\eta(0,x) = \eta_0(x), \quad x \in \Omega;
\end{cases}
\tag{1.3}
$$

$$
\begin{cases}
\alpha_0(w,\eta)\theta_t - \operatorname{div}\left(\alpha(w,\eta)\dfrac{D\theta}{|D\theta|} + 2\nu^2\beta(w,\eta)D\theta\right) = 0 \quad \text{in } Q, \\
(\alpha(w,\eta)\frac{D\theta}{|D\theta|} + 2\nu^2\beta(w,\eta)D\theta) \cdot \boldsymbol{n}_\Gamma = 0 \quad \text{on } \Sigma, \\
\theta(0,x) = \theta_0(x), \quad x \in \Omega.
\end{cases}
\tag{1.4}
$$

Here, Du, Dw, $D\eta$ and $D\theta$ denote, respectively, the (distributional) gradients of the unknowns u, w, η and θ on Ω. $f = f(t,x)$ is the source term on Q, $f_\Gamma = f_\Gamma(t,x)$ is the boundary source on Σ. $u_0 = u_0(x)$, $w_0 = w_0(x)$, $\eta_0 = \eta_0(x)$ and $\theta_0 = \theta_0(x)$ are given initial data on Ω. $\partial\gamma$ is the subdifferential of a proper lower semi-continuous (l.s.c.) and convex function $\gamma = \gamma(w)$ on $\mathbb{R}$. $\lambda = \lambda(w)$, $g = g(w,\eta)$, $\alpha_0 = \alpha_0(w,\eta)$, $\alpha = \alpha(w,\eta)$ and $\beta = \beta(w,\eta)$ are given real-valued functions, and the scripts "$\,'\,$", "$_w$" and "$_\eta$" denote differentials with respect to the corresponding variables. n_0 is a given positive constant, and $\boldsymbol{n}_\Gamma$ is the unit outer normal on Γ.

The system $(\mathrm{S})_\nu$ is based on the non-isothermal model of grain boundary motion by Warren et al. [36], which was derived as an extending version of the "Kobayashi–Warren–Carter model" of grain boundary motion by Kobayashi et al. [22, 23]. Hence, the study of this paper is based on the previous works related to the Kobayashi–Warren–Carter model (e.g., [13, 15, 16, 17, 20, 21, 22, 23, 25, 26, 28, 29, 30, 31, 32, 36, 37, 39]).

According to the modeling method of [36], the system $(\mathrm{S})_\nu$ is roughly configured as a coupled system of the heat equation in (1.1), and a gradient system $\{(1.2)\text{–}(1.4)\}$ of the following governing energy, called *free-energy*:

$$
\begin{aligned}
\mathscr{E}_\nu(u,w,\eta,\theta) := {}& \frac{1}{2}\int_\Omega |Dw|^2\,dx + \int_\Omega \gamma(w)\,dx + \int_\Omega u\lambda(w)\,dx \\
&+ \frac{1}{2}\int_\Omega |D\eta|^2\,dx + \int_\Omega g(w,\eta)\,dx + \int_\Omega \alpha(w,\eta)\,d|D\theta| + \int_\Omega \beta(w,\eta)|D(\nu\theta)|^2\,dx,
\end{aligned}
\tag{1.5}
$$

for $[u,w,\eta,\theta] \in L^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times BV(\Omega)$ with $\nu\theta \in H^1(\Omega)$.

In this context, the unknown $u = u(t,x)$ is the relative temperature with the critical degree 0, and the unknown $w = w(t,x)$ is an order parameter to indicate the solidification order of the polycrystal. The term $u - \lambda(w)$ in (1.1) is the so-called *enthalpy*, and then the term $\lambda(w)$ corresponds to the effect of the *latent heat*. The unknowns $\eta = \eta(t,x)$ and $\theta = \theta(t,x)$ are components of the vector field

$$
(t,x) \in Q \mapsto \eta(t,x)\Big[\cos\theta(t,x), \sin\theta(t,x)\Big] \in \mathbb{R}^2,
$$

which was adopted in [22, 23] as a vectorial phase field to reproduce the crystalline orientation in Q. Here, the components η and θ are order parameters to indicate, respectively, the orientation order and angle of the grain. In particular, w and η are taken to satisfy the constraints $0 \leq w, \eta \leq 1$ in Q, and the cases $[w, \eta] \approx [1, 1]$ and $[w, \eta] \approx [0, 0]$ are respectively assigned to "the solidified-oriented phase" and "the liquefied-disoriented phase" which correspond to two stable phases in physical.

In view of these, we suppose that

(g0) the function $w \in [0, 1] \mapsto \lambda(w) \in \mathbb{R}$ is increasing, and if the temperature u is closed to the critical value, i.e. $u \approx 0$, then the function

$$[u, w, \eta] \in \mathbb{R}^2 \mapsto \gamma(w) + g(w, \eta) - \lambda(w)u \in (-\infty, \infty]$$

has two minimums, around $[1, 1]$ and $[0, 0]$.

Besides, referring to the previous works on phase transitions (e.g., [7, 8, 14, 18, 19, 34, 35]), we can exemplify the following settings as possible expressions of the functions λ, γ and g in the above (g0):

(g1) (constrained setting by logarithmic function; cf. [14, 34, 35])

$$\begin{cases} \lambda(w) = Lw, \quad \gamma(w) := \dfrac{1}{2}\Big(w \log w + (1-w)\log(1-w)\Big) \\ \qquad \text{with } \gamma(0) = \gamma(1) := 1, \\ g(w, \eta) := -\dfrac{L}{2}\left(w - \dfrac{1}{2}\right)^2 + \dfrac{c}{2}(w-\eta)^2, \end{cases} \qquad \text{for } w, \eta \in \mathbb{R},$$

(g2) (setting with non-smooth constraint; cf. [7, 8, 18, 19, 35])

$$\begin{cases} \lambda(w) = Lw, \quad \gamma(w) := I_{[0,1]}(w), \\ g(w, \eta) := -\dfrac{L}{2}\left(w - \dfrac{1}{2}\right)^2 + \dfrac{c}{2}(w-\eta)^2, \end{cases} \qquad \text{for } w, \eta \in \mathbb{R}.$$

Here, L and c are positive constants, and $I_{[0,1]} : \mathbb{R} \to \{0, \infty\}$ is the indicator function on the compact interval $[0, 1]$.

Now, the objective of this study is to generalize the line of recent results [25, 26, 28, 29, 30, 31, 32, 37, 39], and to obtain an enhanced theory which enables the versatile analysis for Kobayashi–Warren–Carter type systems, under various situations. To this end, we set the goal of this paper to specify the assumptions, which can cover the settings as in (g1)–(g2), and can guarantee the validity of the following Main Theorem.

Main Theorem: the existence theorem of the solution $[u, w, \eta, \theta]$ to the systems $(\mathrm{S})_\nu$, for any $\nu \geq 0$, which behaves in the range of $C([0, T]; L^2(\Omega)^4)$, with the L^2-based sources $f \in L^2(0, T; L^2(\Omega))$ and $f_\Gamma \in L^2(0, T; L^2(\Gamma))$.

The main theorem is somehow to enhance the results [25, 31, 32] concerned with qualitative properties of isothermal/non-isothermal Kobayashi–Warren–Carter type systems.

2. Preliminaries

First we elaborate the notations which is used throughout this paper.

Notation 1 (Real analysis)**.** *For arbitrary* $a_0, b_0 \in [-\infty, \infty]$, *we define*

$$a_0 \vee b_0 := \max\{a_0, b_0\} \text{ and } a_0 \wedge b_0 := \min\{a_0, b_0\}.$$

Fix $d \in \mathbb{N}$ *as a constant of dimension. Then, we denote by* $|\boldsymbol{x}|$ *and* $\boldsymbol{x} \cdot \boldsymbol{y}$ *the Euclidean norm of* $\boldsymbol{x} \in \mathbb{R}^d$ *and the standard scalar product of* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, *respectively, as usual, i.e.:*

$$|\boldsymbol{x}| := \sqrt{x_1^2 + \cdots + x_d^2} \text{ and } \boldsymbol{x} \cdot \boldsymbol{y} := x_1 y_1 + \cdots + x_d y_d$$
$$\text{for all } \boldsymbol{x} = [x_1, \ldots, x_d],\ \boldsymbol{y} = [y_1, \ldots, y_d] \in \mathbb{R}^d.$$

The d-dimensional Lebesgue measure is denoted by $\mathscr{L}^d$, *and unless otherwise specified, the measure theoretical phrases, such as "a.e.", "dt", "dx", and so on, are with respect to the Lebesgue measure in each corresponding dimension. Also, in the observations on a smooth surface* $S \subset \mathbb{R}^d$, *the phrase "a.e." is with respect to the Hausdorff measure in each corresponding Hausdorff dimension, and the area element on* S *is denoted by* dS.

For a (Lebesgue) measurable function $f : B \to [-\infty, \infty]$ *on a Borel subset* $B \subset \mathbb{R}^d$, *we denote by* $[f]^+$ *and* $[f]^-$, *respectively, the positive and negative parts of* f, *i.e.,*

$$[f]^+(x) := f(x) \vee 0 \text{ and } [f]^-(x) := -(f(x) \wedge 0), \text{ a.e. } x \in B.$$

Notation 2 (Abstract functional analysis)**.** *For an abstract Banach space* X, *we denote by* $|\cdot|_X$ *the norm of* X, *and when* X *is a Hilbert space, we denote by* $(\cdot, \cdot)_X$ *its inner product. For a subset* A *of a Banach space* X, *we denote by* $\mathrm{int}(A)$ *and* $\overline{A}$ *the interior and the closure of* A, *respectively.*

Fix $1 < d \in \mathbb{N}$. *Then, for a Banach space* X, *the topology of the product Banach space* X^d *is endowed with the norm:*

$$|z|_{X^d} := \sum_{k=1}^{d} |z_k|_X, \text{ for } z = [z_1, \ldots, z_d] \in X^d.$$

However, if X *is a Hilbert space, then the topology of the product Hilbert space* X^d *is endowed with the inner product:*

$$(z, \tilde{z})_{X^d} := \sum_{k=1}^{d} (z_k, \tilde{z}_k)_X, \text{ for } z = [z_1, \ldots, z_d] \in X^d \text{ and } \tilde{z} = [\tilde{z}_1, \ldots, \tilde{z}_d] \in X^d,$$

and hence, the norm in this case is provided by

$$|z|_{X^d} := \sqrt{(z, z)_{X^d}} = \left(\sum_{k=1}^{d} |z_k|_X^2 \right)^{1/2}, \text{ for } z = [z_1, \ldots, z_d] \in X^d.$$

For a Banach space X, *we denote the dual space by* X^*. *For a single-valued operator* $\mathscr{A} : X \to X^*$, *we write*

$$\mathscr{A} z = [\mathscr{A} z_1, \ldots, \mathscr{A} z_d] \in [X^*]^d \text{ for any } z = [z_1, \ldots, z_d] \in X^d.$$

For any proper lower semi-continuous (l.s.c. hereafter) and convex function Ψ defined on a Hilbert space X, we denote by $D(\Psi)$ its effective domain, and denote by $\partial\Psi$ its subdifferential. The subdifferential $\partial\Psi$ is a set-valued map corresponding to a weak differential of Ψ, and it has a maximal monotone graph in the product Hilbert space X^2. More precisely, for each $z_0 \in X$, the value $\partial\Psi(z_0)$ is defined as the set of all elements $z_0^* \in X$ that satisfy the variational inequality

$$(z_0^*, z - z_0)_X \leq \Psi(z) - \Psi(z_0) \ \ \textit{for any } z \in D(\Psi),$$

and the set $D(\partial\Psi) := \{z \in X \mid \partial\Psi(z) \neq \emptyset\}$ is called the domain of $\partial\Psi$. We often use the notation "$[z_0, z_0^*] \in \partial\Psi$ in X^2 " to mean "$z_0^* \in \partial\Psi(z_0)$ in X with $z_0 \in D(\partial\Psi)$" by identifying the operator $\partial\Psi$ with its graph in X^2.

Notation 3 (Basic elliptic operators)**.** *Let $V = H^1(\Omega)$ be a Hilbert space endowed with the inner product:*

$$(w, z)_V := \int_\Omega \nabla w \cdot \nabla z dx + n_0 \int_\Gamma wz\, d\Gamma, \ \ \textit{for } [w, z] \in V^2,$$

and let $C_V > 0$ be the embedding constant of $V \subset L^2(\Omega)$.

Let $\langle \cdot, \cdot \rangle$ be the duality pairing between V and the dual space V^*, and let $F\colon V \to V^*$ be the duality mapping defined by

$$\langle Fw, z\rangle := (w, z)_V, \quad \textit{for } [w, z] \in V^2.$$

Note that V^* forms a Hilbert space endowed with the inner product:

$$(w^*, z^*)_{V^*} := \langle w^*, F^{-1}z^*\rangle, \ \ \textit{for } [w^*, z^*] \in (V^*)^2.$$

For any $\varrho \in L^2(\Omega)$ and any $\varrho_\Gamma \in L^2(\Gamma)$, we can regard the vectorial function $\boldsymbol{\varrho}^* := [\varrho, \varrho_\Gamma] \in L^2(\Omega) \times L^2(\Gamma)$ as an element of V^*, via the following variational form:

$$\langle \boldsymbol{\varrho}^*, z\rangle := (\varrho, z)_{L^2(\Omega)} + n_0(\varrho_\Gamma, z)_{L^2(\Gamma)} \ \ \textit{for } z \in V. \tag{2.1}$$

Note that for any $\boldsymbol{\varrho}^* = [\varrho, \varrho_\Gamma] \in L^2(\Omega) \times L^2(\Gamma)$, the variational form (2.1) *enables the following identification:*

$$F\omega = \boldsymbol{\varrho}^* \textit{ in } V^*, \textit{ iff. } \omega \in H^2(\Omega) \ \textit{ and } \ \begin{cases} -\mathit{\Delta}\omega = \varrho \textit{ in } L^2(\Omega), \\ D\omega \cdot \boldsymbol{n}_\Gamma + n_0(\omega - \varrho_\Gamma) = 0 \textit{ in } L^2(\Gamma). \end{cases}$$

On this basis, the product space $L^2(\Omega) \times L^2(\Gamma)$ can be regarded as a subspace of V^*, and the restriction $F|_{H^2(\Omega)} : H^2(\Omega) \to L^2(\Omega) \times L^2(\Gamma)$ can be regarded as a bijective linear operator associated with the Laplacian, subject to Robin type boundary condition (cf. [24]).

In the meantime, we denote by $\mathit{\Delta}_N$ the Laplacian operator subject to the zero-Neumann boundary condition, i.e.,

$$\mathit{\Delta}_N : z \in W_N := \left\{ \ z \in H^2(\Omega) \ \middle| \ Dz \cdot \boldsymbol{n}_\Gamma = 0 \textit{ in } L^2(\Gamma) \ \right\} \subset L^2(\Omega) \mapsto \mathit{\Delta} z \in L^2(\Omega).$$

Remark 1. *We here show some representative examples of the subdifferentials, which is intimately related to our study.*

(Ex.1) *The quadratic functional* $u \in L^2(\Omega) \mapsto \frac{1}{2}|u|^2_{L^2(\Omega)}$ *can be regarded as a proper l.s.c. and convex function on* V^**, via the standard* ∞*-extension, and then, the* V^**-subdifferential of this function coincides with the duality map* $F : V \to V^*$*, i.e.:*

$$[u, u^*] \in \partial\bigl[\tfrac{1}{2}|\cdot|^2_{L^2(\Omega)}\bigr] \text{ in } [V^*]^2, \text{ iff. } u \in V \text{ and } u^* = Fu \text{ in } V^*.$$

(Ex.2) *Let* $d \in \mathbb{N}$*, and let* $\gamma_0 : \mathbb{R}^d \to \mathbb{R}$ *be a convex function defined as*

$$y = [y_1, \ldots, y_d] \in \mathbb{R}^d \mapsto \gamma_0(y) := \gamma_1(y_1) + \gamma_2(y_2) + \cdots + \gamma_d(y_d),$$

by using proper l.s.c. and convex functions $\gamma_k : \mathbb{R} \to (-\infty, \infty]$*, for* $k = 1, \ldots, d$*. Let* $\Psi^d_{\gamma_0} : L^2(\Omega)^d \to (-\infty, \infty]$ *be a proper l.s.c. and convex function defined as:*

$$z \in L^2(\Omega)^d \mapsto \Psi^d_{\gamma_0}(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |Dz|^2_{\mathbb{R}^{N\times d}}\,dx + \int_\Omega \gamma_0(z)\,dx, \\ \qquad \text{if } z \in H^1(\Omega)^d, \\ \infty, \quad \text{otherwise.} \end{cases}$$

Then, with regard to the subdifferential $\partial\Psi^d_{\gamma_0} \subset [L^2(\Omega)^d]^2$*, it is known (see, e.g., [4, 6]) that*

$$z \in L^2(\Omega)^d \mapsto \partial\Psi^d_{\gamma_0}(z) = \begin{cases} \left\{ z^* \in L^2(\Omega)^d \;\middle|\; \begin{array}{l} z^* + \mathit{\Delta}_N z \in \partial\gamma_0(z) \text{ in} \\ \mathbb{R}^d, \text{ a.e. in } \Omega \end{array} \right\}, \\ \qquad \text{if } z \in W^d_N, \\ \emptyset, \quad \text{otherwise.} \end{cases}$$

This fact is often summarized as $\partial\Psi^d_{\gamma_0} = -\mathit{\Delta}_N + \partial\gamma_0$ *in* $[L^2(\Omega)^d]^2$*.*

Notation 4 (BV theory; cf. [2, 3, 11, 12])**.** *Let* $d \in \mathbb{N}$*, and let* $U \subset \mathbb{R}^d$ *be an open set. We denote by* $\mathcal{M}(U)$ *the space of all finite Radon measures on* U*. The space* $\mathcal{M}(U)$ *is known as the dual space of the Banach space* $C_0(U)$*, i.e.,* $\mathcal{M}(U) = C_0(U)^*$*, where* $C_0(U)$ *is the closure of the class of test functions* $C^\infty_{\rm c}(U)$ *in the topology of* $C(\overline{U})$*.*

A function $z \in L^1(U)$ *is called a function of bounded variation on* U*, iff. its distributional gradient* Dz *is a finite Radon measure on* U*, namely,* $Dz \in \mathcal{M}(U)^d$*. Here, for any* $z \in BV(U)$*, the Radon measure* Dz *is called the variation measure of* z*, and its total variation* $|Dz|$ *is called the total variation measure of* z*. Additionally, for any* $z \in BV(U)$*, it holds that*

$$|Dz|(U) = \sup\left\{ \int_U z \,\mathrm{div}\,\varphi\,dx \;\middle|\; \varphi \in C^1_{\rm c}(U)^d \text{ and } |\varphi| \le 1 \text{ on } U \right\}.$$

The space $BV(U)$ *is a Banach space, endowed with the norm*

$$|z|_{BV(U)} := |z|_{L^1(U)} + |Dz|(U) \text{ for any } z \in BV(U),$$

and we say that $z_n \to z$ *weakly-$*$ in* $BV(U)$*, iff.* $z \in BV(U)$*,* $\{z_n\}_{n=1}^\infty \subset BV(U)$*,* $z_n \to z$ *in* $L^1(U)$ *and* $Dz_n \to Dz$ *weakly-$*$ in* $\mathcal{M}(U)^d$*, as* $n \to \infty$*.*

The space $BV(U)$ has another topology, called "strict topology", which is provided by the following distance (cf. [2, Definition 3.14]):

$$[\varphi, \psi] \in BV(U)^2 \mapsto |\varphi - \psi|_{L^1(U)} + \big||D\varphi|(U) - |D\psi|(U)\big|.$$

In this regard, we say that $z_n \to z$ strictly in $BV(U)$ iff. $z \in BV(U)$, $\{z_n\}_{n=1}^{\infty} \subset BV(U)$, $z_n \to z$ in $L^1(U)$ and $|Dz_n|(U) \to |Dz|(U)$, as $n \to \infty$.

Specifically, when the boundary ∂U is Lipschitz, the Banach space $BV(U)$ is continuously embedded into $L^{d/(d-1)}(U)$ and compactly embedded into $L^p(U)$ for any $1 \leq p < d/(d-1)$ (see, e.g., [2, Corollary 3.49] or [3, Theorems 10.1.3–10.1.4]). Furthermore, if $1 \leq q < \infty$, then the space $C^\infty(\overline{U})$ is dense in $BV(U) \cap L^q(U)$ for the intermediate convergence, i.e., for any $z \in BV(U) \cap L^q(U)$, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subset C^\infty(\overline{U})$ such that $z_n \to z$ in $L^q(U)$ and strictly in $BV(U)$, as $n \to \infty$ (see, e.g., [3, Definition 10.1.3 and Theorem 10.1.2]).

Notation 5 (Weighted total variation; cf. [1, 2])**.** *For any nonnegative $\varrho \in H^1(\Omega) \cap L^\infty(\Omega)$ (i.e. any $0 \leq \varrho \in H^1(\Omega) \cap L^\infty(\Omega)$) and any $z \in L^2(\Omega)$, we call the value $\mathrm{Var}_\varrho(z) \in [0, \infty]$, defined as,*

$$\mathrm{Var}_\varrho(v) := \sup\left\{ \int_\Omega v \,\mathrm{div}\, \boldsymbol{\varpi}\, dx \;\middle|\; \begin{array}{l} \boldsymbol{\varpi} \in L^\infty(\Omega)^N \textit{ with a compact sup-} \\ \textit{port, and } |\boldsymbol{\varpi}| \leq \varrho \textit{ a.e. in } \Omega \end{array} \right\} \in [0, \infty],$$

"the total variation of v weighted by ϱ", or the "weighted total variation" in short.

Remark 2. *Referring to the general theories (e.g., [1, 2, 5]), we can confirm the following facts associated with the weighted total variations.*

(Fact 1) *(Cf. [5, Theorem 5]) For any $0 \leq \varrho \in H^1(\Omega) \cap L^\infty(\Omega)$, the functional $z \in L^2(\Omega) \mapsto \mathrm{Var}_\varrho(z) \in [0, \infty]$ is a proper l.s.c. and convex function that coincides with the lower semi-continuous envelope of*

$$z \in W^{1,1}(\Omega) \cap L^2(\Omega) \mapsto \int_\Omega \varrho |Dz|\, dx \in [0, \infty).$$

(Fact 2) *(Cf. [1, Theorem 4.3] and [2, Proposition 5.48]) If $0 \leq \varrho \in H^1(\Omega) \cap L^\infty(\Omega)$ and $z \in BV(\Omega) \cap L^2(\Omega)$, then there exists a Radon measure $|Dz|_\varrho \in \mathcal{M}(\Omega)$ such that*

$$|Dz|_\varrho(\Omega) = \int_\Omega d|Dz|_\varrho = \mathrm{Var}_\varrho(z),$$

and

$$\begin{cases} |Dz|_\varrho(A) \leq |\varrho|_{L^\infty(\Omega)} |Dz|(A), \\[2ex] |Dz|_\varrho(A) = \inf\left\{ \liminf\limits_{n\to\infty} \displaystyle\int_A \varrho |D\tilde{z}_n|\, dx \;\middle|\; \begin{array}{l} \{\tilde{z}_n\}_{n=1}^{\infty} \subset W^{1,1}(A) \cap L^2(A) \textit{ such} \\ \textit{that } \tilde{z}_n \to z \textit{ in } L^2(A) \textit{ as } n \to \\ \infty \end{array} \right\}, \end{cases} \tag{2.2}$$

for any open set $A \subset \Omega$.

(Fact 3) *If* $\varrho \in H^1(\Omega) \cap L^\infty(\Omega)$, $c_\varrho := \operatorname{ess\,inf}_{x\in\Omega} \varrho > 0$, *and* $z \in BV(\Omega) \cap L^2(\Omega)$, *then for any open set* $A \subset \Omega$, *it follows that*

$$\begin{cases} |Dz|_\varrho(A) \geq c_\varrho |Dz|(A) \ \text{ for any open set } A \subset \Omega, \\ D(\mathrm{Var}_\varrho) = BV(\Omega) \cap L^2(\Omega), \ \text{ and} \\ \mathrm{Var}_\varrho(z) = \sup \left\{ \int_\Omega z \operatorname{div}(\varrho \varpi)\, dx \;\middle|\; \begin{array}{l} \varpi \in L^\infty(\Omega)^N \text{ with a} \\ \text{compact support, and} \\ |\varpi| \leq 1 \text{ a.e. in } \Omega \end{array} \right\}. \end{cases} \tag{2.3}$$

Moreover, the following properties can be inferred from (2.2)–(2.3):

- $|Dz|_c = c|Dz|$ *in* $\mathcal{M}(\Omega)$ *for any constant* $c \geq 0$ *and* $z \in BV(\Omega) \cap L^2(\Omega)$*;*
- $|Dz|_\varrho = \varrho |Dz| \mathscr{L}^N$ *in* $\mathcal{M}(\Omega)$, *if* $0 \leq \varrho \in H^1(\Omega) \cap L^\infty(\Omega)$ *and* $z \in W^{1,1}(\Omega) \cap L^2(\Omega)$.

Notation 6 (Generalized weighted total variation; cf. [25, Section 2])**.** *For any* $\varrho \in H^1(\Omega) \cap L^\infty(\Omega)$ *and any* $z \in BV(\Omega) \cap L^2(\Omega)$, *we define a real-valued Radon measure* $[\varrho|Dz|] \in \mathcal{M}(\Omega)$, *as follows:*

$$[\varrho|Dz|](B) := |Dz|_{[\varrho]^+}(B) - |Dz|_{[\varrho]^-}(B) \ \text{ for any Borel set } B \subset \Omega.$$

Note that $[\varrho|Dz|](\Omega)$ *can be configured as a* generalized total variation *of* $z \in BV(\Omega) \cap L^2(\Omega)$ *by the possibly sign-changing weight* $\varrho \in H^1(\Omega) \cap L^\infty(\Omega)$.

Remark 3. *With regard to the generalized weighted total variations, the following facts are verified in [25, Section 2].*

(Fact 4) *(Strict approximation) Let* $\varrho \in H^1(\Omega) \cap L^\infty(\Omega)$ *and* $z \in BV(\Omega) \cap L^2(\Omega)$ *be arbitrary fixed functions, and let* $\{z_n\}_{n=1}^\infty \subset C^\infty(\overline{\Omega})$ *be a sequence such that*

$$z_n \to z \text{ in } L^2(\Omega) \ \text{ and strictly in } BV(\Omega) \text{ as } n \to \infty.$$

Then

$$\int_\Omega \varrho |Dz_n|\, dx \to \int_\Omega d[\varrho|Dz|] \ \text{ as } n \to \infty.$$

(Fact 5) *For any* $z \in BV(\Omega) \cap L^2(\Omega)$, *the mapping*

$$\varrho \in H^1(\Omega) \cap L^\infty(\Omega) \mapsto \int_\Omega d[\varrho|Dz|] \in \mathbb{R}$$

is a linear functional, and moreover, if $\varphi \in H^1(\Omega) \cap C(\overline{\Omega})$ *and* $\varrho \in H^1(\Omega) \cap L^\infty(\Omega)$, *then*

$$\int_\Omega d[\varphi\varrho|Dz|] = \int_\Omega \varphi\, d[\varrho|Dz|].$$

Finally, we mention the notion of functional convergences.

Definition 1 (Mosco convergence; cf. [27])**.** *Let X be an abstract Hilbert space. Let* $\Psi : X \to (-\infty, \infty]$ *be a proper l.s.c. and convex function, and let* $\{\Psi_n\}_{n=1}^\infty$ *be a sequence of proper l.s.c. and convex functions* $\Psi_n : X \to (-\infty, \infty]$, $n = 1, 2, 3, \ldots$. *We say that* $\Psi_n \to \Psi$ *on X, in the sense of Mosco, as* $n \to \infty$, *iff. the following two conditions are fulfilled.*

The condition of lower bound: $\liminf_{n\to\infty} \Psi_n(z_n^\circ) \geq \Psi(z^\circ)$, *if* $z^\circ \in X$, $\{z_n^\circ\}_{n=1}^\infty \subset X$, *and* $z_n^\circ \to z^\circ$ *weakly in* X *as* $n \to \infty$.

The condition of optimality: *for any* $z^\bullet \in D(\Psi)$, *there exists a sequence* $\{z_n^\bullet\}_{n=1}^\infty \subset X$ *such that* $z_n^\bullet \to z^\bullet$ *in* X *and* $\Psi_n(z_n^\bullet) \to \Psi(z^\bullet)$ *as* $n \to \infty$.

Definition 2 (Γ-convergence; cf. [9])**.** *Let* X *be an abstract Hilbert space,* $\Psi : X \to (-\infty, \infty]$ *be a proper functional, and* $\{\Psi_n\}_{n=1}^\infty$ *be a sequence of proper functionals* $\Psi_n : X \to (-\infty, \infty]$, $n = 1, 2, 3, \ldots$. *We say that* $\Psi_n \to \Psi$ *on* X*, in the sense of* Γ*-convergence, as* $n \to \infty$*, iff. the following two conditions are fulfilled.*

The condition of lower bound: $\liminf_{n\to\infty} \Psi_n(z_n^\circ) \geq \Psi(z^\circ)$, *if* $z^\circ \in X$, $\{z_n^\circ\}_{n=1}^\infty \subset X$, *and* $z_n^\circ \to z^\circ$ *(strongly) in* X *as* $n \to \infty$.

The condition of optimality: *for any* $z^\bullet \in D(\Psi)$, *there exists a sequence* $\{z_n^\bullet\}_{n=1}^\infty \subset X$ *such that* $z_n^\bullet \to z^\bullet$ *in* X *and* $\Psi_n(z_n^\bullet) \to \Psi(z^\bullet)$ *as* $n \to \infty$.

Remark 4. *Note that if the functionals are convex, then Mosco convergence implies* Γ*-convergence, i.e., the* Γ*-convergence of convex functions can be regarded as a weak version of Mosco convergence. Additionally, in the* Γ*-convergence of convex functions, we can see the following:*

(Fact 6) *Let* $\Psi : X \to (-\infty, \infty]$ *and* $\Psi_n : X \to (-\infty, \infty]$ *be proper l.s.c. and convex functions on a Hilbert space* X *such that* $\Psi_n \to \Psi$ *on* X*, in the sense of* Γ*-convergence, as* $n \to \infty$*. If it holds that:*

$$\begin{cases} [z, z^*] \in X^2, \quad [z_n, z_n^*] \in \partial\Psi_n \text{ in } X^2,\ n = 1, 2, 3, \ldots, \\ z_n \to z \text{ in } X \text{ and } z_n^* \to z^* \text{ weakly in } X, \text{ as } n \to \infty, \end{cases}$$

then $[z, z^*] \in \partial\Psi$ *in* X^2 *and* $\Psi_n(z_n) \to \Psi(z)$ *as* $n \to \infty$.

3. Main Theorem and the demonstration scenario

Throughout the paper, we set the following assumptions.

(A1) Let $f \in L^2(0, T; L^2(\Omega))$ and $f_\Gamma \in L^2(0, T; L^2(\Gamma))$ be given functions, and let $\boldsymbol{f}^* := [f, f_\Gamma] \in L^2(0, T; L^2(\Omega) \times L^2(\Gamma))$ be a time-dependent vectorial function which is regarded as $\boldsymbol{f}^* \in L^2(0, T; V^*)$, via (2.1) applied to $\boldsymbol{\varrho}^* = \boldsymbol{f}^*(t)$ for a.e. $t > 0$.

(A2) Let $\lambda \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ be a function, and let $A_* > 0$ be a constant which is defined as:

$$A_* := \frac{1}{4(1 + C_V^2 |\lambda|^2_{W^{2,\infty}(0,1)})},$$

by using the embedding constant $C_V > 0$ of $V \subset L^2(\Omega)$.

(A3) Let $\alpha_0 \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$ and $\alpha, \beta \in C^2(\mathbb{R}^2)$ be functions, such that:

- α and β are convex on $\mathbb{R}^2$;
- $\delta_* := \inf[\alpha_0(\mathbb{R}^2) \cup \alpha(\mathbb{R}^2) \cup \beta(\mathbb{R}^2)] > 0$;

- $\alpha_\eta(w,0) \le 0, \beta_\eta(w,0) \le 0, \alpha_\eta(w,1) \ge 0$, and $\beta_\eta(w,1) \ge 0$, for any $w \in [0,1]$.

(A4) Let $\gamma : \mathbb{R} \to (-\infty, \infty]$ be a proper l.s.c. and convex function, such that $D(\gamma) = [0,1]$.

(A5) Let $g \in C^2(\mathbb{R}^2)$ be a function such that

$$g_\eta(w,0) \le 0 \text{ and } g_\eta(w,1) \ge 0, \text{ for any } w \in [0,1].$$

(A6) There exists a constant c_* such that $\gamma(w) + g(\boldsymbol{v}) \ge c_*$, for any $\boldsymbol{v} = [w,\eta] \in \mathbb{R}^2$.

(A7) Let $[u_0, \boldsymbol{v}_0, \theta_0] = [u_0, w_0, \eta_0, \theta_0]$ is a quartet of initial data, such that:

$$[u_0, w_0, \eta_0, \theta_0] \in \begin{cases} D_0 := \left\{ [\tilde{u}, \tilde{w}, \tilde{\eta}, \tilde{\theta}] \;\middle|\; \begin{array}{l} \tilde{u} \in L^2(\Omega),\ \tilde{w}, \tilde{\eta} \in H^1(\Omega), \\ \tilde{\theta} \in BV(\Omega) \cap L^\infty(\Omega), \text{ and} \\ 0 \le \tilde{w}, \tilde{\eta} \le 1 \text{ a.e. in } \Omega \end{array} \right\}, \\ \quad \text{if } \nu = 0, \\ D_1 := D_0 \cap [L^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)], \\ \quad \text{if } \nu > 0. \end{cases}$$

Now, for simplicity of description, we prepare the following notations:

$$\begin{cases} G(u; \boldsymbol{v}) = G(u; w, \eta) := g(w,\eta) + u\lambda(w), \\ [\nabla g](\boldsymbol{v}) = [\nabla g](w,\eta) := [g_w(w,\eta), g_\eta(w,\eta)], \\ [\nabla G](u; \boldsymbol{v}) = [\nabla G](u; w, \eta) := [g_w(w,\eta) + u\lambda'(w), g_\eta(w,\eta)], \end{cases}$$

and

$$\begin{cases} [\nabla \alpha](\boldsymbol{v}) = [\nabla \alpha](w,\eta) := [\alpha_w(w,\eta), \alpha_\eta(w,\eta)], \\ [\nabla \beta](\boldsymbol{v}) = [\nabla \beta](w,\eta) := [\beta_w(w,\eta), \beta_\eta(w,\eta)], \end{cases}$$
$$\text{for } u \in \mathbb{R} \text{ and } \boldsymbol{v} = [w,\eta] \in \mathbb{R}^2.$$

For any $\nu \ge 0$ and any $\boldsymbol{v} = [w,\eta] \in [H^1(\Omega) \cap L^\infty(\Omega)]^2$, we define a proper l.s.c. and convex function $\Phi_\nu(\boldsymbol{v}; \cdot)$ on $L^2(\Omega)$ by letting:

$$\theta \in L^2(\Omega) \mapsto \Phi_\nu(\boldsymbol{v}; \theta) = \Phi_\nu(w, \eta; \theta) := \begin{cases} \displaystyle\int_\Omega d[\alpha(\boldsymbol{v})|D\theta|] + \int_\Omega \beta(\boldsymbol{v})|D(\nu\theta)|^2\, dx, \\ \qquad \text{if } \theta \in BV(\Omega) \text{ and } \nu\theta \in H^1(\Omega), \\ \infty, \quad \text{otherwise.} \end{cases}$$

Additionally, we set:

$$B_* := \frac{1 + A_*}{2}, \text{ by using the constant } A_* \text{ as in (A2)}, \tag{3.1}$$

and define a functional $\mathscr{F}_\nu$ on $L^2(\Omega)^4$ by letting:

$$\begin{aligned} [u, \boldsymbol{v}, \theta] = [u, w, \eta, \theta] \in L^2(\Omega)^4 \mapsto \mathscr{F}_\nu(u, \boldsymbol{v}, \theta) &= \mathscr{F}_\nu(u, w, \eta, \theta) \\ := B_*|u|^2_{L^2(\Omega)} + \Psi^2_\gamma(\boldsymbol{v}) + \int_\Omega (g(\boldsymbol{v}) - c_*)\, dx + \Phi_\nu(\boldsymbol{v}; \theta)&, \end{aligned} \tag{3.2}$$

where Ψ^2_γ is the convex function $\Psi^d_{\gamma_0}$ in Remark 1 in the case when $d = 2$ and $\gamma_0 = \gamma$. The above functional $\mathscr{F}_\nu$ is a modified version of the free-energy as in (1.5), and the assumptions (A3)–(A6) guarantee the non-negativity of this functional, i.e. $\mathscr{F}_\nu \geq 0$ on $L^2(\Omega)^4$.

Based on these, we define the solutions to the systems $(\mathrm{S})_\nu$, for $\nu \geq 0$, as follows.

Definition 3. *For any $\nu \geq 0$, a quartet $[u, \boldsymbol{v}, \theta] = [u, w, \eta, \theta] \in L^2(0, T; L^2(\Omega)^4)$ with $\boldsymbol{v} = [w, \eta]$ is called a solution to $(S)_\nu$, iff. $[u, \boldsymbol{v}, \theta]$ fulfills the following (S1)–(S6).*

(S1) $u \in W^{1,2}(0, T; V^*) \cap L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V) \subset C([0, T]; L^2(\Omega))$.

(S2) $\boldsymbol{v} = [w, \eta] \in W^{1,2}(0, T; L^2(\Omega)^2) \cap L^\infty(0, T; H^1(\Omega)^2)$, *and*

$$0 \leq w(t, x) \leq 1 \text{ and } 0 \leq \eta(t, x) \leq 1, \text{ a.e. } (t, x) \in Q.$$

(S3) $\theta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(Q)$, $|D\theta(\,\cdot\,)|(\Omega) \in L^\infty(0, T)$, $\nu\theta \in L^\infty(0, T; H^1(\Omega))$, *and* $|\theta| \leq |\theta_0|_{L^\infty(\Omega)}$ *a.e. in Q.*

(S4) u satisfies the following variational form:

$$\langle [u - \lambda(w)]_t(t), z\rangle + (Du(t), Dz)_{L^2(\Omega)^N} + n_0(u(t), z)_{L^2(\Gamma)}$$
$$= (f(t), z)_{L^2(\Omega)} + n_0(f_\Gamma(t), z)_{L^2(\Gamma)}, \text{ for any } z \in V, \text{ and a.e. } t \in (0, T),$$

with the initial condition $u(0) = u_0$ in $L^2(\Omega)$.

(S5) $\boldsymbol{v} = [w, \eta]$ satisfies the following two variational forms:

$$\big(w_t(t) + g_w(\boldsymbol{v})(t) + u(t)\lambda'(w(t)), w(t) - \varphi\big)_{L^2(\Omega)} + (Dw(t), D(w(t) - \varphi))_{L^2(\Omega)^N}$$
$$+ \int_\Omega d[(w(t) - \varphi)\alpha_w(\boldsymbol{v}(t))|D\theta(t)|] + \int_\Omega (w(t) - \varphi)\beta_w(\boldsymbol{v}(t))|D(\nu\theta)(t)|^2\,dx$$
$$+ \int_\Omega \gamma(w(t))\,dx \leq \int_\Omega \gamma(\varphi)dx, \text{ for any } \varphi \in H^1(\Omega) \cap L^\infty(\Omega) \text{ and a.e. } t \in (0, T),$$

and

$$\big(\eta_t(t) + g_\eta(\boldsymbol{v})(t), \psi\big)_{L^2(\Omega)} + (D\eta(t), D\psi)_{L^2(\Omega)^N}$$
$$+ \int_\Omega d[\psi\alpha_\eta(\boldsymbol{v}(t))|D\theta(t)|] + \int_\Omega \psi\beta_\eta(\boldsymbol{v}(t))|D(\nu\theta)(t)|^2\,dx = 0,$$
$$\text{for any } \psi \in H^1(\Omega) \cap L^\infty(\Omega) \text{ and a.e. } t \in (0, T),$$

with the initial condition $\boldsymbol{v}(0) = [w(0), \eta(0)] = \boldsymbol{v}_0 = [w_0, \eta_0]$ in $L^2(\Omega)^2$.

(S6) θ satisfies the following variational form:

$$(\alpha_0(\boldsymbol{v}(t))\theta_t(t), \theta(t) - \omega)_{L^2(\Omega)} + \Phi_\nu(\boldsymbol{v}(t); \theta(t)) \leq \Phi_\nu(\boldsymbol{v}(t); \omega),$$
$$\text{for any } \omega \in D(\Phi_\nu(\boldsymbol{v}(t); \cdot\,)) \text{ and a.e. } t \in (0, T),$$

with the initial condition $\theta(0) = \theta_0$ in $L^2(\Omega)$.

Remark 5. *The variational identity in the above (S4) can be reformulated as:*

$$[u - \lambda(w)]_t(t) + Fu(t) = \boldsymbol{f}^*(t) \quad \text{in } V^*, \ \text{for a.e. } t \in (0, T). \tag{3.3}$$

Also, two variational forms in (S5) can be reduced to:

$$\begin{aligned}
&(\boldsymbol{v}_t(t) + [\nabla G](u; \boldsymbol{v}(t)), \boldsymbol{v}(t) - \boldsymbol{\varpi})_{L^2(\Omega)^2} \\
&\quad + (D\boldsymbol{v}(t), D(\boldsymbol{v}(t) - \boldsymbol{\varpi}))_{L^2(\Omega)^{N\times 2}} \\
&\quad + \int_\Omega d[|D\theta(t)|(\boldsymbol{v}(t) - \boldsymbol{\varpi}) \cdot [\nabla\alpha](\boldsymbol{v}(t))] \\
&\quad + \int_\Omega |D(\nu\theta)(t)|^2 (\boldsymbol{v}(t) - \boldsymbol{\varpi}) \cdot [\nabla\beta](\boldsymbol{v}(t))\, dx \\
&\quad + \int_\Omega \gamma(\boldsymbol{v}(t))\, dx \leq \int_\Omega \gamma(\boldsymbol{\varpi})\, dx, \\
&\text{for any } \boldsymbol{\varpi} = [\varphi, \psi] \in [H^1(\Omega) \cap L^\infty(\Omega)]^2 \text{ and a.e. } t \in (0, T),
\end{aligned} \tag{3.4}$$

by using the identification

$$\gamma(\tilde{\boldsymbol{v}}) := \gamma(\tilde{w}), \quad \text{for all } \tilde{\boldsymbol{v}} = [\tilde{w}, \tilde{\eta}] \in \mathbb{R}^2,$$

and by using the abbreviation:

$$\begin{aligned}
&\int_\Omega d[|D\tilde{\theta}|\boldsymbol{\varpi} \cdot \tilde{\boldsymbol{v}}] := \int_\Omega d[\varphi\tilde{w}|D\tilde{\theta}|] + \int_\Omega d[\psi\tilde{\eta}|D\tilde{\theta}|], \\
&\text{for } \tilde{\boldsymbol{v}} = [\tilde{w}, \tilde{\eta}],\ \boldsymbol{\varpi} = [\varphi, \psi] \in [H^1(\Omega) \cap L^\infty(\Omega)]^2 \text{ and } \tilde{\theta} \in BV(\Omega) \cap L^2(\Omega).
\end{aligned} \tag{3.5}$$

Furthermore, the variational form in (S6) is equivalent to the following evolution equation:

$$\alpha_0(\boldsymbol{v}(t))\theta_t(t) + \partial\Phi_\nu(\boldsymbol{v}(t); \theta(t)) \ni 0 \text{ in } L^2(\Omega), \text{ a.e. } t \in (0, T), \tag{3.6}$$

governed by the subdifferential $\partial\Phi_\nu(\boldsymbol{v}(t); \cdot\,) \subset L^2(\Omega)^2$ of the time-dependent convex function $\Phi_\nu(\boldsymbol{v}(t); \cdot\,)$, for $t \in (0, T)$.

Now, our Main Theorem is stated as follows.

Main Theorem *Let $\nu \geq 0$ be a fixed constant. Then, under (A1)-(A7), the system $(S)_\nu$ admits at least one solution $[u, \boldsymbol{v}, \theta] = [u, w, \eta, \theta] \in L^2(0, T; L^2(\Omega)^4)$ with $\boldsymbol{v} = [w, \eta]$.*

Remark 6. *Note that the presence of mobilities $\alpha_0 = \alpha_0(w, \eta)$, $\alpha = \alpha(w, \eta)$ and $\beta = \beta(w, \eta)$ makes the uniqueness problems for the systems $(S)_\nu$, $\nu \geq 0$, be quite tough. In fact, even if we overview the kindred works to this study, we can find only two cases [15, Theorem 2.2] and [40, Theorem 2.2] that obtained the uniqueness results under some restricted situations.*

Finally, we devote the remaining part of this Section to show the sketch of the demonstration scenario, since the proof of the Main Theorem is going to be extended.

In this paper, the Main Theorem will be obtained as a consequence of some approximating approaches, and then, the approximating problems will be associated with the time-discretization versions of (3.3)–(3.6), under positive setting of the constant ν. Hence, when we consider the approximating problems, we suppose $\nu > 0$, and fix the constant of time-step $h \in (0, 1]$. Also, we denote by $[f]_0^{\rm ex} \in L^2(\mathbb{R}; L^2(\Omega))$, $[f_\Gamma]_0^{\rm ex} \in L^2(\mathbb{R}; L^2(\Gamma))$ and $[\boldsymbol{f}^*]_0^{\rm ex} \in L^2(\mathbb{R}; V^*)$ the zero-extensions of f, f_Γ and $\boldsymbol{f}^*$ $(= [f, f_\Gamma])$, respectively.

On this basis, the approximating problem for our system $(\mathrm{S})_\nu$ is denoted by $(\mathrm{AP})_h^\nu$, and stated as follows.

$(\mathrm{AP})_h^\nu$: to find a sequence $\{[u_i^\nu, \boldsymbol{v}_i^\nu, \theta_i^\nu]\}_{i=1}^\infty \subset D_1$ with $\{\boldsymbol{v}_i^\nu\}_{i=1}^\infty = \{[w_i^\nu, \eta_i^\nu]\}_{i=1}^\infty$, which fulfills that

$$\frac{u_i^\nu - u_{i-1}^\nu}{h} - \lambda'(w_i^\nu)\frac{w_i^\nu - w_{i-1}^\nu}{h} + F u_i^\nu = [\boldsymbol{f}_i^*]^h \ \text{ in } V^*, \tag{3.7}$$

$$\begin{aligned}
&\frac{1}{h}(\boldsymbol{v}_i^\nu - \boldsymbol{v}_{i-1}^\nu, \boldsymbol{v}_i^\nu - \boldsymbol{\varpi})_{L^2(\Omega)^2} + (D\boldsymbol{v}_i^\nu, D(\boldsymbol{v}_i^\nu - \boldsymbol{\varpi}))_{L^2(\Omega)^{N\times 2}} \\
&\quad + ([\nabla G](u_i^\nu; \boldsymbol{v}_i^\nu), \boldsymbol{v}_i^\nu - \boldsymbol{\varpi})_{L^2(\Omega)^2} + \int_\Omega \gamma(\boldsymbol{v}_i^\nu)\, dx \\
&\quad + \int_\Omega (\boldsymbol{v}_i^\nu - \boldsymbol{\varpi}) \cdot (|D\theta_{i-1}^\nu|[\nabla\alpha](\boldsymbol{v}_i^\nu) + \nu^2 |D\theta_{i-1}^\nu|^2 [\nabla\beta](\boldsymbol{v}_i^\nu))\, dx \\
&\quad \leq \int_\Omega \gamma(\boldsymbol{\varpi})\, dx, \ \text{ for any } \boldsymbol{\varpi} \in [H^1(\Omega) \cap L^\infty(\Omega)]^2,
\end{aligned} \tag{3.8}$$

$$\alpha_0(\boldsymbol{v}_i^\nu)\frac{\theta_i^\nu - \theta_{i-1}^\nu}{h} + \partial\Phi_\nu(\boldsymbol{v}_i^\nu; \theta_i^\nu) \ni 0 \text{ in } L^2(\Omega), \tag{3.9}$$

for $i = 1, 2, 3, \ldots$, starting from the initial data:

$$[u_0^\nu, \boldsymbol{v}_0^\nu, \theta_0^\nu] \in D_1 \text{ with } \boldsymbol{v}_0^\nu = [w_0^\nu, \eta_0^\nu].$$

In the context, for any $i \in \mathbb{N}$, $[\boldsymbol{f}_i^*]^h = [f_i^h, f_{\Gamma,i}^h] \in L^2(\Omega) \times L^2(\Gamma)$ $(\subset V^*)$, consists of the components:

$$f_i^h := \frac{1}{h}\int_{(i-1)h}^{ih} [f]_0^{\rm ex}(\tau)\, d\tau \text{ in } L^2(\Omega) \ \text{ and } \ f_{\Gamma,i}^h := \frac{1}{h}\int_{(i-1)h}^{ih} [f_\Gamma]_0^{\rm ex}(\tau)\, d\tau \text{ in } L^2(\Gamma).$$

Hence, before the proof of Main Theorem, it will be needed to verify the following theorem.

Theorem 1 (Solvability of the approximating problem)**.** *There exists a small constant $h_1^\circ \in (0, 1]$ such that if $\nu > 0$ and $h \in (0, h_1^\circ]$, then the approximating problem $(\mathrm{AP})_h^\nu$ admits a unique solution $\{[u_i^\nu, \boldsymbol{v}_i^\nu, \theta_i^\nu]\}_{i=1}^\infty \subset D_1$, and moreover,*

$$\begin{aligned}
&\frac{A_*}{2h}|u_i^\nu - u_{i-1}^\nu|_{V^*}^2 + \frac{1}{2h}|\boldsymbol{v}_i^\nu - \boldsymbol{v}_{i-1}^\nu|_{L^2(\Omega)^2}^2 + \frac{1}{h}\left|\sqrt{\alpha_0(\boldsymbol{v}_i^\nu)}(\theta_i^\nu - \theta_{i-1}^\nu)\right|_{L^2(\Omega)}^2 + \frac{h}{2}|u_i^\nu|_V^2 \\
&\quad + \mathscr{F}_\nu(u_i^\nu, \boldsymbol{v}_i^\nu, \theta_i^\nu) \leq \mathscr{F}_\nu(u_{i-1}^\nu, \boldsymbol{v}_{i-1}^\nu, \theta_{i-1}^\nu) + h|[\boldsymbol{f}_i^*]^h|_{V^*}^2, \ \textit{for } i = 1, 2, 3, \ldots,
\end{aligned} \tag{3.10}$$

where A_* is the constant as in (A2).

However, due to the presence of L^1-terms $\nu^2|D\theta_{i-1}|^2[\nabla\beta](\boldsymbol{v}_i^\nu) \in L^1(\Omega)^2$, $i = 1, 2, 3, \dots$, in (3.8), the above Theorem 1 will not be a straightforward consequence of standard variational method, and in fact, this theorem will be obtained via further approximating approach by means of some relaxed systems for $(\mathrm{AP})_h^\nu$.

In the observation of the relaxed system, we first fix a large constant $M > (N+2)/2$, and fix a small constant $\varepsilon \in (0, 1]$ as the relaxation index. Besides, we define

$$D_M := D_1 \cap [L^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times H^M(\Omega)],$$

and for any $\tilde{\boldsymbol{v}} \in L^2(\Omega)^2$, we define a relaxed functional $\Phi_\varepsilon^\nu(\tilde{\boldsymbol{v}}; \cdot)$ for $\Phi_\nu(\tilde{\boldsymbol{v}}; \cdot)$, by letting:

$$\theta \in L^2(\Omega) \mapsto \Phi_\varepsilon^\nu(\tilde{\boldsymbol{v}}; \theta) := \begin{cases} \Phi_\nu(\tilde{\boldsymbol{v}}; \theta) + \dfrac{\varepsilon^2}{2}|\theta|^2_{H^M(\Omega)}, \text{ if } \theta \in H^M(\Omega), \\ \infty, \quad \text{otherwise.} \end{cases}$$

Note that for any $\tilde{\boldsymbol{v}} \in L^2(\Omega)^2$, the functional $\Phi_\varepsilon^\nu(\tilde{\boldsymbol{v}}; \cdot)$ is proper l.s.c. and convex on $L^2(\Omega)$, such that:

$$D(\Phi_\varepsilon^\nu(\tilde{\boldsymbol{v}}; \cdot)) = H^M(\Omega) \subset W^{1,\infty}(\Omega),$$

and hence, the L^2-subdifferential $\partial\Phi_\varepsilon^\nu(\tilde{\boldsymbol{v}}; \cdot)$ is a maximal monotone graph in $L^2(\Omega)^2$.

On this basis, we denote by $(\mathrm{RX})_\varepsilon$ the relaxed system for $(\mathrm{AP})_h^\nu$, and prescribe the system $(\mathrm{RX})_\varepsilon$ as follows.

$(\mathrm{RX})_\varepsilon$: to find a sequence $\{[u_{\varepsilon,i}^\nu, \boldsymbol{v}_{\varepsilon,i}^\nu, \theta_{\varepsilon,i}^\nu]\}_{i=1}^\infty \subset D_M$ with $\{\boldsymbol{v}_{\varepsilon,i}^\nu\}_{i=1}^\infty = \{[w_{\varepsilon,i}^\nu, \eta_{\varepsilon,i}^\nu]\}_{i=1}^\infty$, which fulfills that

$$\frac{u_{\varepsilon,i}^\nu - u_{\varepsilon,i-1}^\nu}{h} - \lambda'(w_{\varepsilon,i}^\nu)\frac{w_{\varepsilon,i}^\nu - w_{\varepsilon,i-1}^\nu}{h} + Fu_{\varepsilon,i}^\nu = [\boldsymbol{f}_i^*]^h \text{ in } V^*, \tag{3.11}$$

$$\begin{aligned} &\frac{\boldsymbol{v}_{\varepsilon,i}^\nu - \boldsymbol{v}_{\varepsilon,i-1}^\nu}{h} - \varDelta_N \boldsymbol{v}_{\varepsilon,i}^\nu + \partial\gamma(\boldsymbol{v}_{\varepsilon,i}^\nu) + [\nabla G](u_{\varepsilon,i}^\nu; \boldsymbol{v}_{\varepsilon,i}^\nu) \\ &\qquad + |D\theta_{\varepsilon,i-1}^\nu|[\nabla\alpha](\boldsymbol{v}_{\varepsilon,i}^\nu) + \nu^2|D\theta_{\varepsilon,i-1}^\nu|^2[\nabla\beta](\boldsymbol{v}_{\varepsilon,i}^\nu) \ni 0 \text{ in } L^2(\Omega)^2, \end{aligned} \tag{3.12}$$

$$\alpha_0(\boldsymbol{v}_{\varepsilon,i}^\nu)\frac{\theta_{\varepsilon,i}^\nu - \theta_{\varepsilon,i-1}^\nu}{h} + \partial\Phi_\varepsilon^\nu(\boldsymbol{v}_{\varepsilon,i}^\nu; \theta_{\varepsilon,i}^\nu) \ni 0 \text{ in } L^2(\Omega), \tag{3.13}$$

for $i = 1, 2, 3, \dots$, starting from the initial data:

$$[u_{\varepsilon,0}^\nu, \boldsymbol{v}_{\varepsilon,0}^\nu, \theta_{\varepsilon,0}^\nu] \in D_M \text{ with } \boldsymbol{v}_{\varepsilon,0}^\nu = [w_{\varepsilon,0}^\nu, \eta_{\varepsilon,0}^\nu].$$

Then, we can see that

$$|D\theta_{\varepsilon,i-1}^\nu| \in L^\infty(\Omega) \text{ and } \nu^2|D\theta_{\varepsilon,i-1}^\nu|^2[\nabla\beta](\boldsymbol{v}_{\varepsilon,i}^\nu) \in L^\infty(\Omega)^2, \ i = 1, 2, 3, \dots.$$

It implies that the general theories of L^2-subdifferentials will be available for the relaxed system $(\mathrm{RX})_\varepsilon$.

Thus, it will be needed to verify the following proposition, as the first task to proving the Main Theorem.

Proposition 1. *There exists a small constant $h_0^\circ \in (0, 1]$, such that if $h \in (0, h_0^\circ]$, then the system $(\mathrm{RX})_\varepsilon$ admits a unique solution $\{[u_{\varepsilon,i}^\nu, \boldsymbol{v}_{\varepsilon,i}^\nu, \theta_{\varepsilon,i}^\nu]\}_{i=1}^\infty \subset D_M$ with $\{\boldsymbol{v}_{\varepsilon,i}^\nu\}_{i=1}^\infty = \{[w_{\varepsilon,i}^\nu, \eta_{\varepsilon,i}]^\nu\}_{i=1}^\infty$.*

In view of these, we set the demonstration scenario of the Main Theorem, by assigning the proofs of Proposition 1, Theorem 1 and Main Theorem to Sections 4, 5 and 6, respectively.

4. Proof of Proposition 1

Before we start the proof, we need to show some lemmas.

Lemma 1. *Let us put* $\Delta^\bullet := [0,1] \times [-1,2] \subset \mathbb{R}^2$*, and let us assume*

$$0 < h \leq h_2^\circ := \frac{1}{2(1 + |g|_{C^2(\Delta^\circ)} + 5|\lambda|^2_{W^{2,\infty}(0,1)})}. \tag{4.1}$$

Let us fix $f_0^* \in V^*$, $[u_0^\circ, \eta_0^\circ, w_0^\circ, \theta_0^\circ] \in L^2(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times W^{1,\infty}(\Omega)$ *and* $w^\circ \in H^1(\Omega)$*, and let us assume that* $0 \leq w_0^\circ, w^\circ \leq 1$ *a.e. in* Ω*. Then, the following auxiliary system:*

$$\frac{u - u_0^\circ}{h} - \lambda'(w^\circ)\frac{w - w_0^\circ}{h} + Fu = f_0^* \text{ in } V^*, \tag{4.2}$$

$$\begin{aligned} &\frac{w - w_0^\circ}{h} - \mathit{\Delta}_N w + \partial\gamma(w) + g_w(w,\eta) \\ &\qquad +\alpha_w(w,\eta)|D\theta_0^\circ| + \nu^2\beta_w(w,\eta)|D\theta_0^\circ|^2 \ni -\lambda'(w^\circ)u \text{ in } L^2(\Omega), \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\frac{\eta - \eta_0^\circ}{h} - \mathit{\Delta}_N \eta + \partial I_{[-1,2]}(\eta) + g_\eta(w,\eta) \\ &\qquad +\alpha_\eta(w,\eta)|D\theta_0^\circ| + \nu^2\beta_\eta(w,\eta)|D\theta_0^\circ|^2 = 0 \text{ in } L^2(\Omega), \end{aligned} \tag{4.4}$$

admits a unique solution $[u, w, \eta] \in V \times H^1(\Omega)^2$*, where* $\partial I_{[-1,2]}$ *is the subdifferential of the indicator function* $I_{[-1,2]} : \mathbb{R} \to \{0, \infty\}$ *on the compact interval* $[-1,2]$*, and this is an additional term to guarantee the boundedness of the range* $\eta(\Omega)$ *for the component* η*.*

Proof. First, we put:

$$e := u - \lambda'(w^\circ)w,\ e_0^\circ := u_0^\circ - \lambda'(w^\circ)w_0^\circ, \text{ and } \boldsymbol{v}_0^\circ = [w_0^\circ, \eta_0^\circ],$$

$$[\tilde{w}, \tilde{\eta}] \in \mathbb{R} \mapsto \gamma^\bullet(\tilde{w}, \tilde{\eta}) := \gamma(\tilde{w}) + I_{[-1,2]}(\tilde{\eta}),$$

and reformulate the system {(4.2)–(4.4)} as follows:

$$\frac{e - e_0^\circ}{h} + F(e + \lambda'(w^\circ)w) = f_0^* \text{ in } V^*, \tag{4.5}$$

$$\begin{aligned} &\frac{\boldsymbol{v} - \boldsymbol{v}_0^\circ}{h} + \partial\Psi^2_{\gamma^\bullet}(\boldsymbol{v}) + [\nabla g](w,\eta) \\ &\qquad +|D\theta_0^\circ|[\nabla\alpha](\boldsymbol{v}) + \nu^2|D\theta_0^\circ|^2[\nabla\beta](\boldsymbol{v}) \ni \begin{bmatrix} -\lambda'(w^\circ)(e + \lambda'(w^\circ)w) \\ 0 \end{bmatrix} \text{ in } L^2(\Omega)^2, \end{aligned} \tag{4.6}$$

where $\Psi^2_{\gamma^\bullet}$ is the functional $\Psi^d_{\gamma_0}$ as in Remark 1 (Ex.2), in the case when $d = 2$ and $\gamma_0 = \gamma^\bullet$ on $\mathbb{R}^2$, and $\partial\Psi^2_{\gamma^\bullet}$ is the subdifferential of $\Psi^2_{\gamma^\bullet}$ in $L^2(\Omega)^2$. Then, in the light of Remark 1, we can associate the auxiliary system {(4.2)–(4.4)} with a minimization problem for the following functional:

$$
\begin{aligned}
[e,\boldsymbol{v}] \ &= \ [e,w,\eta] \in V^* \times L^2(\Omega)^2 \mapsto \Psi_0^\bullet(w^\circ; e,\boldsymbol{v}) = \Psi_0^\bullet(w^\circ; e,w,\eta) \\
&:= \begin{cases}
\frac{1}{2h}|e-e_0^\circ|_{V^*}^2 + \frac{1}{2h}|\boldsymbol{v}-\boldsymbol{v}_0^\circ|_{L^2(\Omega)^2}^2 + \frac{1}{2}|e+\lambda'(w^\circ)w|_{L^2(\Omega)}^2 \\
\qquad + \Psi_{\gamma^\bullet}^2(\boldsymbol{v}) + \int_\Omega \left(\alpha(\boldsymbol{v})|D\theta_0^\circ| + \nu^2\beta(\boldsymbol{v})|D\theta_0^\circ|^2\right) dx \\
\qquad + \int_\Omega g(\boldsymbol{v})\,dx - (f_0^*, e)_{V^*}, \\
\qquad \text{if } [e,\boldsymbol{v}] = [e,w,\eta] \in L^2(\Omega) \times D(\Psi_{\gamma^\bullet}^2), \\
\infty, \quad \text{otherwise},
\end{cases}
\end{aligned}
\tag{4.7}
$$

via its stationary system {(4.5)–(4.6)}. Then, taking into account (A2)–(A6), (4.1) and (4.7), we find a positive constant C_0°, independent of the variables $[e,\boldsymbol{v}] = [e,w,\eta]$ and w°, such that:

$$
\Psi_0^\bullet(w^\circ; e,\boldsymbol{v}) \geq C_0^\circ(|e|_{L^2(\Omega)}^2 + |\boldsymbol{v}|_{H^1(\Omega)^2}^2 - 1), \text{ for all } [e,\boldsymbol{v}] \in D(\Psi_0^\bullet(w^\circ;\cdot)). \tag{4.8}
$$

Now, the above coercivity enables us to apply the standard minimization argument to $\Psi_0^\bullet$ (cf. [3, 10]), and to obtain the solution $[u,w,\eta] = [e+\lambda'(w^\circ)w, w, \eta]$ to {(4.2)–(4.4)}, via the minimizer $[e,\boldsymbol{v}] = [e,w,\eta] \in V \times H^1(\Omega)^2$ of $\Psi_0^\bullet(w^\circ;\cdot)$, with $\boldsymbol{v} = [w,\eta] \in D(\Psi_{\gamma^\bullet}^2)$.

In the meantime, the uniqueness can be seen by using the standard method, i.e. by taking the difference of two solutions $[e_k,\boldsymbol{v}_k] = [e_k,w_k,\eta_k] \in V^* \times L^2(\Omega)^2$ with $\boldsymbol{v}_k = [w_k,\eta_k] \in D(\Psi_\gamma^2)$, $k=1,2$, to the stationary system {(4.5)–(4.6)}. In fact, multiplying the both sides of the subtraction of (4.5) by e_1-e_2 in V^*, multiplying the both sides of the subtraction of (4.6) by $\boldsymbol{v}_1-\boldsymbol{v}_2$ in $L^2(\Omega)^2$, and using (A2)–(A5), (4.8) and Schwartz's inequality, we have:

$$
\begin{aligned}
&\frac{1}{h}|e_1-e_2|_{V^*}^2 + \frac{1}{h}\left(1 - h|[\nabla g]|_{W^{1,\infty}(\Delta^\bullet)^2}\right)|\boldsymbol{v}_1-\boldsymbol{v}_2|_{L^2(\Omega)^2}^2 \\
&+|D(\boldsymbol{v}_1-\boldsymbol{v}_2)|_{L^2(\Omega)^{N\times 2}}^2 + |(e_1-e_2)+\lambda'(w^\circ)(w_1-w_2)|_{L^2(\Omega)}^2 \leq 0.
\end{aligned}
\tag{4.9}
$$

Since the assumption (4.1) implies $(1-h|[\nabla g]|_{W^{1,\infty}(\Delta^\bullet)^2}) \geq \frac{1}{2}$, we can deduce from (4.9) the uniqueness for the system {(4.2)–(4.4)}. □ □

Lemma 2. *Let $w^\circ \in H^1(\Omega)$ be the function as in Lemma 1, and let $\Psi_0^\bullet(w^\circ;\cdot)$ be the functional on $V^* \times L^2(\Omega)^2$ given in* (4.7)*. Also, let us take a sequence $\{w_n^\circ\}_{n=1}^\infty \subset H^1(\Omega)$ such that* $0 \leq w_n^\circ \leq 1$ *a.e. in Ω, for $n=1,2,3,\ldots$, and let us define a sequence $\{\Psi_0^\bullet(w_n^\circ;\cdot)\}_{n=1}^\infty$ of functionals on $V^*\times L^2(\Omega)^2$, by putting $w^\circ = w_n^\circ$ in* (4.7)*, for $n=1,2,3,\ldots$. Besides, let us assume that:*

$$
w_n^\circ \to w^\circ \text{ in the pointwise sense a.e. in } \Omega, \text{ as } n \to \infty. \tag{4.10}
$$

Then, $\Psi_0^\bullet(w_n^\circ;\cdot) \to \Psi_0^\bullet(w^\circ;\cdot)$ on $V^*\times L^2(\Omega)^2$, in the sense of Γ*-convergence, as $n\to\infty$.*

Proof. The condition of lower-bound will be seen, immediately, from the lower semi-continuity of the following functional (of 4-variables):

$$
[w^\circ, e, \boldsymbol{v}] \in L^2(\Omega) \times V^* \times L^2(\Omega)^2 \mapsto \Psi_0^\bullet(w^\circ; e,\boldsymbol{v}) \in (-\infty,\infty].
$$

The condition of optimality will be verified by taking the singleton $\{[e,\boldsymbol{v}]\}$ for any $[e,\boldsymbol{v}] \in D(\Psi_0^\bullet(w^\circ;\cdot)) = D(\Psi_0^\bullet(w_n^\circ;\cdot))$ for all $n \geq 1$ as the sequence corresponding to $\{z_n^\bullet\}_{n=1}^\infty$ in Definition 2. □ □

Lemma 3. *Under the assumptions as in the previous Lemmas 1–2, let us take the solution* $[e, \boldsymbol{v}] = [e, w, \eta] \in V \times H^1(\Omega)^2$ *to the stationary system* $\{(4.5)–(4.6)\}$ *with* $\boldsymbol{v} = [w, \eta]$*, and for any* $n \in \mathbb{N}$*, let us denote by* $[e_n, \boldsymbol{v}_n] = [e_n, w_n, \eta_n] \in V \times H^1(\Omega)^2$ *the solution to* $\{(4.5)–(4.6)\}$ *with* $\boldsymbol{v}_n = [w_n, \eta_n]$*, when* $w^\circ = w_n^\circ$*. Then, the assumption* (4.10) *implies that:*

$$\begin{gathered}[e_n, \boldsymbol{v}_n] = [e_n, w_n, \eta_n] \to [e, \boldsymbol{v}] = [e, w, \eta] \text{ in } V^* \times L^2(\Omega)^2, \\ \text{and weakly in } L^2(\Omega) \times H^1(\Omega)^2, \text{ as } n \to \infty.\end{gathered} \tag{4.11}$$

Proof. In the light of Lemma 1 (including the proof), we can see that:

$$\begin{aligned}&\Psi_0^\bullet(\tilde{w}^\circ; \tilde{e}, \tilde{\boldsymbol{v}}) = \Psi_0^\bullet(\tilde{w}^\circ; \tilde{e}, \tilde{w}, \tilde{\eta}) \le \Psi_0^\bullet(\tilde{w}^\circ; 0, 0, 0) \\ &\le C_1^\circ := \frac{1}{2h}(|e_0^\circ|_{V^*}^2 + |\boldsymbol{v}_0^\circ|_{L^2(\Omega)^2}^2) + \mathscr{L}^N(\Omega)(|\gamma(0)| + |g(0, 0)|) \\ &\qquad +\alpha(0, 0)|\theta_0^\circ|_{W^{1,1}(\Omega)} + \nu^2\beta(0, 0)|\theta_0^\circ|_{H^1(\Omega)}^2,\end{aligned} \tag{4.12}$$

$$\begin{gathered}\text{for any } \tilde{w}^\circ \in H^1(\Omega) \text{ with } 0 \le \tilde{w}^\circ \le 1 \text{ a.e. in } \Omega, \text{ and} \\ \text{any solution } [\tilde{e}, \tilde{\boldsymbol{v}}] = [\tilde{e}, \tilde{w}, \tilde{\eta}] \text{ to } \{(4.5)–(4.6)\} \text{ with } \tilde{\boldsymbol{v}} = [\tilde{w}, \tilde{\eta}] \text{ when } w^\circ = \tilde{w}^\circ.\end{gathered}$$

Since the constant C_1° is independent of the choice of $\tilde{w}^\circ$, the convergence (4.11) will be observed by taking into account (4.8), (4.12) and the uniqueness for $\{(4.5)–(4.6)\}$, and by applying Lemma 2, and the general theories of the compact embeddings (cf. [3, 11]) and the Γ convergence (cf. [9]). □ ◻

Lemma 4. *Let* h_2° *be the constant as in* (4.1)*. Let* $f_0^* \in V^*$*,* $u_0^\circ \in L^2(\Omega)$*,* $\boldsymbol{v}_0^\circ = [w_0^\circ, \eta_0^\circ] \in H^1(\Omega)^2$ *and* $\theta_0^\circ \in W^{1,\infty}(\Omega)$ *be the functions as in Lemma 1. Then, for any* $h \in (0, h_2^\circ]$*, the following system:*

$$\frac{u - u_0^\circ}{h} - \lambda'(w)\frac{w - w_0^\circ}{h} + Fu = f_0^* \text{ in } V^*, \tag{4.13}$$

$$\begin{aligned}&\frac{\boldsymbol{v} - \boldsymbol{v}_0^\circ}{h} - \varDelta_N \boldsymbol{v} + \begin{bmatrix} \partial\gamma(w) \\ 0 \end{bmatrix} + [\nabla g](\boldsymbol{v}) \\ &\qquad +|D\theta_0^\circ|[\nabla\alpha](\boldsymbol{v}) + \nu^2|D\theta_0^\circ|^2[\nabla\beta](\boldsymbol{v}) \ni \begin{bmatrix} -\lambda'(w)u \\ 0 \end{bmatrix} \text{ in } L^2(\Omega)^2,\end{aligned} \tag{4.14}$$

admits at least one solution $[u, \boldsymbol{v}] = [u, w, \eta] \in V \times H^1(\Omega)^2$ *with* $\boldsymbol{v} = [w, \eta]$.

Proof. Let us set a compact set $K_1^\bullet$ in $L^2(\Omega)$, by letting:

$$K_1^\bullet := \left\{ \tilde{w} \in H^1(\Omega) \ \middle| \ \begin{aligned}&0 \le \tilde{w} \le 1 \text{ a.e. in } \Omega, \text{ and} \\ &\frac{1}{2h}|\tilde{w} - w_0^\circ|_{L^2(\Omega)}^2 + \frac{1}{2}|D\tilde{w}|_{L^2(\Omega)^N}^2 \\ &\le C_1^\circ + |c_*|\mathscr{L}^N(\Omega) + \frac{1}{2h}|e_0^\circ|_{V^*}^2 + h|f_0^*|_{V^*}^2\end{aligned} \right\},$$

and let us consider an operator $P_1^\bullet : K_1^\bullet \to L^2(\Omega)$, which maps any $w^\circ \in K_1^\bullet$ to the component w of the solution $[u, w, \eta] \in V \times H^1(\Omega) \times H^1(\Omega)$ to $\{(4.2)–(4.4)\}$. Then, on account of (A3), (A6), Lemma 3, (4.7) and (4.12), it will be seen that $P_1^\bullet K_1^\bullet \subset K_1^\bullet$ and $P_1^\bullet$ is a continuous operator in the topology of $L^2(\Omega)$.

So, applying Schauder's fixed point theorem, we find a fixed point $w^\bullet \in K_1^\bullet$ for $P_1^\bullet$, i.e. $w^\bullet = P_1^\bullet w^\bullet$ in $L^2(\Omega)$.

Now, let us denote by $[u^\bullet, w^\bullet, \eta^\bullet] \in V \times H^1(\Omega) \times H^1(\Omega)$ the solution to {(4.2)–(4.4)}, involved in the fixed point $w^\bullet$. Then, this triplet $[u^\bullet, w^\bullet, \eta^\bullet]$ must be a special solution to {(4.2)–(4.4)} such that $w^\bullet = w^\circ$. Hence, our remaining task will be to show that

$$0 \leq \eta^\bullet \leq 1 \text{ a.e. in } \Omega, \tag{4.15}$$

namely, the subdifferential $\partial I_{[-1,2]}$ in (4.4) will not affect for $\eta^\bullet$. To this end, we check two inequalities:

$$\frac{0 - \eta_0^\circ}{h} + g_\eta(w^\bullet, 0) + |D\theta_0^\circ|\alpha_\eta(w^\bullet, 0) + \nu^2|D\theta_0^\circ|^2\beta_\eta(w^\bullet, 0) \leq 0 \text{ in } L^2(\Omega), \tag{4.16}$$

$$\frac{1 - \eta_0^\circ}{h} + g_\eta(w^\bullet, 1) + |D\theta_0^\circ|\alpha_\eta(w^\bullet, 1) + \nu^2|D\theta_0^\circ|^2\beta_\eta(w^\bullet, 1) \geq 0 \text{ in } L^2(\Omega), \tag{4.17}$$

with use of the assumptions (A3), (A5) and $0 \leq \eta_0^\circ \leq 1$ a.e. in Ω.

On this basis, let us take the difference from (4.16) to (4.4) when $\eta = \eta^\bullet$ and $w = w^\circ = w^\bullet$ (resp. from (4.4) to (4.17) when $\eta = \eta^\bullet$ and $w = w^\circ = w^\bullet$), and multiply the both sides by $[-\eta^\bullet]^+$ (resp. by $[\eta^\bullet - 1]^+$). Then, with the assumptions (A3), (A5) and $\partial I_{[-1,2]}(0) = \{0\}$ (resp. $\partial I_{[-1,2]}(1) = \{0\}$) in mind, it is inferred that:

$$\begin{gathered}\frac{1}{h}\left(1 - h|g_{\eta\eta}|_{C(\Delta^\bullet)}\right)|[-\eta^\bullet]^+|^2_{L^2(\Omega)} + |D[-\eta^\bullet]^+|^2_{L^2(\Omega)^N} \leq 0 \\ \left(\text{resp. } \frac{1}{h}\left(1 - h|g_{\eta\eta}|_{C(\Delta^\bullet)}\right)|[\eta^\bullet - 1]^+|^2_{L^2(\Omega)} + |D[\eta^\bullet - 1]^+|^2_{L^2(\Omega)^N} \leq 0\right).\end{gathered} \tag{4.18}$$

Since the assumption (4.1) implies $1 - h|g_{\eta\eta}|_{C(\Delta^\bullet)} \geq \frac{1}{2}$, we can deduce (4.15) from (4.18), and conclude that the triplet $[u^\bullet, \boldsymbol{v}^\bullet] = [u^\bullet, \eta^\bullet, w^\bullet]$ with $\boldsymbol{v}^\bullet := [w^\bullet, \eta^\bullet]$ solves the system {(4.13)–(4.14)}. □ □

Lemma 5. *Let $f_0^* \in V^*$ and $\theta_0^\circ \in H^M(\Omega)$ be fixed functions, and let $[u, \boldsymbol{v}] = [u, w, \eta] \in V \times H^1(\Omega)^2$ be a solution to the system* {(4.13)–(4.14)} *with $\boldsymbol{v} = [w, \eta]$. Then, the inclusion*

$$\alpha_0(\boldsymbol{v})\frac{\theta - \theta_0^\circ}{h} + \partial\Phi_\varepsilon^\nu(\boldsymbol{v}; \theta) \ni 0 \text{ in } L^2(\Omega) \tag{4.19}$$

admits a unique solution $\theta \in H^M(\Omega)$.

Proof. We omit the proof, because this lemma is obtained, immediately, just as a direct consequence of [31, Lemma 3.4]. □ □

Lemma 6. *Under the assumption* (4.1)*, let us take a quartet $[u, \boldsymbol{v}, \theta] = [u, w, \eta, \theta] \in D_M$ with $\boldsymbol{v} = [w, \eta] \in H^1(\Omega)^2$, which solves the coupled system* {(4.13)–(4.14),(4.19)}*. Then, the following energy-inequality holds:*

$$\begin{gathered}\frac{A_*}{2h}|u - u_0^\circ|^2_{V^*} + \frac{1}{2h}|\boldsymbol{v} - \boldsymbol{v}_0^\circ|^2_{L^2(\Omega)^2} + \frac{1}{h}|\sqrt{\alpha_0(\boldsymbol{v})}(\theta - \theta_0^\circ)|^2_{L^2(\Omega)} \\ + \frac{h}{2}|u|^2_V + \mathscr{F}_\varepsilon^\nu(u, \boldsymbol{v}, \theta) \leq \mathscr{F}_\varepsilon^\nu(u_0^\circ, \boldsymbol{v}_0^\circ, \theta_0^\circ) + h|f_0^*|^2_{V^*},\end{gathered} \tag{4.20}$$

where $A_* > 0$ is the constant as in (A2), and $\mathscr{F}_\varepsilon^\nu$ is the relaxed version of the functional $\mathscr{F}_\nu$, defined as:

$$
\begin{aligned}
[u, \boldsymbol{v}, \theta] = [u, w, \eta, \theta] \in L^2(\Omega)^4 \mapsto \mathscr{F}_\varepsilon^\nu(u, \boldsymbol{v}, \theta) = \mathscr{F}_\varepsilon^\nu(u, w, \eta, \theta) \\
= B_* |u|_{L^2(\Omega)}^2 + \Psi_\gamma^2(\boldsymbol{v}) + \int_\Omega (g(\boldsymbol{v}) - c_*)\, dx + \Phi_\varepsilon^\nu(\boldsymbol{v}; \theta),
\end{aligned}
\tag{4.21}
$$

with the constant $B_* = (1 + A_*)/2$ as in (3.1).

Proof. First, let us rewrite the equation (4.13) as follows:

$$
\begin{aligned}
(u - u_0^\circ, z)_{L^2(\Omega)} + h\langle Fu, z\rangle = h\langle f_0^*, z\rangle \\
+(\lambda'(w)(w - w_0^\circ), z)_{L^2(\Omega)}, \text{ for any } z \in V,
\end{aligned}
\tag{4.22}
$$

and let us put $z = u$. Then, by using Schwarz's inequality, we have:

$$
\frac{1}{2}|u|_{L^2(\Omega)}^2 + \frac{h}{2}|u|_V^2 \le \frac{1}{2}|u_0^\circ|_{L^2(\Omega)}^2 + \frac{h}{2}|f_0^*|_{V^*}^2 + (\lambda'(w)(w - w_0^\circ), u)_{L^2(\Omega)}. \tag{4.23}
$$

Alternatively, if we rewrite the equation (4.13) to:

$$
\begin{aligned}
\frac{1}{h}(u - u_0^\circ, z^*)_{V^*} + \langle z^*, u\rangle = (f_0^*, z^*)_{V^*} + \frac{1}{h}(\lambda'(w)(w - w_0^\circ), z^*)_{V^*}, \\
\text{for any } z^* \in V^*,
\end{aligned}
$$

and put $z^* = A_*(u - u_0^\circ) \in V$, then we also see that:

$$
\frac{A_*}{2h}|u - u_0^\circ|_{V^*}^2 + \frac{A_*}{2}|u|_{L^2(\Omega)}^2 \le \frac{A_*}{2}|u_0^\circ|_{L^2(\Omega)}^2 + A_* h |f_0^*|_{V^*}^2 + \frac{1}{4h}|w - w_0^\circ|_{L^2(\Omega)}^2. \tag{4.24}
$$

Next, let us multiply the both sides of the inclusion (4.14) by $\boldsymbol{v} - \boldsymbol{v}_0^\circ$. Then, in the light of (A2)–(A5) and Taylor's theorem, we infer that:

$$
\begin{aligned}
&\frac{1}{h}\left(1 - \frac{h}{2}|g|_{C^2([0,1]^2)}\right)|\boldsymbol{v} - \boldsymbol{v}_0^\circ|_{L^2(\Omega)^2}^2 + \frac{1}{2}|D\boldsymbol{v}|_{L^2(\Omega)^{N\times 2}}^2 + \int_\Omega \gamma(w)\, dx + \int_\Omega g(\boldsymbol{v})\, dx \\
&\qquad + \int_\Omega \alpha(\boldsymbol{v})|D\theta_0^\circ|\, dx + \nu^2 \int_\Omega \beta(\boldsymbol{v})|D\theta_0^\circ|^2\, dx \\
&\le \frac{1}{2}|D\boldsymbol{v}_0^\circ|_{L^2(\Omega)^{N\times 2}}^2 + \int_\Omega \gamma(w_0^\circ)\, dx + \int_\Omega g(\boldsymbol{v}_0^\circ)\, dx \\
&\qquad + \int_\Omega \alpha(\boldsymbol{v}_0^\circ)|D\theta_0^\circ|\, dx + \nu^2 \int_\Omega \beta(\boldsymbol{v}_0^\circ)|D\theta_0^\circ|^2\, dx - (\lambda'(w)(w - w_0^\circ), u)_{L^2(\Omega)}.
\end{aligned}
\tag{4.25}
$$

Furthermore, applying the both sides of (4.19) by $\theta - \theta_0^\circ$, it follows that:

$$
\frac{1}{h}|\sqrt{\alpha_0(\boldsymbol{v})}(\theta - \theta_0^\circ)|_{L^2(\Omega)}^2 + \Phi_\varepsilon^\nu(\boldsymbol{v}; \theta) \le \Phi_\varepsilon^\nu(\boldsymbol{v}; \theta_0^\circ). \tag{4.26}
$$

Now, since (4.1) implies $1 - \frac{h}{2}|g|_{C^2([0,1]^2)^2} \ge \frac{3}{4}$, the energy-inequality (4.20) can be obtained by taking the sum of (4.23)–(4.26) with (A2) in mind. □ □

Lemma 7. *By the restriction $1 \leq N \leq 3$ of the spatial dimension, there exists a positive constant C_2°, such that under the notations and assumptions as in Lemma 6, the condition:*

$$C_2^\circ h^{\frac{1}{3}}(1 + 2(\mathscr{F}_\varepsilon^\gamma(u_0^\circ, \boldsymbol{v}_0^\circ, \theta_0^\circ) + h|f_0^*|_{V^*}^2)^{\frac{2}{3}}) \leq \frac{1}{2}, \text{ and } 0 < h \leq h_2^\circ, \tag{4.27}$$

implies the uniqueness of the solution $[u, \boldsymbol{v}, \theta] = [u, w, \eta, \theta] \in D_M$ to the system {(4.13)–(4.14), (4.19)} *with $\boldsymbol{v} = [w, \eta]$.*

Proof. In the light of the uniqueness of θ as in Lemma 5, it is enough to focus only on the uniqueness for the component $[u, \boldsymbol{v}] = [u, w, \eta] \in V \times H^1(\Omega)^2$ with $\boldsymbol{v} = [w, \eta]$. To this end, we take two triplets $[u_k, \boldsymbol{v}_k] = [u_k, w_k, \eta_k] \in D_M$ with $\boldsymbol{v}_k = [w_k, \eta_k]$, $k = 1, 2$, that fulfill (4.13)–(4.14).

First, with the equivalence of (4.13) and (4.22) in mind, we take the difference between two variational forms (4.22) for u_k, $k = 1, 2$, and put $z = u_1 - u_2$ in V. Then:

$$\begin{aligned}|u_1 - u_2|_{L^2(\Omega)}^2 + h|u_1 - u_2|_V^2 = {}& (\lambda'(w_1)w_1 - \lambda'(w_2)w_2, u_1 - u_2)_{L^2(\Omega)} \\ & -((\lambda'(w_1) - \lambda'(w_2))w_0^\circ, u_1 - u_2)_{L^2(\Omega)},\end{aligned}$$

so that by using (A2) and Schwarz's inequality, we have:

$$\frac{1}{4}|u_1 - u_2|_{L^2(\Omega)}^2 + h|u_1 - u_2|_V^2 \leq 3|\lambda'|_{W^{1,\infty}(0,1)}^2 |w_1 - w_2|_{L^2(\Omega)}^2. \tag{4.28}$$

Secondly, let us take the difference between two inclusions (4.14) for $\boldsymbol{v}_k = [w_k, \eta_k]$, $k = 1, 2$, and multiply the both sides by $\boldsymbol{v}_1 - \boldsymbol{v}_2$ in $L^2(\Omega)^2$. Then, by using (A2)–(A5) and Schwarz's inequality, it is computed that:

$$\begin{aligned}&\frac{1}{h}\left(1 - h|[\nabla g]|_{C^1([0,1]^2)^2}\right)|\boldsymbol{v}_1 - \boldsymbol{v}_2|_{L^2(\Omega)^2}^2 + |D(\boldsymbol{v}_1 - \boldsymbol{v}_2)|_{L^2(\Omega)^{N\times 2}}^2 \\ \leq {}& -(\lambda'(w_1)u_1 - \lambda'(w_2)u_2, w_1 - w_2)_{L^2(\Omega)} \\ \leq {}& |\lambda'|_{L^\infty(0,1)}|u_1 - u_2|_{L^2(\Omega)}|w_1 - w_2|_{L^2(\Omega)} + \left(u_1(\lambda'(w_1) - \lambda'(w_2)), w_1 - w_2\right)_{L^2(\Omega)} \\ \leq {}& \frac{1}{8}|u_1 - u_2|_{L^2(\Omega)}^2 + 2|\lambda'|_{L^\infty(0,1)}^2|w_1 - w_2|_{L^2(\Omega)}^2 + |\lambda''|_{L^\infty(0,1)}\int_\Omega |u_1||w_1 - w_2|^2\, dx.\end{aligned} \tag{4.29}$$

Here, the dimensional restriction $1 \leq N \leq 3$ and the assumption (4.27) enable to apply the analytic technique as in [19, Lemma 3.1], and to find a constant $C_2^\circ > 0$, independent of h and triplets $[u_0^\circ, \boldsymbol{v}_0^\circ]$ and $[u_k, \boldsymbol{v}_k]$, $k = 1, 2$, such that:

$$|\lambda''|_{L^\infty(0,1)}\int_\Omega |u_1||w_1 - w_2|^2 dx \leq \frac{1}{2}|D(w_1 - w_2)|_{L^2(\Omega)}^2 + C_2^\circ(1 + |u_1|_V^{\frac{4}{3}})|w_1 - w_2|_{L^2(\Omega)}^2. \tag{4.30}$$

Furthermore, under (4.27), the inequality (4.20) enables to derive that:

$$\begin{aligned}C_2^\circ(1 + |u_1|_V^{\frac{4}{3}})|w_1 - w_2|_{L^2(\Omega)}^2 &= C_2^\circ h^{\frac{1}{3}}(h^{\frac{2}{3}} + (h|u_1|_V^2)^{\frac{2}{3}}) \cdot \frac{1}{h}|w_1 - w_2|_{L^2(\Omega)}^2 \\ &\leq C_2^\circ h^{\frac{1}{3}}(1 + 2(\mathscr{F}_\varepsilon^\gamma(u_0^\circ, \boldsymbol{v}_0^\circ, \theta_0^\circ) + h|f_0^*|_{V^*}^2)^{\frac{2}{3}}) \cdot \frac{1}{h}|w_1 - w_2|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2h}|w_1 - w_2|_{L^2(\Omega)}^2.\end{aligned} \tag{4.31}$$

Now, taking sum of (4.28)–(4.29) with (4.30)–(4.31) in mind, we obtain that:

$$\begin{aligned}&\frac{1}{8}|u_1-u_2|^2_{L^2(\Omega)}+h|u_1-u_2|^2_V\\&\quad+\frac{1}{h}\left(\frac{1}{2}-h\left(|g|_{C^2([0,1]^2)}+5|\lambda|^2_{W^{2,\infty}(0,1)}\right)\right)|\boldsymbol{v}_1-\boldsymbol{v}_2|^2_{L^2(\Omega)^2}\\&\quad+\frac{1}{2}|D(\boldsymbol{v}_1-\boldsymbol{v}_2)|^2_{L^2(\Omega)^{N\times 2}}\leq 0.\end{aligned}\tag{4.32}$$

This implies the required uniqueness, because $\frac{1}{2}-h(|g|_{C^2([0,1]^2)^2}+5|\lambda|^2_{W^{2,\infty}(0,1)})>0$ follows from the assumption (4.1) and (4.27). □ □

Proof of Proposition 1. Let us take a positive constant h_2° defined by (4.1). Let us set a positive constant h_0°, so small to satisfy that:

$$C_2^\circ(h_0^\circ)^{\frac{1}{3}}(1+2(\mathscr{F}_\varepsilon^\nu(u_0^\circ,\boldsymbol{v}_0^\circ,\theta_0^\circ)+h_0^\circ|[f^*]_0^{\mathrm{ex}}|^2_{L^2(0,T;V^*)})^{\frac{2}{3}})\leq\frac{1}{2},\text{ and }0<h_0^\circ\leq h_2^\circ.$$

Then, from (4.20), it will be observed that:

$$\begin{aligned}&C_2^\circ h^{\frac{1}{3}}(1+2(\mathscr{F}_\varepsilon^\nu(u_{\varepsilon,i-1}^\nu,\boldsymbol{v}_{\varepsilon,i-1}^\nu,\theta_{\varepsilon,i-1}^\nu)+h|[f_i^*]^h|^2_{V^*})^{\frac{2}{3}})\\&\quad\leq C_2^\circ h^{\frac{1}{3}}(1+2(\mathscr{F}_\varepsilon^\nu(u_{\varepsilon,i-2}^\nu,\boldsymbol{v}_{\varepsilon,i-2}^\nu,\theta_{\varepsilon,i-2}^\nu)+h(|[f_i^*]^h|^2_{V^*}+|[f_{i-1}^*]^h|^2_{V^*}))^{\frac{2}{3}})\\&\quad\leq\cdots\leq C_2^\circ h^{\frac{1}{3}}(1+2(\mathscr{F}_\varepsilon^\nu(u_{\varepsilon,0}^\nu,\boldsymbol{v}_{\varepsilon,0}^\nu,\theta_{\varepsilon,0}^\nu)+|[f^*]_0^{\mathrm{ex}}|^2_{L^2(0,T;V^*)})^{\frac{2}{3}})\\&\quad\leq\frac{1}{2},\quad\text{for all }0<h\leq h_0^\circ\ (\leq h_2^\circ)\text{ and }i=1,2,3,\ldots.\end{aligned}\tag{4.33}$$

In view of this, the Proposition 1 will be concluded by means of the following algorithm.

(Step 0) Let $h\in(0,h_0^\circ]$, let $i=1$, and let $[u_{\varepsilon,0}^\nu,\boldsymbol{v}_{\varepsilon,0}^\nu,\theta_{\varepsilon,0}^\nu]\in D_M$.

(Step 1) Obtain the quartet $[u_{\varepsilon,i}^\nu,\boldsymbol{v}_{\varepsilon,i}^\nu,\theta_{\varepsilon,i}^\nu]\in D_M$ as the unique solution to the system {(4.13)–(4.14), (4.19)}, by applying Lemmas 4–7 to the case when:

$$f_0^*=[\boldsymbol{f}_{i-1}^*]^h\text{ in }V^*,\ u_0^\circ=u_{\varepsilon,i-1}^\nu\text{ in }L^2(\Omega),$$
$$\boldsymbol{v}_0^\circ=\boldsymbol{v}_{\varepsilon,i-1}^\nu\text{ in }H^1(\Omega)^2\text{ and }\theta_0^\circ=\theta_{\varepsilon,i-1}^\nu\text{ in }H^M(\Omega).$$

(Step 2) Iterate the value of i and return to (Step 1).

Note that (4.33) let the assumption $h\in(0,h_0^\circ]$ be a uniform condition to make sense (Step 1), for all $i=1,2,3,\ldots$. □

5. Proof of Theorem 1

Let us set $h_1^\circ:=h_0^\circ$ i.e. the constant as in Proposition 1, and let us fix $\nu>0$, $h\in(0,h_1^\circ]$ and the initial value $[u_0^\nu,\boldsymbol{v}_0^\nu,\theta_0^\nu]=[u_0^\nu,w_0^\nu,\eta_0^\nu,\theta_0^\nu]\in D_1$ with $\boldsymbol{v}_0^\nu=[w_0^\nu,\eta_0^\nu]$. Besides, we recall the following lemmas obtained in [31].

Lemma 8. (cf. [31, Lemma 4.1]) *Assume* $\boldsymbol{v}^\circ \in [H^1(\Omega) \cap L^\infty(\Omega)]^2$, $\{\boldsymbol{v}_\varepsilon^\circ\}_{0<\varepsilon\leq 1} \subset [H^1(\Omega) \cap L^\infty(\Omega)]^2$, *and*

$$\begin{cases} \boldsymbol{v}_\varepsilon^\circ \to \boldsymbol{v}^\circ \text{ in the pointwise sense a.e. in } \Omega \text{ as } \varepsilon \downarrow 0, \\ \{\boldsymbol{v}_\varepsilon^\circ\}_{0<\varepsilon\leq 1} \text{ is bounded in } L^\infty(\Omega)^2. \end{cases}$$

Then, for the sequence of convex functions $\{\Phi_\varepsilon^\nu(\boldsymbol{v}_\varepsilon^\circ; \cdot)\}_{0<\varepsilon\leq 1}$, *it holds that* $\Phi_\varepsilon^\nu(\boldsymbol{v}_\varepsilon^\circ; \cdot) \to \Phi_\nu(\boldsymbol{v}^\circ; \cdot)$ *on* $L^2(\Omega)$, *in the sense of Mosco, as* $\varepsilon \downarrow 0$.

Lemma 9. (cf. [31, Lemma 4.2]) *Assume that*

$$\begin{cases} \boldsymbol{v}^\circ \in [H^1(\Omega) \cap L^\infty(\Omega)]^2, \ \{\boldsymbol{v}_\varepsilon^\circ\}_{0<\varepsilon\leq 1} \subset [H^1(\Omega) \cap L^\infty(\Omega)]^2, \\ \{\boldsymbol{v}_\varepsilon^\circ\}_{0<\varepsilon\leq 1} \text{ is bounded in } L^\infty(\Omega)^2, \\ \boldsymbol{v}_\varepsilon^\circ \to \boldsymbol{v}^\circ \text{ in the pointwise sense, a.e. in } \Omega, \text{ as } \varepsilon \downarrow 0, \end{cases}$$

and

$$\begin{cases} \theta^\circ \in H^1(\Omega), \ \{\theta_\varepsilon^\circ\}_{0<\varepsilon\leq 1} \subset H^1(\Omega), \\ \theta_\varepsilon^\circ \to \theta^\circ \text{ in } L^2(\Omega) \text{ and } \Phi_\varepsilon^\nu(\boldsymbol{v}_\varepsilon^\circ; \theta_\varepsilon^\circ) \to \Phi_\nu(\boldsymbol{v}^\circ; \theta^\circ), \text{ as } \varepsilon \downarrow 0. \end{cases}$$

Then, $\theta_\varepsilon^\circ \to \theta^\circ$ *in* $H^1(\Omega)$ *and* $\varepsilon\theta_\varepsilon^\circ \to 0$ *in* $H^M(\Omega)$, *as* $\varepsilon \downarrow 0$.

Lemma 10. (cf. [31, Lemma 4.4]) *Let* $\boldsymbol{v}^\circ \in [H^1(\Omega) \cap L^\infty(\Omega)]^2$ *and* $\check{\theta}_0^\circ, \hat{\theta}_0^\circ \in H^1(\Omega)$ *be fixed functions, and let* $[\check{\theta}, \check{\theta}^*]$, $[\hat{\theta}, \hat{\theta}^*] \in L^2(\Omega)^2$ *be pairs of functions such that*

$$\begin{cases} [\check{\theta}, \check{\theta}^*] \in \partial\Phi_\nu(\boldsymbol{v}^\circ; \cdot) \text{ in } L^2(\Omega)^2 \text{ and } \dfrac{1}{h}\alpha_0(\boldsymbol{v}^\circ)(\check{\theta} - \check{\theta}_0^\circ) + \check{\theta}^* \leq 0 \text{ a.e. in } \Omega, \\ [\hat{\theta}, \hat{\theta}^*] \in \partial\Phi_\nu(\boldsymbol{v}^\circ; \cdot) \text{ in } L^2(\Omega)^2 \text{ and } \dfrac{1}{h}\alpha_0(\boldsymbol{v}^\circ)(\hat{\theta} - \hat{\theta}_0^\circ) + \hat{\theta}^* \geq 0 \text{ a.e. in } \Omega, \end{cases} \tag{5.1}$$

respectively. Then, it follows that

$$|\sqrt{\alpha_0(\boldsymbol{v}^\circ)}[\check{\theta} - \hat{\theta}]^+|_{L^2(\Omega)}^2 \leq |\sqrt{\alpha_0(\boldsymbol{v}^\circ)}[\check{\theta}_0^\circ - \hat{\theta}_0^\circ]^+|_{L^2(\Omega)}^2,$$

and therefore, if $\check{\theta}_0^\circ \leq \hat{\theta}_0^\circ$ *in* Ω, *then the inequality* $\check{\theta} \leq \hat{\theta}$ *a.e. in* Ω *also follows from (A3).*

Moreover, if the both inequalities in (5.1) *hold as equalities, then:*

$$|\sqrt{\alpha_0(\boldsymbol{v}^\circ)}(\check{\theta} - \hat{\theta})|_{L^2(\Omega)}^2 \leq |\sqrt{\alpha_0(\boldsymbol{v}^\circ)}(\check{\theta}_0^\circ - \hat{\theta}_0^\circ)|_{L^2(\Omega)}^2,$$
$$\text{i.e. } \check{\theta}_0^\circ = \hat{\theta}_0^\circ \text{ implies } \check{\theta} = \hat{\theta} \text{ in } L^2(\Omega).$$

Based on these, we divide the proof of Theorem 1 in two parts: the part of existence; the part of uniqueness and energy inequality.

The part of existence. Let $\nu > 0$ be a fixed constant. By Lemma 8, there exists a sequence $\{\tilde{\theta}_{\varepsilon,0}^\nu\}_{0<\varepsilon\leq 1} \subset H^M(\Omega)$ such that

$$\tilde{\theta}_{\varepsilon,0}^\nu \to \theta_0^\nu \text{ in } H^1(\Omega) \text{ and } \Phi_\varepsilon^\nu(\boldsymbol{v}_0^\nu; \tilde{\theta}_{\varepsilon,0}^\nu) \to \Phi_\nu(\boldsymbol{v}_0^\nu; \theta_0^\nu) \text{ as } \varepsilon \downarrow 0.$$

So, by virtue of Proposition 1 we can take a class $\{[\tilde{u}_{\varepsilon,i}^\nu, \tilde{\boldsymbol{v}}_{\varepsilon,i}^\nu, \tilde{\theta}_{\varepsilon,i}^\nu] \,|\, i \in \mathbb{N}, \ \varepsilon \in (0, 1]\}$ consisting of solutions $\{[\tilde{u}_{\varepsilon,i}^\nu, \tilde{\boldsymbol{v}}_{\varepsilon,i}^\nu, \tilde{\theta}_{\varepsilon,i}^\nu]\}_{i=1}^\infty = \{[\tilde{u}_{\varepsilon,i}^\nu, \tilde{w}_{\varepsilon,i}^\nu, \tilde{\eta}_{\varepsilon,i}^\nu, \tilde{\theta}_{\varepsilon,i}^\nu]\}_{i=1}^\infty \subset D_M$ to $(\mathrm{RX})_\varepsilon$ with $\{\tilde{\boldsymbol{v}}_{\varepsilon,i}^\nu\}_{i=1}^\infty = \{[\tilde{w}_{\varepsilon,i}^\nu, \tilde{\eta}_{\varepsilon,i}^\nu]\}_{i=1}^\infty$, starting from the initial data $[u^\nu_{\varepsilon,0}, \boldsymbol{v}^\nu_{\varepsilon,0}, \theta^\nu_{\varepsilon,0}] = [u^\nu_0, \boldsymbol{v}^\nu_0, \tilde{\theta}^\nu_{\varepsilon,0}]$ for $0 < \varepsilon \leq 1$. Then, with Lemma 6 and the algorithm (Step 0)–(Step 2) in mind, we remark the following energy-inequality:

$$\begin{aligned} &\frac{A_*}{2h}|\tilde{u}^\nu_{\varepsilon,i} - \tilde{u}^\nu_{\varepsilon,i-1}|^2_{V^*} + \frac{1}{2h}|\tilde{\boldsymbol{v}}^\nu_{\varepsilon,i} - \tilde{\boldsymbol{v}}^\nu_{\varepsilon,i-1}|^2_{L^2(\Omega)^2} + \frac{1}{h}\left|\sqrt{\alpha_0(\tilde{\boldsymbol{v}}^\nu_{\varepsilon,i})}(\tilde{\theta}^\nu_{\varepsilon,i} - \tilde{\theta}^\nu_{\varepsilon,i-1})\right|^2_{L^2(\Omega)} \\ &\quad + \frac{h}{2}|\tilde{u}^\nu_{\varepsilon,i}|^2_V + \mathscr{F}^\nu_\varepsilon(\tilde{u}^\nu_{\varepsilon,i}, \tilde{\boldsymbol{v}}^\nu_{\varepsilon,i}, \tilde{\theta}^\nu_{\varepsilon,i}) \leq \mathscr{F}^\nu_\varepsilon(\tilde{u}^\nu_{\varepsilon,i-1}, \tilde{\boldsymbol{v}}^\nu_{\varepsilon,i-1}, \tilde{\theta}^\nu_{\varepsilon,i-1}) + h|[\boldsymbol{f}^*_i]^h|^2_{V^*}, \\ &\qquad\qquad \text{for all } 0 < \varepsilon \leq 1 \text{ and } i = 1, 2, 3, \ldots. \end{aligned} \tag{5.2}$$

In the light of (A3)–(A6), (4.21) and (5.2), the class $\{[\tilde{u}^\nu_{\varepsilon,i}, \tilde{\boldsymbol{v}}^\nu_{\varepsilon,i}, \tilde{\theta}^\nu_{\varepsilon,i}] \mid i \in \mathbb{N},\ \varepsilon \in (0,1]\}$ is bounded in $V \times H^1(\Omega)^3$. Therefore, applying a diagonal argument and the general theories of compactness (cf. [3, 11]), we find sequences $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1]$, $\{[u^\nu_i, \boldsymbol{v}^\nu_i, \theta^\nu_i]\}_{i=1}^\infty = \{[u^\nu_i, w^\nu_i, \eta^\nu_i, \theta^\nu_i]\}_{i=1}^\infty \subset V \times H^1(\Omega)^2 \times H^1(\Omega)$, with $\{\boldsymbol{v}^\nu_i\}_{i=1}^\infty = \{[w^\nu_i, \eta^\nu_i]\}_{i=1}^\infty$, such that

$$\begin{cases} 1 \geq \varepsilon_1 > \cdots > \varepsilon_n \downarrow 0 \text{ as } n \to \infty, \\ \tilde{u}^\nu_{i,n} := \tilde{u}^\nu_{\varepsilon_n,i} \to u^\nu_i \text{ in } L^2(\Omega), \text{ weakly in } V \text{ as } n \to \infty, \\ \tilde{\boldsymbol{v}}^\nu_{i,n} := \tilde{\boldsymbol{v}}^\nu_{\varepsilon_n,i} \to \boldsymbol{v}^\nu_i \text{ in } L^2(\Omega)^2, \text{ weakly in } H^1(\Omega)^2, \text{ weakly-}* \text{ in } L^\infty(\Omega)^2, \\ \qquad\qquad \text{and in the pointwise sense a.e. in } \Omega, \text{ as } n \to \infty, \\ \tilde{\theta}^\nu_{i,n} \equiv \tilde{\theta}^\nu_{\varepsilon_n,i} \to \theta^\nu_i \text{ in } L^2(\Omega), \text{ weakly in } H^1(\Omega) \\ \qquad\qquad \text{and in the pointwise sense a.e. in } \Omega, \text{ as } n \to \infty, \\ 0 \leq w^\nu_i \leq 1 \text{ and } 0 \leq \eta^\nu_i \leq 1 \text{ a.e. in } \Omega; \text{ for all } i = 0, 1, 2, \ldots. \end{cases} \tag{5.3}$$

Moreover, by (3.13), (5.3), Lemmas 8–9 and Remark 4 (Fact 6), we infer that

$$\left.\begin{cases} [\theta^\nu_i, -\frac{1}{h}\alpha_0(\boldsymbol{v}^\nu_i)(\theta^\nu_i - \theta^\nu_{i-1})] \in \partial\Phi_\nu(\boldsymbol{v}^\nu_i; \cdot) \text{ in } L^2(\Omega)^2, \\ \Phi^\nu_{\varepsilon_n}(\tilde{\boldsymbol{v}}^\nu_{i,n}; \tilde{\theta}^\nu_{i,n}) \to \Phi_\nu(\boldsymbol{v}^\nu_i; \theta^\nu_i),\ \tilde{\theta}^\nu_{i,n} \to \theta^\nu_i \text{ in } H^1(\Omega) \\ \qquad \text{and } \varepsilon_n\tilde{\theta}^\nu_{i,n} \to 0 \text{ in } H^M(\Omega), \text{ as } n \to \infty, \end{cases}\right. \quad \text{for } i = 0, 1, 2, \ldots. \tag{5.4}$$

Also, since

$$[c, 0] \in \partial\Phi_\nu(\boldsymbol{v}^\nu_i; \cdot) \quad \text{in } L^2(\Omega)^2, \text{ for all } c \in \mathbb{R} \text{ and } i = 0, 1, 2, \ldots,$$

it is observed that

$$\theta^\nu_i \leq |\theta^\nu_{i-1}|_{L^\infty(\Omega)} \text{ (resp. } \theta^\nu_i \geq -|\theta^\nu_{i-1}|_{L^\infty(\Omega)}) \text{ a.e. in } \Omega, \text{ for any } i \in \mathbb{N},$$

by applying Lemma 10 as the case when

$$\begin{cases} \boldsymbol{v}^\circ = \boldsymbol{v}^\nu_i, \\ \check{\theta}^\circ_0 = \theta^\nu_{i-1},\ \hat{\theta}^\circ_0 = |\theta^\nu_{i-1}|_{L^\infty(\Omega)} \text{ (resp. } \check{\theta}^\circ_0 = -|\theta^\nu_{i-1}|_{L^\infty(\Omega)},\ \hat{\theta}^\circ_0 = \theta^\nu_{i-1}), \\ [\check{\theta}, \check{\theta}^*] = [\theta^\nu_i, -\frac{1}{h}\alpha_0(\boldsymbol{v}^\nu_i)(\theta^\nu_i - \theta^\nu_{i-1})] \text{ (resp. } [\check{\theta}, \check{\theta}^*] = [-|\theta^\nu_{i-1}|_{L^\infty(\Omega)}, 0]), \\ [\hat{\theta}, \hat{\theta}^*] = [|\theta^\nu_{i-1}|_{L^\infty(\Omega)}, 0] \left(\text{resp. } [\hat{\theta}, \hat{\theta}^*] = [\theta^\nu_i, -\frac{1}{h}\alpha_0(\boldsymbol{v}^\nu_i)(\theta^\nu_i - \theta^\nu_{i-1})]\right). \end{cases}$$

Having in mind (A2)–(A5), (3.11)–(3.12) and (5.3)–(5.4), we can see that

$$
\begin{aligned}
&\frac{1}{h}(u_i^\nu - u_{i-1}^\nu, z)_{L^2(\Omega)} - \frac{1}{h}(\lambda'(w_i^\nu)(w_i^\nu - w_{i-1}^\nu), z)_{L^2(\Omega)} + (u_i^\nu, z)_V \\
&= \lim_{n\to\infty}\left[\frac{1}{h}(\tilde{u}_{i,n}^\nu - \tilde{u}_{i-1,n}^\nu, z)_{L^2(\Omega)} - \frac{1}{h}(\lambda'(\tilde{w}_{i,n}^\nu)(\tilde{w}_{i,n}^\nu - \tilde{w}_{i-1,n}^\nu), z)_{L^2(\Omega)} + (\tilde{u}_{i,n}^\nu, z)_V\right] \\
&= \langle [\boldsymbol{f}_i^*]^h, z\rangle, \text{ for any } z \in V \text{ and } i = 1, 2, 3, \ldots,
\end{aligned}
$$

and

$$
\begin{aligned}
&(D\boldsymbol{v}_i^\nu, D(\boldsymbol{v}_i^\nu - \boldsymbol{\varpi}))_{L^2(\Omega)^{N\times 2}} + \int_\Omega \gamma(w_i^\nu)\,dx - \int_\Omega \gamma(\varphi)\,dx \\
&\leq \liminf_{n\to\infty}(D\tilde{\boldsymbol{v}}_{i,n}^\nu, D(\tilde{\boldsymbol{v}}_{i,n}^\nu - \boldsymbol{\varpi}))_{L^2(\Omega)^{N\times 2}} + \liminf_{n\to\infty}\int_\Omega \gamma(\tilde{w}_{i,n}^\nu)dx - \int_\Omega \gamma(\varphi)\,dx \\
&\leq \limsup_{n\to\infty}(D\tilde{\boldsymbol{v}}_{i,n}^\nu, D(\tilde{\boldsymbol{v}}_{i,n}^\nu - \boldsymbol{\varpi}))_{L^2(\Omega)^{N\times 2}} + \liminf_{n\to\infty}\int_\Omega \gamma(\tilde{w}_{i,n}^\nu)dx - \int_\Omega \gamma(\varphi)\,dx \\
&\leq -\lim_{n\to\infty}\left(\frac{1}{h}(\tilde{\boldsymbol{v}}_{i,n}^\nu - \tilde{\boldsymbol{v}}_{i-1,n}^\nu, \tilde{\boldsymbol{v}}_{i,n}^\nu - \boldsymbol{\varpi})_{L^2(\Omega)^2} + \int_\Omega [\nabla G](\tilde{u}_{i,n}^\nu; \tilde{\boldsymbol{v}}_{i,n}^\nu)\cdot(\tilde{\boldsymbol{v}}_{i,n}^\nu - \boldsymbol{\varpi})dx\right) \\
&\qquad - \lim_{n\to\infty}\int_\Omega ([\nabla\alpha](\tilde{\boldsymbol{v}}_{i,n}^\nu)|D\tilde{\theta}_{i-1,n}^\nu| + \nu^2[\nabla\beta](\tilde{\boldsymbol{v}}_{i,n}^\nu)|D\tilde{\theta}_{i-1,n}^\nu|^2)\cdot(\tilde{\boldsymbol{v}}_{i,n}^\nu - \boldsymbol{\varpi})dx \\
&\leq -\frac{1}{h}(\boldsymbol{v}_i^\nu - \boldsymbol{v}_{i-1}^\nu, \boldsymbol{v}_i^\nu - \boldsymbol{\varpi})_{L^2(\Omega)^2} - \int_\Omega [\nabla G](u_i^\nu; \boldsymbol{v}_i^\nu)\cdot(\boldsymbol{v}_i^\nu - \boldsymbol{\varpi})dx \\
&\qquad - \int_\Omega ([\nabla\alpha](\boldsymbol{v}_i^\nu)|D\theta_{i-1}^\nu| + \nu^2[\nabla\beta](\boldsymbol{v}_i^\nu)|D\theta_{i-1}^\nu|^2)\cdot(\boldsymbol{v}_i^\nu - \boldsymbol{\varpi})dx, \\
&\quad \text{for any } \boldsymbol{\varpi} = [\varphi, \psi] \in [H^1(\Omega)\cap L^\infty(\Omega)]^2, \text{ and } i = 1, 2, 3, \ldots.
\end{aligned}
\tag{5.5}
$$

The above calculations imply that the limiting sequence $\{[u_i^\nu, \boldsymbol{v}_i^\nu, \theta_i^\nu]\}_{i=1}^\infty$ is a solution to the approximating system $(\mathrm{AP})_h^\nu$. □

The part of uniqueness and energy inequality. By putting $\boldsymbol{\varpi} = \boldsymbol{v}_i^\nu$ in (5.5), for $i \in \mathbb{N}$, one can see from (5.3) that:

$$
\begin{aligned}
&|D\boldsymbol{v}_i^\nu|_{L^2(\Omega)^{N\times 2}}^2 \leq \liminf_{n\to\infty}|D\tilde{\boldsymbol{v}}_{i,n}^\nu|_{L^2(\Omega)^{N\times 2}}^2 \leq \limsup_{n\to\infty}|D\tilde{\boldsymbol{v}}_{i,n}^\nu|_{L^2(\Omega)^{N\times 2}}^2 \\
\leq\ &\lim_{n\to\infty}(D\tilde{\boldsymbol{v}}_{i,n}^\nu, D\boldsymbol{v}_i^\nu)_{L^2(\Omega)^{N\times 2}} + \int_\Omega \gamma(w_i^\nu)\,dx - \liminf_{n\to\infty}\int_\Omega \gamma(\tilde{w}_{i,n}^\nu)\,dx \\
\leq\ &|D\boldsymbol{v}_i^\nu|_{L^2(\Omega)^{N\times 2}}^2, \text{ for } i = 1, 2, 3, \ldots.
\end{aligned}
\tag{5.6}
$$

By the convergences (5.3) and (5.6), the uniform convexity of L^2-based topologies enable to say:

$$
\tilde{\boldsymbol{v}}_{i,n}^\nu \to \boldsymbol{v}_i^\nu \text{ in } H^1(\Omega)^2 \text{ as } n \to \infty, \text{ for } i = 1, 2, 3, \ldots. \tag{5.7}
$$

Hence, the energy-inequality (3.10) will be obtained, immediately, by putting $\varepsilon = \varepsilon_n$ in (5.2), for $n \in \mathbb{N}$, and letting $n \to \infty$ with (5.3) and (5.7) in mind.

In the meantime, we note that the condition (4.33) is still available in the proof of Theorem 1. Also, the regularity $\theta_0^\circ \in H^M(\Omega)$ will not necessary in the calculations (4.29)–(4.32), and the line of these calculations will work even if $\theta_0^\circ \in H^1(\Omega)$.

In view of these, the verification part of the uniqueness for $(\mathrm{AP})_h^\nu$ will be a slight modification of that as in (Step 1) in the previous section. Then, the principal modifications will be to replace the application parts of Lemma 5 and the energy-inequality (4.20), by those of Lemma 10 and (3.10), respectively. $\square$

6. Proof of Main Theorem

Let $\nu \geq 0$ be a fixed constant, and let $h_1^\circ \in (0, 1]$ be the constant as in Theorem 1. Also, we refer to [31] to recall the following lemma.

Lemma 11. (Γ-convergence; [31, Lemma 6.2]) *Assume $\boldsymbol{v}^\bullet \in [H^1(\Omega) \cap L^\infty(\Omega)]^2$, $\{\boldsymbol{v}_{\tilde{\nu}}^\bullet\}_{\tilde{\nu}>0} \subset [H^1(\Omega) \cap L^\infty(\Omega)]^2$, and*

$$\begin{cases} \boldsymbol{v}_{\tilde{\nu}}^\bullet \to \boldsymbol{v}^\bullet \text{ in the pointwise sense, a.e. in } \Omega, \text{ as } \tilde{\nu} \downarrow 0, \\ \{\boldsymbol{v}_{\tilde{\nu}}^\bullet\}_{\tilde{\nu}>0} \text{ is bounded in } L^\infty(\Omega)^2. \end{cases}$$

Then, for the sequence of convex functions $\{\Phi_{\tilde{\nu}}(\boldsymbol{v}_{\tilde{\nu}}^\bullet; \cdot\,)\}_{\tilde{\nu}>0}$, it holds that $\Phi_{\tilde{\nu}}(\boldsymbol{v}_{\tilde{\nu}}^\bullet; \cdot\,) \to \Phi_0(\boldsymbol{v}^\bullet; \cdot\,)$ on $L^2(\Omega)$, in the sense of Γ-convergence, as $\tilde{\nu} \downarrow 0$.

Based on Lemma 11 and [31, Remark 6.1], we take a sequence $\{\vartheta_0^{\tilde{\nu}}\}_{\tilde{\nu}>0} \subset H^1(\Omega)$, such that:

$$|\theta_0^{\tilde{\nu}}| \leq |\theta_0|_{L^\infty(\Omega)} \text{ a.e. in } \Omega, \text{ for any } \tilde{\nu} > 0,$$

and

$$\begin{cases} \vartheta_0^{\tilde{\nu}} \to \theta_0 \text{ in } L^2(\Omega) \text{ and } \Phi_{\tilde{\nu}}(\boldsymbol{v}_0; \vartheta_0^{\tilde{\nu}}) \to \Phi_0(\boldsymbol{v}_0; \theta_0), \text{ as } \tilde{\nu} \downarrow 0, \text{ if } \nu = 0, \\ \vartheta_0^{\tilde{\nu}} = \theta_0 \text{ in } H^1(\Omega) \text{ for } \tilde{\nu} > 0, \text{ if } \nu > 0, \end{cases}$$

and for any $h \in (0, h_1^\circ]$ and any $\tilde{\nu} \in (0, \nu + 1]$, let us take the solution $\{[u_i^{\tilde{\nu}}, \boldsymbol{v}_i^{\tilde{\nu}}, \theta_i^{\tilde{\nu}}]\}_{i=0}^\infty$ to $(\mathrm{AP})_h^{\tilde{\nu}}$ with $\{\boldsymbol{v}_i^{\tilde{\nu}}\}_{i=1}^\infty = \{[w_i^{\tilde{\nu}}, \eta_i^{\tilde{\nu}}]\}_{i=1}^\infty$, under the initial condition $[u_0^{\tilde{\nu}}, \boldsymbol{v}_0^{\tilde{\nu}}, \theta_0^{\tilde{\nu}}] = [u_0, \boldsymbol{v}_0, \vartheta_0^{\tilde{\nu}}] \in D_1$ with $\boldsymbol{v}_0^{\tilde{\nu}} = [w_0^{\tilde{\nu}}, \eta_0^{\tilde{\nu}}] = [w_0, \eta_0]$. As is easily seen,

$$F_0^\nu := \sup_{0<\tilde{\nu}\leq\nu+1} \mathscr{F}_{\tilde{\nu}}(u_0, \boldsymbol{v}_0, \vartheta_0^{\tilde{\nu}}) < \infty.$$

For any $h \in (0, h_1^\circ]$ and any $\tilde{\nu} \in (0, \nu + 1]$, we define the following time-interpolations:

$$\boldsymbol{f}_h^*(t) = [f_h(t), f_{\Gamma,h}(t)] := [\boldsymbol{f}_i^*]^h = [f_i^h, f_{\Gamma,i}^h] \text{ in } V^* \text{ (in } L^2(\Omega) \times L^2(\Gamma)),$$
$$\text{for all } t \geq 0 \text{ and } 0 \leq i \in \mathbb{Z} \text{ satisfying } t \in ((i-1)h, ih],$$

and

$$\begin{cases} [\overline{u}_h^{\tilde{\nu}}(t), \overline{\boldsymbol{v}}_h^{\tilde{\nu}}(t), \overline{\theta}_h^{\tilde{\nu}}(t)] = [\overline{u}_h^{\tilde{\nu}}(t), \overline{w}_h^{\tilde{\nu}}(t), \overline{\eta}_h^{\tilde{\nu}}(t), \overline{\theta}_h^{\tilde{\nu}}(t)] := [u_i^{\tilde{\nu}}, \boldsymbol{v}_i^{\tilde{\nu}}, \theta_i^{\tilde{\nu}}] \text{ in } L^2(\Omega)^4, \\ \qquad \text{for all } t \geq 0 \text{ and } 0 \leq i \in \mathbb{Z} \text{ satisfying } t \in ((i-1)h, ih], \\ [\underline{u}_h^{\tilde{\nu}}(t), \underline{\boldsymbol{v}}_h^{\tilde{\nu}}(t), \underline{\theta}_h^{\tilde{\nu}}(t)] = [\underline{u}_h^{\tilde{\nu}}(t), \underline{w}_h^{\tilde{\nu}}(t), \underline{\eta}_h^{\tilde{\nu}}(t), \underline{\theta}_h^{\tilde{\nu}}(t)] := [u_{i-1}^{\tilde{\nu}}, \boldsymbol{v}_{i-1}^{\tilde{\nu}}, \theta_{i-1}^{\tilde{\nu}}] \text{ in } L^2(\Omega)^4, \\ \qquad \text{for all } t \geq 0 \text{ and } 0 \leq i \in \mathbb{Z} \text{ satisfying } t \in [(i-1)h, ih), \\ [\widehat{u}_h^{\tilde{\nu}}(t), \widehat{\boldsymbol{v}}_h^{\tilde{\nu}}(t), \widehat{\theta}_h^{\tilde{\nu}}(t)] = [\widehat{u}_h^{\tilde{\nu}}(t), \widehat{w}_h^{\tilde{\nu}}(t), \widehat{\eta}_h^{\tilde{\nu}}(t), \widehat{\theta}_h^{\tilde{\nu}}(t)] \\ \quad := \dfrac{ih - t}{h}[u_{i-1}^{\tilde{\nu}}(t), \boldsymbol{v}_{i-1}^{\tilde{\nu}}(t), \theta_{i-1}^{\tilde{\nu}}(t)] + \dfrac{t - (i-1)h}{h}[u_i^{\tilde{\nu}}, \boldsymbol{v}_i^{\tilde{\nu}}, \theta_i^{\tilde{\nu}}] \text{ in } L^2(\Omega)^4, \\ \qquad \text{for all } t \geq 0 \text{ and } 0 \leq i \in \mathbb{Z} \text{ satisfying } t \in [(i-1)h, ih). \end{cases} \tag{6.1}$$

Besides, we define:

$$D_\nu(\theta_0) := \begin{cases} \left\{ [\tilde{u}, \tilde{\boldsymbol{v}}, \tilde{\theta}] \in D_0 \;\middle|\; |\tilde{\theta}|_{L^\infty(\Omega)} \leq |\theta_0|_{L^\infty(\Omega)} \right\}, \text{ if } \nu = 0, \\ \left\{ [\tilde{u}, \tilde{\boldsymbol{v}}, \tilde{\theta}] \in D_1 \;\middle|\; |\tilde{\theta}|_{L^\infty(\Omega)} \leq |\theta_0|_{L^\infty(\Omega)} \right\}, \text{ if } \nu > 0, \end{cases}$$

and we note that:

$$\begin{gathered} \{[\overline{u}_h^{\tilde{\nu}}(t), \overline{\boldsymbol{v}}_h^{\tilde{\nu}}(t), \overline{\theta}_h^{\tilde{\nu}}(t)], [\underline{u}_h^{\tilde{\nu}}(t), \underline{\boldsymbol{v}}_h^{\tilde{\nu}}(t), \underline{\theta}_h^{\tilde{\nu}}(t)], [\widehat{u}_h^{\tilde{\nu}}(t), \widehat{\boldsymbol{v}}_h^{\tilde{\nu}}(t), \widehat{\theta}_h^{\tilde{\nu}}(t)]\} \\ \subset D_\nu(\theta_0), \quad \text{for all } t \geq 0,\ 0 < h \leq h_1^\circ \text{ and } 0 < \tilde{\nu} \leq \nu + 1. \end{gathered}$$

Then, from the energy-inequality (3.10) in Theorem 1, it is checked that

$$\begin{gathered} \frac{A_*}{2}\int_s^t |(\widehat{u}_h^{\tilde{\nu}})_t|_{V^*}\, d\tau + \frac{1}{2}\int_s^t |(\widehat{\boldsymbol{v}}_h^{\tilde{\nu}})_t(\tau)|_{L^2(\Omega)^2}^2 d\tau + \int_s^t \left| \sqrt{\alpha_0(\overline{\boldsymbol{v}}_h^{\tilde{\nu}})}(\widehat{\theta}_h^{\tilde{\nu}})_t(\tau) \right|_{L^2(\Omega)}^2 d\tau \\ + \frac{1}{2}\int_s^t |\overline{u}_h^{\tilde{\nu}}(\tau)|_V^2 d\tau + \mathscr{F}_{\tilde{\nu}}(\overline{u}_h^{\tilde{\nu}}, \overline{\boldsymbol{v}}_h^{\tilde{\nu}}, \overline{\theta}_h^{\tilde{\nu}})(t) \leq \mathscr{F}_{\tilde{\nu}}(\underline{u}_h^{\tilde{\nu}}, \underline{\boldsymbol{v}}_h^{\tilde{\nu}}, \underline{\theta}_h^{\tilde{\nu}})(s) + \int_s^t |\boldsymbol{f}_h^*(\tau)|_{V^*}^2 d\tau \\ \text{for all } 0 \leq s \leq t \leq T,\ 0 < h \leq h_1^\circ \text{ and } 0 < \tilde{\nu} \leq \nu + 1, \end{gathered}$$

and additionally, from (A1)–(A6) and (3.2), it follows that

$$\begin{aligned} & B_*|\overline{u}_h^{\tilde{\nu}}(t)|_{L^2(\Omega)}^2 + \frac{1}{2}|D\overline{\boldsymbol{v}}_h^{\tilde{\nu}}(t)|_{L^2(\Omega)^{N\times 2}}^2 + \delta_*(|D\overline{\theta}_h^{\tilde{\nu}}(t)|(\Omega) + |D(\tilde{\nu}\overline{\theta}_h^{\tilde{\nu}})(t)|_{L^2(\Omega)^{N\times 2}}^2) \\ & \leq \mathscr{F}_{\tilde{\nu}}(\overline{u}_h^{\tilde{\nu}}, \overline{\boldsymbol{v}}_h^{\tilde{\nu}}, \overline{\theta}_h^{\tilde{\nu}})(t) \vee \mathscr{F}_{\tilde{\nu}}(\underline{u}_h^{\tilde{\nu}}, \underline{\boldsymbol{v}}_h^{\tilde{\nu}}, \underline{\theta}_h^{\tilde{\nu}})(t) \\ & \leq F_*^\nu := F_0^\nu + |\boldsymbol{f}^*|_{L^2(0,T;V^*)}^2, \quad \text{for all } 0 \leq t \leq T \text{ and } 0 < \tilde{\nu} \leq \nu + 1. \end{aligned} \tag{6.2}$$

Based on these, one can see that:

(♯1) the class $\{\widehat{u}_h^{\tilde{\nu}} \,|\, h \in (0, h_1^\circ],\ \tilde{\nu} \in (0, \nu + 1]\}$ is bounded in the space $W^{1,2}(0, T; V^*) \cap C([0, T]; L^2(\Omega)) \cap L^2(0, T; V)$.

(♯2) the class $\{\widehat{\boldsymbol{v}}_h^{\tilde{\nu}} \,|\, h \in (0, h_1^\circ],\ \tilde{\nu} \in (0, \nu + 1]\}$ is bounded in the space $W^{1,2}(0, T; L^2(\Omega)^2) \cap L^\infty(0, T; H^1(\Omega)^2) \cap L^\infty(Q)^2$.

($\sharp$3) the class $\{\widehat{\theta}_h^{\tilde{\nu}} \,|\, h \in (0, h_1^\circ],\ \tilde{\nu} \in (0, \nu + 1]\}$ is bounded in the space $W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(Q)$, and $\{\Phi_{\tilde{\nu}}(\boldsymbol{v}_h^{\tilde{\nu}}; \theta_h^{\tilde{\nu}}) \,|\, h \in (0, h_1^\circ],\ \tilde{\nu} \in (0, \nu + 1]\}$ is bounded in $L^\infty(0, T)$, i.e. $\{|D\overline{\theta}_h^{\tilde{\nu}}(\,\cdot\,)|(\Omega) \,|\, h \in (0, h_1^\circ],\ \tilde{\nu} \in (0, \nu + 1]\}$ is bounded in $L^\infty(0, T)$, and $\{D(\tilde{\nu}\overline{\theta}_h^{\tilde{\nu}}) \,|\, h \in (0, h_1^\circ],\ \tilde{\nu} \in (0, \nu + 1]\}$ is bounded in $L^\infty(0, T; L^2(\Omega)^N)$.

Hence, by applying the general theories of compactness, as in [2, 3, 11, 33], we find a quartet of functions $[u, \boldsymbol{v}, \theta] = [u, w, \eta, \theta] \in L^2(0, T; L^2(\Omega)^4)$ with $\boldsymbol{v} = [w, \eta]$ and sequences $\{h_n\}_{n=1}^\infty \subset (0, h_1^\circ]$ and $\{\nu_n\}_{n=1}^\infty \subset (0, \nu + 1]$, with the subsequences:

$$\begin{cases} \{[\overline{u}_n, \overline{\boldsymbol{v}}_n, \overline{\theta}_n]\}_{n=1}^\infty = \{[\overline{u}_n, \overline{w}_n, \overline{\eta}_n, \overline{\theta}_n]\}_{n=1}^\infty := \{[\overline{u}_{h_n}^{\nu_n}, \overline{\boldsymbol{v}}_{h_n}^{\nu_n}, \overline{\theta}_{h_n}^{\nu_n}]\}_{n=1}^\infty, \\[1ex] \{[\underline{u}_n, \underline{\boldsymbol{v}}_n, \underline{\theta}_n]\}_{n=1}^\infty = \{[\underline{u}_n, \underline{w}_n, \underline{\eta}_n, \underline{\theta}_n]\}_{n=1}^\infty := \{[\underline{u}_{h_n}^{\nu_n}, \underline{\boldsymbol{v}}_{h_n}^{\nu_n}, \underline{\theta}_{h_n}^{\nu_n}]\}_{n=1}^\infty, \\[1ex] \{[\widehat{u}_n, \widehat{\boldsymbol{v}}_n, \widehat{\theta}_n]\}_{n=1}^\infty = \{[\widehat{u}_n, \widehat{w}_n, \widehat{\eta}_n, \widehat{\theta}_n]\}_{n=1}^\infty := \{[\widehat{u}_{h_n}^{\nu_n}, \widehat{\boldsymbol{v}}_{h_n}^{\nu_n}, \widehat{\theta}_{h_n}^{\nu_n}]\}_{n=1}^\infty, \end{cases}$$

such that:

$$h_1^\circ \geq h_1 > h_2 > \cdots > h_n \downarrow 0 \text{ and } \nu_n \to \nu, \text{ as } n \to \infty, \tag{6.3}$$

$$\begin{cases} u \in W^{1,2}(0, T; V^*) \cap L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V) \subset C([0, T]; L^2(\Omega)), \\[1ex] \boldsymbol{v} \in W^{1,2}(0, T; L^2(\Omega)^2) \cap L^\infty(0, T; H^1(\Omega)^2) \cap L^\infty(Q)^2, \\[1ex] \theta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(Q),\ \Phi_\nu(\boldsymbol{v}; \theta) \in L^\infty(0, T), \\[1ex] [u(t), \boldsymbol{v}(t), \theta(t)] \in D_\nu(\theta_0) \text{ for all } t \geq 0, \\[1ex] [u(0), \boldsymbol{v}(0), \theta(0)] = [u_0, \boldsymbol{v}_0, \theta_0] \text{ in } L^2(\Omega)^4, \end{cases} \tag{6.4}$$

$$\begin{cases} \widehat{u}_n \to u \text{ in } L^2(I; L^2(\Omega)),\ \text{weakly in } W^{1,2}(I; V^*), \\ \qquad \text{weakly-}* \text{ in } L^\infty(I; V), \\[1ex] \widehat{\boldsymbol{v}}_n \to \boldsymbol{v} \text{ in } C(\overline{I}; L^2(\Omega)^2), \text{ weakly in } W^{1,2}(I; L^2(\Omega)^2), \\ \qquad \text{weakly-}* \text{ in } L^\infty(I; H^1(\Omega)^2) \text{ and weakly-}* \text{ in } L^\infty(Q)^2, \\[1ex] \widehat{\theta}_n \to \theta \text{ in } C(\overline{I}; L^2(\Omega)), \text{ weakly in } W^{1,2}(I; L^2(\Omega)), \\ \qquad \text{weakly-}* \text{ in } L^\infty(Q), \\[1ex] \nu_n\widehat{\theta}_n \to \nu\theta \text{ weakly in } L^2(I; H^1(\Omega)), \end{cases} \tag{6.5}$$

$$\boldsymbol{f}_{h_n}^* \to \boldsymbol{f}^* \text{ in } L^2(I; V^*) \quad ([f_{h_n}, f_{\Gamma,h_n}] \to [f, f_\Gamma] \text{ in } L^2(I; L^2(\Omega) \times L^2(\Gamma))), \tag{6.6}$$

as $n \to \infty$, for any open interval $I \subset (0, T)$, and

$$\begin{cases} \overline{u}_n(t) \to u(t) \text{ and } \underline{u}_n(t) \to u(t) \text{ in } L^2(\Omega), \text{ weakly in } V, \\[1ex] \overline{\boldsymbol{v}}_n(t) \to \boldsymbol{v}(t) \text{ and } \underline{\boldsymbol{v}}_n(t) \to \boldsymbol{v}(t) \text{ in } L^2(\Omega)^2, \text{ weakly in } H^1(\Omega)^2 \\ \qquad \text{and weakly-}* \text{ in } L^\infty(\Omega)^2, \\[1ex] \overline{\theta}_n(t) \to \theta(t) \text{ in } L^2(\Omega), \text{ weakly-}* \text{ in } BV(\Omega), \\[1ex] \nu_n\overline{\theta}_n(t) \to \nu\theta(t) \text{ weakly in } H^1(\Omega), \end{cases} \tag{6.7}$$

as $n \to \infty$ for a.e. $t \in (0, T)$.

Now, we recall some lemmas which will act key-roles in the proof of Main Theorem.

Lemma 12. *Let $I \subset (0,T)$ be an open interval, and let $\nu \geq 0$ and $\{\nu_n\}_{n=1}^\infty$ be the sequence as in* (6.3). *Let $\zeta \in L^2(I;L^2(\Omega))$ be a function such that*

$$|D\zeta(\,\cdot\,)|(\Omega) \in L^1(I) \ \text{ and } \ \nu\zeta \in L^2(I;H^1(\Omega)).$$

Then, there exists a sequence $\{\tilde{\zeta}_n\}_{n=1}^\infty \subset C^\infty(\overline{Q})$, such that:

$$\tilde{\zeta}_n \to \zeta \text{ in } L^2(I;L^2(\Omega)), \ \int_I \left| \int_\Omega |D\tilde{\zeta}_n(t)|\,dx - \int_\Omega d|D\zeta(t)| \right| dt \to 0,$$
$$\text{and } \ \nu_n\tilde{\zeta}_n \to \nu\zeta \text{ in } L^2(I;H^1(\Omega)), \ \text{ as } n \to \infty.$$

Proof. When $\nu > 0$, the standard C^∞-approximation in $L^2(I;H^1(\Omega))$ will correspond to the required sequence. Meanwhile, when $\nu = 0$, this lemma is verified by taking the C^∞-approximation as in [25, Lemma 5] and [29, Remark 2], and by applying the diagonal argument as in [25, Lemma 8]. □ □

Lemma 13. *Let $I \subset (0,T)$ be any open interval. Assume that*

$$\begin{cases} \varrho \in C(\overline{I};L^2(\Omega)) \cap L^\infty(I;H^1(\Omega)), \ \log\varrho \in L^\infty(I\times\Omega), \\ \varrho_n \in C(\overline{I};L^2(\Omega)) \cap L^\infty(I;H^1(\Omega)), \ \log\varrho_n \in L^\infty(I\times\Omega), \text{ for } n = 1,2,3,\ldots, \\ \varrho_n(t) \to \varrho(t) \text{ in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } n \to \infty, \text{ for a.e. } t \in I, \end{cases}$$

and

$$\begin{cases} \zeta \in L^2(I;L^2(\Omega)) \text{ with } |D\zeta(\,\cdot\,)|(\Omega) \in L^1(I), \\ \{\zeta_n\}_{n=1}^\infty \subset L^2(I;L^2(\Omega)) \text{ with } \{|D\zeta_n(\,\cdot\,)|(\Omega)\}_{n=1}^\infty \subset L^1(I), \\ \zeta_n(t) \to \zeta(t) \text{ in } L^2(\Omega) \text{ as } n \to \infty, \ \text{ a.e. } t \in I. \end{cases}$$

Then the following items hold.

(I) The functions:

$$t \in I \mapsto \int_\Omega d[\varrho(t)|D\zeta(t)|]\,dt \text{ and } t \in I \mapsto \int_\Omega d[\varrho_n(t)|D\zeta_n(t)|]\,dt, \text{ for } n = 1,2,3,\ldots,$$

are integrable, and

$$\liminf_{n\to\infty} \int_I \int_\Omega d[\varrho_n(t)|D\zeta_n(t)|]\,dt \geq \int_I \int_\Omega d[\varrho(t)|D\zeta(t)|]\,dt.$$

(II) If:

$$\int_I \int_\Omega d[\varrho_n(t)|D\zeta_n(t)|]\,dt \to \int_I \int_\Omega d[\varrho(t)|D\zeta(t)|]\,dt \text{ as } n \to \infty$$

and

$$\begin{cases} \omega \in L^\infty(I;H^1(\Omega)) \cap L^\infty(I\times\Omega), \ \{\omega_n\}_{n=1}^\infty \subset L^\infty(I;H^1(\Omega)) \cap L^\infty(I\times\Omega), \\ \{\omega_n\}_{n=1}^\infty \text{ is a bounded sequence in } L^\infty(I\times\Omega), \\ \omega_n(t) \to \omega(t) \text{ in } L^2(\Omega) \text{ and weakly in } H^1(\Omega) \text{ as } n \to \infty, \text{ a.e. } t \in I, \end{cases}$$

then

$$\int_I \int_\Omega \omega_n(t)|D\zeta_n(t)|\,dx\,dt \to \int_I \int_\Omega d[\omega(t)|D\zeta(t)|] \text{ as } n \to \infty.$$

Proof. This lemma is verified, immediately, as a consequence of [26, Lemmas 4.2–4.4] (see also [25, Section 2]). □ □

Proof of Main Theorem. We show that the quartet $[u, \boldsymbol{v}, \theta] = [u, w, \eta, \theta] \in L^2(0, T; L^2(\Omega)^4)$ as in (6.4) fulfills the conditions (S1)–(S6) in Definition 3. Then, since (6.4) directly guarantees the conditions (S1)–(S3), we focus on the verifications of remaining (S4)–(S6).

To this end, let us fix arbitrary open interval $I \subset (0, T)$, and let us review (3.7)–(3.9) and (6.1), to check that:

$$\begin{aligned}\int_I \langle (\widehat{u}_n)_t(t), z\rangle\, dt + \int_I (\overline{u}_n(t), z)_V\, dt &= \int_I (\lambda'(\overline{w}_n(t))(\widehat{w}_n)_t(t), z)_{L^2(\Omega)}\, dt \\ &\quad + \int_I \langle \boldsymbol{f}^*_{h_n}(t), z\rangle\, dt, \quad \text{for any } z \in V \text{ and } n = 1, 2, 3, \ldots,\end{aligned} \tag{6.8}$$

$$\begin{aligned}&\int_I ((\widehat{\boldsymbol{v}}_n)_t(t), \overline{\boldsymbol{v}}_n(t) - \boldsymbol{\varpi})_{L^2(\Omega)^2} dt \\ &\quad + \int_I (D\overline{\boldsymbol{v}}_n(t), D(\overline{\boldsymbol{v}}_n(t) - \boldsymbol{\varpi}))_{L^2(\Omega)^{N\times 2}}\, dt \\ &\quad + \int_I ([\nabla G](\overline{u}_n; \overline{\boldsymbol{v}}_n)(t), \overline{\boldsymbol{v}}_n(t) - \boldsymbol{\varpi})_{L^2(\Omega)^2} dt \\ &\quad + \int_I \int_\Omega [\nabla\alpha](\overline{\boldsymbol{v}}_n(t)) \cdot (\overline{\boldsymbol{v}}_n(t) - \boldsymbol{\varpi}) |D\underline{\theta}_n(t)| dx dt \\ &\quad + \nu_n^2 \int_I \int_\Omega [\nabla\beta](\overline{\boldsymbol{v}}_n(t)) \cdot (\overline{\boldsymbol{v}}_n(t) - \boldsymbol{\varpi}) |D\underline{\theta}_n(t)|^2 dx dt \\ &\quad + \int_I \int_\Omega \gamma(\overline{w}_n(t)) dx dt \leq \int_I \int_\Omega \gamma(\varphi) dx dt, \\ &\qquad \text{for any } \boldsymbol{\varpi} = [\varphi, \psi] \in [H^1(\Omega) \cap L^\infty(\Omega)]^2 \text{ and } n = 1, 2, 3, \ldots,\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}&\int_I (\alpha_0(\overline{\boldsymbol{v}}_n(t))(\widehat{\theta}_n)_t(t), \overline{\theta}_n(t) - \zeta(t))_{L^2(\Omega)} dt \\ &\qquad + \int_I \int_\Omega \alpha(\overline{\boldsymbol{v}}_n(t)) |D\overline{\theta}_n(t)|\, dx dt + \nu_n^2 \int_I \int_\Omega \beta(\overline{\boldsymbol{v}}_n(t)) |D\overline{\theta}_n(t)|^2\, dx dt \\ &\leq \int_I \int_\Omega \alpha(\overline{\boldsymbol{v}}_n(t)) |D\zeta(t)|\, dx dt + \nu_n^2 \int_I \int_\Omega \beta(\overline{\boldsymbol{v}}_n(t)) |D\zeta(t)|^2\, dx dt \\ &\qquad \text{for any } \zeta \in L^2(I; H^1(\Omega)) \text{ and } n = 1, 2, 3, \ldots.\end{aligned} \tag{6.10}$$

Now, let us first take the limit of (6.10) as $n \to \infty$. Then, from (A3), (♯2)–(♯3), (6.4)–(6.5), (6.7) and Lemma 13 (I), it is seen that

$$\begin{aligned}
&\int_I (\alpha_0(\boldsymbol{v}(t))\theta_t(t), \theta(t) - \zeta(t))_{L^2(\Omega)} dt + \int_I \Phi_\nu(\boldsymbol{v}(t); \theta(t))\, dt \\
&\quad \leq \lim_{n\to\infty} \int_I (\alpha_0(\overline{\boldsymbol{v}}_n)(\widehat{\theta}_n)_t(t), \overline{\theta}_n(t) - \zeta(t))_{L^2(\Omega)}\, dt \\
&\qquad + \liminf_{n\to\infty} \left[\int_I \int_\Omega \alpha(\overline{\boldsymbol{v}}_n(t))|D\overline{\theta}_n(t)|\, dxdt + \int_I \int_\Omega \beta(\overline{\boldsymbol{v}}_n(t))|D(\nu_n\overline{\theta}_n)(t)|^2\, dxdt \right] \\
&\quad \leq \lim_{n\to\infty} \left[\int_I \int_\Omega \alpha(\overline{\boldsymbol{v}}_n(t))|D\zeta(t)|\, dxdt + \int_I \int_\Omega \beta(\overline{\boldsymbol{v}}_n(t))|D(\nu_n\zeta)(t)|^2\, dxdt \right] \\
&\quad = \int_I \Phi_\nu(\boldsymbol{v}(t); \zeta(t))\, dt, \text{ for any } \zeta \in L^2(I; H^1(\Omega)).
\end{aligned}$$

Since the open interval $I \subset (0, T)$ is arbitrary, the above inequality implies that

$$\begin{gathered}
(\alpha_0(\boldsymbol{v}(t))\theta_t(t), \theta(t) - \omega)_{L^2(\Omega)} + \Phi_\nu(\boldsymbol{v}(t); \theta(t)) \leq \Phi_\nu(\boldsymbol{v}(t); \omega) \\
\text{for any } \omega \in H^1(\Omega) \text{ and a.e. } t \in (0, T).
\end{gathered}$$

Additionally, in the light of Remark 3 (Fact 4), we can say the above inequality holds for $\omega \in BV(\Omega) \cap L^2(\Omega)$. Thus, (S6) is verified.

Next, with (6.4) and Lemma 12 in mind, let us take a sequence $\{\tilde{\theta}_n\}_{n=1}^\infty \subset C^\infty(\overline{I \times \Omega})$ such that

$$\begin{gathered}
\tilde{\theta}_n \to \theta \quad \text{in } L^2(I; L^2(\Omega)), \quad \int_I |D\tilde{\theta}_n|\, dxdt \to \int_I d|D\theta(t)|\, dt, \\
\nu_n\tilde{\theta}_n \to \nu\theta \text{ in } L^2(I; H^1(\Omega)), \text{ as } n \to \infty.
\end{gathered}$$

Then, putting $\zeta = \tilde{\theta}_n$ in (6.10) and letting $n \to \infty$, it is observed from (♯2)–(♯3), (6.4)–(6.5), (6.7) and Lemma 13 that:

$$\begin{aligned}
&\int_I \int_\Omega d[\alpha(\boldsymbol{v}(t))|D\theta(t)|]\, dt + \int_I \int_\Omega \beta(\boldsymbol{v}(t))|D(\nu\theta)(t)|^2\, dxdt \\
&\quad \leq \liminf_{n\to\infty} \int_I \int_\Omega \alpha(\overline{\boldsymbol{v}}_n(t))|D\overline{\theta}_n(t)|\, dxdt + \liminf_{n\to\infty} \int_I \int_\Omega \beta(\overline{\boldsymbol{v}}_n(t))|D(\nu_n\overline{\theta}_n)(t)|^2\, dxdt \\
&\quad \leq \limsup_{n\to\infty} \left[\int_I \int_\Omega \alpha(\overline{\boldsymbol{v}}_n(t))|D\overline{\theta}_n(t)| dx\, dt + \int_I \int_\Omega \beta(\overline{\boldsymbol{v}}_n(t))|D(\nu_n\overline{\theta}_n)(t)|^2\, dxdt \right] \\
&\quad \leq \lim_{n\to\infty} \left[\int_I \int_\Omega \alpha(\overline{\boldsymbol{v}}_n(t))|D\tilde{\theta}_n(t)| dx\, dt + \int_I \int_\Omega \beta(\overline{\boldsymbol{v}}_n(t))|D(\nu_n\tilde{\theta}_n)(t)|^2\, dxdt \right] \\
&\qquad - \lim_{n\to\infty} \int_I (\alpha_0(\overline{\boldsymbol{v}}_n)(\widehat{\theta}_n)_t(t), \overline{\theta}_n(t) - \tilde{\theta}_n(t))_{L^2(\Omega)}\, dt \\
&\quad = \int_I \int_\Omega d[\alpha(\boldsymbol{v}(t))|D\theta(t)|]\, dt + \int_I \int_\Omega \beta(\boldsymbol{v}(t))|D(\nu\theta)(t)|^2\, dxdt.
\end{aligned}$$

The above inequality implies that:

$$\lim_{n\to\infty} \int_I \int_\Omega \alpha(\overline{\boldsymbol{v}}_n(t))|D\overline{\theta}_n(t)| dxdt = \int_I \int_\Omega d[\alpha(\boldsymbol{v}(t))|D\theta(t)|] dt, \tag{6.11}$$

and
$$\lim_{n\to\infty}\int_I\int_\Omega \beta(\overline{\boldsymbol{v}}_n(t))|D(\nu_n\overline{\theta}_n)(t)|^2dxdt = \int_I\int_\Omega \beta(\boldsymbol{v}(t))|D(\nu\theta)(t)|^2\,dt. \tag{6.12}$$
By virtue of (♯2)–(♯3), (6.4)–(6.5), (6.7) and (6.11), we can apply Lemma 13 to see that:
$$\int_I\int_\Omega |D\overline{\theta}_n(t)|dxdt \to \int_I\int_\Omega d|D\theta(t)|\,dt,\ \text{as } n\to\infty.$$
Besides, (6.1)–(6.2) and (6.5) enable to check:
$$\left|\int_I\int_\Omega |D\overline{\theta}_n|dxdt - \int_I\int_\Omega |D\underline{\theta}_n|dxdt\right| \le \frac{2F_*^\nu}{\delta_*}h_n \to 0,\ \text{as } n\to\infty,$$
and (6.5), (6.7) and the above convergence further enable to show that:
$$\begin{aligned}&\lim_{n\to\infty}\int_I\int_\Omega (\overline{\boldsymbol{v}}_n(t)-\boldsymbol{\varpi})\cdot[\nabla\alpha](\overline{\boldsymbol{v}}_n(t))|D\underline{\theta}_n(t)|dxdt\\ &= \int_I\int_\Omega d[(\overline{\boldsymbol{v}}_n(t)-\boldsymbol{\varpi})\cdot[\nabla\alpha](\boldsymbol{v}(t))|D\theta(t)|]dt\ \text{ for any } \boldsymbol{\varpi}\in[H^1(\Omega)\cap L^\infty(\Omega)]^2,\end{aligned} \tag{6.13}$$
by applying Lemma 13 (II).

Similarly, from (6.12) and the uniform convexity of L^2-based topology, one can see that
$$\begin{cases}\sqrt{\beta(\overline{\boldsymbol{v}}_n)}D(\nu_n\overline{\theta}_n) \to \sqrt{\beta(\boldsymbol{v})}D(\nu\theta)\ \text{ in } L^2(I;L^2(\Omega)^N),\text{ and hence}\\ D(\nu_n\overline{\theta}_n)\to D(\nu\theta) \text{ in } L^2(I;L^2(\Omega)^N),\ \text{ as } n\to\infty.\end{cases}$$
Besides, (6.1)–(6.2) and (6.5) enable to check:
$$\left|\int_I\int_\Omega |D(\nu_n\overline{\theta}_n)|^2dxdt - \int_I\int_\Omega |D(\nu_n\underline{\theta}_n)|^2\,dxdt\right| \le \frac{2F_*^\nu}{\delta_*}h_n \to 0,\ \text{as } n\to\infty,$$
and the above convergence further enables to show that:
$$\begin{cases}D(\nu_n\underline{\theta}_n)\to D(\nu\theta)\ \text{ in } L^2(I;L^2(\Omega)^N),\text{ and hence}\\ (\overline{\boldsymbol{v}}_n-\boldsymbol{\varpi})\cdot[\nabla\beta](\overline{\boldsymbol{v}}_n)D(\nu_n\underline{\theta}_n)\\ \qquad\to(\boldsymbol{v}-\boldsymbol{\varpi})\cdot[\nabla\beta](\boldsymbol{v})D(\nu\theta) \text{ in } L^2(I;L^2(\Omega)^N),\\ \text{for any } \boldsymbol{\varpi}\in[H^1(\Omega)\cap L^\infty(\Omega)]^2, \text{ as } n\to\infty.\end{cases} \tag{6.14}$$

With (A2)–(A5), (♯1)–(♯3), (6.4)–(6.5), (6.7), (6.13)–(6.14) and lower semi-continuity of L^2-norm in mind, letting $n\to\infty$ in (6.9) yields that:
$$\begin{aligned}&\int_I(\boldsymbol{v}_t(t),\boldsymbol{v}(t)-\boldsymbol{\varpi})_{L^2(\Omega)^2}dt + \int_I(D\boldsymbol{v}(t),D(\boldsymbol{v}(t)-\boldsymbol{\varpi}))_{L^2(\Omega)^{N\times2}}dt\\ &\quad+\int_I\int_\Omega\gamma(w(t))dxdt + \int_I([\nabla G](u(t);\boldsymbol{v}(t)),\boldsymbol{v}(t)-\boldsymbol{\varpi})_{L^2(\Omega)^2}dt\\ &\quad+\int_I\int_\Omega d[(\boldsymbol{v}(t)-\boldsymbol{\varpi})\cdot[\nabla\alpha](\boldsymbol{v}(t))|D\theta(t)|]dt\\ &\quad+\int_I\int_\Omega[\nabla\beta](\boldsymbol{v}(t))\cdot(\boldsymbol{v}(t)-\boldsymbol{\varpi})|\nabla(\nu\theta)|^2dxdt\\ &\le\int_I\int_\Omega\gamma(\varphi)dxdt,\ \text{ for any } \boldsymbol{\varpi}=[\varphi,\psi]\in[H^1(\Omega)\cap L^\infty(\Omega)]^2.\end{aligned} \tag{6.15}$$

Finally, taking the limit of (6.8), and applying (6.5)–(6.7), one can see that:

$$\begin{aligned}\int_I \langle u_t(t), z\rangle\, dt + \int_I (u(t), z)_V\, dt = \int_I (\lambda'(w(t))w_t(t), z)_{L^2(\Omega)}\, dt \\ + \int_I \langle \boldsymbol{f}^*(t), z\rangle\, dt, \quad \text{for any } z \in V.\end{aligned} \tag{6.16}$$

Since the open interval $I \subset (0, T)$ is arbitrary, the conditions (S4)–(S5) will be verified by taking into account (6.4) and (6.15)–(6.16). □

Acknowledgments

This research was supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (C), 16K05224, No. 26400138 and Young Scientists (B), No. 25800086. The authors express their gratitude to an anonymous referees for reviewing the original manuscript and for many valuable comments that helped clarify and refine this paper.

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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