Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Hybrid iterative algorithms for two families of finite maximal monotone mappings

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Abstract
In this paper, we introduce and analyze a new general hybrid iterative algorithm for finding a common element of the set of common zeros of two families of finite maximal monotone mappings, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping in a real Hilbert space. Our algorithm is based on four well-known methods: Mann's iteration method, composite method, outer-approximation method and extragradient method. We prove the strong convergence theorem for the proposed algorithm. The results presented in this paper extend and improve the corresponding results of Wei and Tan (Fixed Point Theory Appl. 2014:77, 2014). Some special cases are also discussed.

MSC: 47H05; 47H10; 47J25; 49J40

Keywords: hybrid iterative algorithms; maximal monotone mappings; fixed points; strong convergence

1 Introduction

Let H be a real Hilbert space and C be a nonempty, closed, and convex subset of H. A mapping $T : C \to C$ is said to be monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A multi-valued mapping $T : H \to 2^H$ is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone mapping. Lots of researches are focused on the maximal monotone mapping due to its importance.

In 1976, to solve the inclusion problem $0 \in Ax$, Rockafellar [1] introduced the following proximal point method:

$$x_0 \in H, \quad x_{n+1} = J_{r_n} x_n, \quad n = 0, 1, 2, \ldots,$$

where $J_{r_n} = (I + r_n A)^{-1}$ and $A : H \to 2^H$ is a maximal monotone mapping. It is shown that the iterative sequence $\{x_n\}$ converges weakly to a zero of A under some appropriate conditions. The strong convergence of the sequence has been extensively discussed by Zegeye and Shahzad [2] and Hu and Liu [3] in Banach spaces.

In 2014, Wei and Tan [4] introduced the following Mann-type composite viscosity iterative scheme for finding the common zeros of two families of finite maximal monotone mappings. In particular, they proved the following theorem.

Theorem 1.1 ([4], Theorem 2.2) *Let H be a real Hilbert space, C be a nonempty closed and convex subset of H, A_i, B_j $(i = 1,2,\ldots,k; j = 1,2,\ldots,l) : C \to C$ be two families of m-accretive mappings. Suppose*

$$S_{r_n}^{A_kA_{k-1}\cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l},$$

with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $\sum_{m=0}^{l} a_m = 1$ and $a_m \in (0,1)$. The sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ are generated by

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n f(x_n) + (1-\beta_n) S_{r_n}^{A_kA_{k-1}\cdots A_1} x_n, \\ u_n = \nu_n f(y_n) + (1-\nu_n) T_{r_n} y_n, \\ x_{n+1} = \alpha_n f(u_n) + (1-\alpha_n) u_n, \end{cases}$$

for every $n = 0,1,2,\ldots$, where $f : C \to C$ is a contraction. If $D = (\bigcap_{i=1}^{k} A_i^{-1}0) \cap (\bigcap_{j=1}^{l} B_j^{-1}0)$ is nonempty, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\nu_n\}$ are three sequences in $(0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$, *and* $\alpha_n \to 0$ *as* $n \to \infty$;
(ii) $\sum_{n=1}^{\infty} \beta_n = +\infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < +\infty$, *and* $\beta_n \to 0$ *as* $n \to \infty$;
(iii) $\sum_{n=1}^{\infty} |\nu_{n+1} - \nu_n| < +\infty$, *and* $\nu_n \to 0$ *as* $n \to \infty$;
(iv) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < +\infty$, *and* $r_n \to r^* \geq \varepsilon > 0$ *as* $n \to \infty$.

Then $\{x_n\}$ converges strongly to a point $p_0 \in D$, which is the unique solution of the following variational inequality:

$$\langle f(p_0) - p_0, p_0 - q \rangle \geq 0, \quad \forall q \in D.$$

Remark 1.1 Actually, the m-accretive mapping in a Hilbert space defined in Wei and Tan [4] is a maximal monotone mapping.

Theorem 1.1 gives rise naturally to the question we concerned: the real sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\nu_n\}$, and $\{r_n\}$ are satisfying the conditions (i), (ii), (iii), and (iv). When is the restrictions of the real sequences relaxed? The purpose of our paper is to give an affirmative answer to the question. Moreover, a new algorithm and an extensive problem are considered.

On the other hand, the variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C,$$

where $A : C \to H$ is a nonlinear mapping. The set of solutions of the variational inequality problem is denoted by $VI(C,A)$. The variational inequality problem was first discussed by Lions [5]. For finding a solution of the variational inequality problem in Euclidean space R^n, Korpelevich [6] introduced the following extragradient method:

$$\begin{cases} x_0 = x \in C, \\ \overline{x}_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A \overline{x}_n), \end{cases}$$

for every $n = 0, 1, 2, \ldots$ where C is a nonempty, closed, and convex subset of R^n, $A : C \to R^n$ is a monotone and ρ-Lipschitz-continuous mapping, $\lambda \in (0, \frac{1}{\rho})$. She showed that if $VI(C, A)$ is nonempty, then the generated sequences $\{x_n\}$ and $\{\overline{x}_n\}$ converge to the same point $z \in VI(C, A)$. The extragradient iterative process was successfully generalized and extended not only to Euclidean but also to Hilbert and Banach spaces; see, *e.g.*, the recent references of [7–9].

Furthermore, Iiduka and Takahashi [10] introduced the following outer-approximation method:

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

for every $n = 0, 1, 2, \ldots$, where $A : C \to H$ is an ρ-inverse strongly monotone mapping, $S : C \to C$ is a nonexpansive mapping, $0 < a \leq \lambda_n \leq b < 2\rho$ and $0 \leq \alpha_n \leq c < 1$. They showed that if $F(S) \cap VI(C, A)$ is nonempty, then the generated sequence $\{x_n\}$ converges to $P_{F(S) \cap VI(C,A)} x$. The outer-approximation method was originally introduced by Haugazeau in 1968 and was successfully generalized and extended in recent papers [11–14].

In this paper, inspired and motivated by the above work, we introduce the following general hybrid iterative algorithm, which is based on four well-known methods: Mann's iteration method, the composite method, the outer-approximation method, and the extragradient method.

Algorithm 1.1 Let H be a real Hilbert space, C be a nonempty, closed, and convex subset of H, $A : C \to H$ be a monotone and ρ-Lipschitz-continuous mapping, $S : C \to C$ be a nonexpansive mapping, A_i, B_j ($i = 1, 2, \ldots, k; j = 1, 2, \ldots, l$) $: C \to C$ be two families of finite maximal monotone mappings. Suppose

$$S_{r_n}^{A_k A_{k-1} \cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l},$$

with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $\sum_{m=0}^{l} a_m = 1$ and $a_m \in (0, 1)$. The sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \\ w_n = (1 - \beta_n - \gamma_n) z_n + \beta_n S_{r_n}^{A_k A_{k-1} \cdots A_1} z_n + \gamma_n T_{r_n} z_n, \\ C_n = \{z \in C : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

for every $n = 0, 1, 2, \ldots$.

Our algorithm is first used for finding the common zeros of two families of finite maximal monotone mappings. By using Algorithm 1.1, we will find a common element of the set of common zeros of two families of finite maximal monotone mappings, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping. We will prove the strong convergence theorem for the proposed algorithm, which extends and improves the corresponding results in the early and recent literature; see, *e.g.*, [3, 4, 6, 10, 14].

2 Preliminaries

Throughout our paper, let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty, closed, and convex subset of H, and I be the identity mapping on H. We write $x_n \stackrel{w}{\longrightarrow} x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists an unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is said to be the metric projection of H onto C. A mapping $T : C \to C$ is said to be ρ-Lipschitz-continuous if

$$\|Tx - Ty\| \leq \rho\|x - y\|, \quad \forall x, y \in C.$$

$T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Obviously, the 1-Lipschitz-continuous mapping is a nonexpansive mapping. It is well known that P_C is nonexpansive.

We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. We use $T^{-1}0$ to denote the set of zeros of T, that is, $T^{-1}0 = \{x \in C : Tx = 0\}$. We use J_r^T $(r > 0)$ to denote the resolvent operator of T, that is, $J_r^T = (I + rT)^{-1}$. As is well known, J_r^T is nonexpansive and $F(J_r^T) = T^{-1}0$.

It is well known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y,g) \in \operatorname{Graph}(T)$ implies $f \in Tx$. Next we provide an example to illustrate the concept of maximal monotone mapping. Let $A : C \to H$ be a monotone, ρ-Lipschitz-continuous mapping and $N_C v$ be the normal cone to C at $v \in C$, *i.e.*, $N_C v = \{\omega \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \phi, & \text{if } v \notin C, \end{cases}$$

it is well known that in this case T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [15]. At the same time, it is well known that H satisfies Opial's condition [16], *i.e.*, for any sequence $\{x_n\}$ with $x_n \stackrel{w}{\longrightarrow} x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. H also admits Kadec-Klee property, *i.e.*, sequential weak convergence on the unit sphere coincides with norm convergence; see [17].

In order to prove our main results, we need the following lemmas.

Lemma 2.1 [18] *For $\forall x \in H$ and $\forall y \in C$, P_C is characterized by the properties*:
 (i) $\langle x - P_C x, P_C x - y\rangle \geq 0$;
 (ii) $\|P_C x - y\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2$.

Lemma 2.2 [4] *For $\forall x \in H$, $\forall y \in A^{-1}0$ and $r > 0$,*

$$\|J_r^A x - y\|^2 + \|x - J_r^A x\|^2 \leq \|x - y\|^2.$$

Lemma 2.3 [4] *Let H be a real Hilbert space, C be a nonempty, closed, and convex subset of H, A_i, B_j $(i = 1,2,\dots,k; j = 1,2,\dots,l) : C \to C$ be two families of finite maximal monotone mappings such that $(\bigcap_{i=1}^k A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0)$ is nonempty. Suppose*

$$S_{r_n}^{A_k A_{k-1} \cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l},$$

with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $\sum_{m=0}^l a_m = 1$, $a_m \in (0,1)$ and $r_n > 0$. Then $S_{r_n}^{A_k A_{k-1} \cdots A_1} : C \to C$ and $T_{r_n} : C \to C$ are nonexpansive.

Lemma 2.4 [4] *Let $H, C, A_i, B_j, S_{r_n}^{A_k A_{k-1} \cdots A_1}$, and T_{r_n} be the same as those in Lemma 2.3, suppose $(\bigcap_{i=1}^k A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0)$ is nonempty, then $F(S_{r_n}^{A_k A_{k-1} \cdots A_1}) = \bigcap_{i=1}^k A_i^{-1}0$ and $F(T_{r_n}) = \bigcap_{j=1}^l B_j^{-1}0$.*

3 Strong convergence theorems

Theorem 3.1 *Let H be a real Hilbert space, C be a nonempty, closed, and convex subset of H, $A : C \to H$ be a monotone and ρ-Lipschitz-continuous mapping, $S : C \to C$ be a nonexpansive mapping, A_i, B_j $(i = 1,2,\dots,k; j = 1,2,\dots,l) : C \to C$ be two families of finite maximal monotone mappings such that $D = (\bigcap_{i=1}^k A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0) \cap F(S) \cap VI(C,A)$ is nonempty. Suppose*

$$S_{r_n}^{A_k A_{k-1} \cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l},$$

with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $\sum_{m=0}^l a_m = 1$ and $a_m \in (0,1)$.

If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, and $\{r_n\}$ satisfy the following conditions:
 (i) $0 \leq \alpha_n \leq b < 1$;
 (ii) $0 < c \leq \beta_n \leq 1$;
 (iii) $0 < d \leq \gamma_n \leq 1$, $\beta_n + \gamma_n \leq 1$;
 (iv) $0 < p \leq \lambda_n \leq q < \frac{1}{\rho}$;
 (v) $0 < \eta \leq r_n < +\infty$,

where b, c, d, p, q and η are constants, then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ generated by Algorithm 1.1 converge strongly to $P_D x$.

Proof We will split the proof into five steps.

Step 1. $D = (\bigcap_{i=1}^k A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0) \cap F(S) \cap VI(C,A) \subset C_n \cap Q_n$.

First, we show $D \subset C_n$.

Put $t_n = P_C(x_n - \lambda_n Ay_n)$. Take a fixed $p \in D$ arbitrarily, then we get $p \in (\bigcap_{i=1}^k A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0)$, $p \in F(S)$ and $p \in VI(C,A)$. From Lemma 2.1(ii) and the monotonicity of A, we have

$$\begin{aligned}
&\|t_n - p\|^2 \\
&\quad \le \|x_n - \lambda_n Ay_n - p\|^2 - \|x_n - \lambda_n Ay_n - t_n\|^2 \\
&\quad = \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle Ay_n, p - t_n\rangle \\
&\quad = \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \bigl(\langle Ay_n - Ap, p - y_n\rangle + \langle Ap, p - y_n\rangle + \langle Ay_n, y_n - t_n\rangle\bigr) \\
&\quad \le \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\
&\quad = \|x_n - p\|^2 - \bigl(\|x_n - y_n\|^2 + 2\langle x_n - y_n, y_n - t_n\rangle + \|y_n - t_n\|^2\bigr) + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\
&\quad = \|x_n - p\|^2 - \bigl(\|x_n - y_n\|^2 + \|y_n - t_n\|^2\bigr) + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.
\end{aligned} \tag{3.1}$$

Since $y_n = P_C(x_n - \lambda_n Ax_n)$, A is ρ-Lipschitz-continuous and by Lemma 2.1(i), we have

$$\begin{aligned}
&\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\
&\quad = \langle x_n - \lambda_n Ax_n - y_n, t_n - y_n\rangle + \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\quad \le \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\quad \le \lambda_n \rho \|x_n - y_n\| \|t_n - y_n\|.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1) and by condition (iv), we have

$$\begin{aligned}
&\|t_n - p\|^2 \\
&\quad \le \|x_n - p\|^2 - \bigl(\|x_n - y_n\|^2 + \|y_n - t_n\|^2\bigr) + 2\lambda_n \rho \|x_n - y_n\| \|t_n - y_n\| \\
&\quad \le \|x_n - p\|^2 - \bigl(\|x_n - y_n\|^2 + \|y_n - t_n\|^2\bigr) + \bigl(\lambda_n^2 \rho^2 \|x_n - y_n\|^2 + \|t_n - y_n\|^2\bigr) \\
&\quad \le \|x_n - p\|^2 + \bigl(\lambda_n^2 \rho^2 - 1\bigr)\|x_n - y_n\|^2 \\
&\quad \le \|x_n - p\|^2.
\end{aligned} \tag{3.3}$$

From (3.3), Algorithm 1.1, the convexity of $\|\cdot\|^2$ and the nonexpansiveness of S, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \bigl\|\alpha_n x_n + (1 - \alpha_n) St_n - p\bigr\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|St_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|t_n - p\|^2 \\
&\le \|x_n - p\|^2 + (1 - \alpha_n)\bigl(\lambda_n^2 \rho^2 - 1\bigr)\|x_n - y_n\|^2 \\
&\le \|x_n - p\|^2.
\end{aligned} \tag{3.4}$$

By (3.4), Algorithm 1.1, Lemma 2.3, and Lemma 2.4, we have

$$\begin{aligned}
\|w_n - p\|^2 &= \bigl\|(1 - \beta_n - \gamma_n) z_n + \beta_n S_{r_n}^{A_k A_{k-1} \cdots A_1} z_n + \gamma_n T_{r_n} z_n - p\bigr\|^2 \\
&\le (1 - \beta_n - \gamma_n)\|z_n - p\|^2 + \beta_n \bigl\|S_{r_n}^{A_k A_{k-1} \cdots A_1} z_n - p\bigr\|^2 + \gamma_n \|T_{r_n} z_n - p\|^2
\end{aligned}$$

$$\begin{aligned}&\le (1-\beta_n-\gamma_n)\|z_n-p\|^2+\beta_n\|z_n-p\|^2+\gamma_n\|z_n-p\|^2\\&=\|z_n-p\|^2\\&\le \|x_n-p\|^2+(1-\alpha_n)\big(\lambda_n^2\rho^2-1\big)\|x_n-y_n\|^2\\&\le \|x_n-p\|^2.\end{aligned}\tag{3.5}$$

By virtue of the definition of C_n and (3.5), we have $p\in C_n$. So, $D\subset C_n$, for every $n=0,1,2,\ldots$.

Next, through the mathematical induction method, we will prove that $\{x_n\}$ is well defined and $D\subset C_n\cap Q_n$, $n=0,1,2,\ldots$. For $n=0$, $x_0=x\in C$, $Q_0=C$. Hence $D\subset C_0\cap Q_0$. Suppose that $\{x_n\}$ is given and $D\subset C_k\cap Q_k$ for some $k\in N$. Because D is nonempty, $C_k\cap Q_k$ is nonempty. It is obvious that C_n is closed and Q_n is closed and convex. As $C_n=\{z\in C:\|z_n-x_n\|^2+2\langle z_n-x_n,x_n-z\rangle\le 0\}$, we also have C_n is convex, for every $n=0,1,2,\ldots$. Thus, $C_k\cap Q_k$ is a nonempty closed convex subset of C, so there exists an unique element $x_{k+1}\in C_k\cap Q_k$ such that $x_{k+1}=P_{C_k\cap Q_k}x$. It is obvious that

$$\langle x_{k+1}-z,x-x_{k+1}\rangle\ge 0,\quad \forall z\in C_k\cap Q_k.$$

Since $D\subset C_k\cap Q_k$, we have

$$\langle x_{k+1}-z,x-x_{k+1}\rangle\ge 0,\quad \forall z\in D.$$

That is, $z\in Q_{k+1}$. Hence $D\subset Q_{k+1}$. Therefore, we get $D\subset C_{k+1}\cap Q_{k+1}$. Thus, $D\subset C_n\cap Q_n$, for every $n=0,1,2,\ldots$.

Step 2. $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{w_n\}$ are all bounded.

Let $p_0=P_Dx$, then $p_0\in D\subset C_n\cap Q_n$ from step 1. From $x_{n+1}=P_{C_n\cap Q_n}x$ and the definition of the metric projection, we have

$$\|x_{n+1}-x\|\le\|p_0-x\|,\tag{3.6}$$

for every $n=0,1,2,\ldots$. Therefore, $\{x_n\}$ is bounded. By virtue of (3.3), (3.4) and (3.5), we also obtain $\{t_n\}$, $\{z_n\}$ and $\{w_n\}$ are also bounded.

Again from (3.5), conditions (i) and (v), we have

$$\begin{aligned}\|x_n-y_n\|^2&\le\frac{1}{(1-\alpha_n)(1-\lambda_n^2\rho^2)}\big(\|x_n-p\|^2-\|w_n-p\|^2\big)\\&\le\frac{1}{(1-\alpha_n)(1-\lambda_n^2\rho^2)}\big(\|x_n-p\|+\|w_n-p\|\big)\big(\|x_n-p\|-\|w_n-p\|\big)\\&\le\frac{1}{(1-\alpha_n)(1-\lambda_n^2\rho^2)}\big(\|x_n-p\|+\|w_n-p\|\big)\|x_n-w_n\|\\&\le\frac{1}{(1-b)(1-q^2\rho^2)}\big(\|x_n-p\|+\|w_n-p\|\big)\|x_n-w_n\|.\end{aligned}\tag{3.7}$$

So, $\{y_n\}$ is bounded.

Step 3. $\lim_{n\to\infty}\|x_n-y_n\|=0$, $\lim_{n\to\infty}\|x_n-z_n\|=0$, $\lim_{n\to\infty}\|x_n-w_n\|=0$ and $\lim_{n\to\infty}\|x_n-t_n\|=0$.

As $Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}$, we have $\langle x_n - z, x - x_n\rangle \geq 0$ for all $z \in Q_n$ and by the definition of the metric projection, we have $x_n = P_{Q_n}x$. Because of $x_{n+1} = P_{C_n\cap Q_n}x \in C_n \cap Q_n \subset Q_n$ and (3.6), we have

$$\|x_n - x\| \leq \|x_{n+1} - x\| \leq \|p_0 - x\|,$$

for every $n = 0, 1, 2, \ldots$. Therefore there exists $\lim_{n\to\infty} \|x_n - x\|$. By Lemma 2.1(ii), we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2,$$

this implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}$$

By $x_{n+1} \in C_n$ and the definition of C_n, we have $\|x_{n+1} - w_n\| \leq \|x_{n+1} - x_n\|$ and hence

$$\|w_n - x_n\| \leq \|w_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\|. \tag{3.9}$$

From (3.8) and (3.9), we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.10}$$

From conditions (i), (iv), (3.7) and (3.10), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.11}$$

In (3.3), using another technique, by condition (iv), we get

$$\begin{aligned}\|t_n - p\|^2 &\leq \|x_n - p\|^2 - \left(\|x_n - y_n\|^2 + \|y_n - t_n\|^2\right) + 2\lambda_n\rho\|x_n - y_n\|\|t_n - y_n\| \\ &\leq \|x_n - p\|^2 - \left(\|x_n - y_n\|^2 + \|y_n - t_n\|^2\right) + \left(\|x_n - y_n\|^2 + \lambda_n^2\rho^2\|t_n - y_n\|^2\right) \\ &= \|x_n - p\|^2 + \left(\lambda_n^2\rho^2 - 1\right)\|t_n - y_n\|^2. \end{aligned} \tag{3.12}$$

From (3.4) and (3.12), we have

$$\begin{aligned}\|z_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|t_n - p\|^2 \\ &\leq \|x_n - p\|^2 + (1 - \alpha_n)\left(\lambda_n^2\rho^2 - 1\right)\|t_n - y_n\|^2. \end{aligned} \tag{3.13}$$

By (3.5) and (3.13), we have

$$\|w_n - p\|^2 \leq \|z_n - p\|^2 \leq \|x_n - p\|^2 + (1 - \alpha_n)\left(\lambda_n^2\rho^2 - 1\right)\|t_n - y_n\|^2. \tag{3.14}$$

So, (3.14) implies that

$$\begin{aligned}\|t_n - y_n\|^2 &\leq \frac{1}{(1 - \alpha_n)(1 - \lambda_n^2\rho^2)}\left(\|x_n - p\|^2 - \|w_n - p\|^2\right) \\ &\leq \frac{1}{(1 - \alpha_n)(1 - \lambda_n^2\rho^2)}\left(\|x_n - p\| + \|w_n - p\|\right)\|x_n - w_n\|. \end{aligned} \tag{3.15}$$

Since $\{x_n\}$, $\{w_n\}$ are bounded, and by conditions (i), (iv), (3.10), and (3.15), we have

$$\lim_{n\to\infty} \|y_n - t_n\| = 0. \tag{3.16}$$

By $\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|$, (3.11) and (3.16), we get

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{3.17}$$

From (3.4), (3.5), Lemma 2.2, Lemma 2.3 and Lemma 2.4, we have, for $\forall p \in D$,

$$\begin{aligned}
\|w_n - p\|^2 &\le (1-\beta_n-\gamma_n)\|z_n - p\|^2 + \beta_n \big\|S_{r_n}^{A_kA_{k-1}\cdots A_1} z_n - p\big\|^2 + \gamma_n \|T_{r_n} z_n - p\|^2 \\
&\le (1-\beta_n)\|z_n - p\|^2 + \beta_n \big\|S_{r_n}^{A_kA_{k-1}\cdots A_1} z_n - p\big\|^2 \\
&\le (1-\beta_n)\|z_n - p\|^2 + \beta_n \big(\big\|S_{r_n}^{A_{k-1}\cdots A_1} z_n - p\big\|^2 \\
&\quad - \big\|S_{r_n}^{A_kA_{k-1}\cdots A_1} z_n - S_{r_n}^{A_{k-1}\cdots A_1} z_n\big\|^2\big) \\
&\le \|z_n - p\|^2 - \beta_n \big\|S_{r_n}^{A_kA_{k-1}\cdots A_1} z_n - S_{r_n}^{A_{k-1}\cdots A_1} z_n\big\|^2 \\
&\le \|x_n - p\|^2 - \beta_n \big\|S_{r_n}^{A_kA_{k-1}\cdots A_1} z_n - S_{r_n}^{A_{k-1}\cdots A_1} z_n\big\|^2.
\end{aligned} \tag{3.18}$$

Then (3.18) implies that

$$\begin{aligned}
\big\|S_{r_n}^{A_kA_{k-1}\cdots A_1} z_n - S_{r_n}^{A_{k-1}\cdots A_1} z_n\big\|^2 &\le \frac{1}{\beta_n}\big(\|x_n - p\|^2 - \|w_n - p\|^2\big) \\
&\le \frac{1}{\beta_n}\big(\|x_n - p\| + \|w_n - p\|\big)\|x_n - w_n\|.
\end{aligned} \tag{3.19}$$

From (3.10), (3.19), condition (ii), we have

$$S_{r_n}^{A_kA_{k-1}\cdots A_1} z_n - S_{r_n}^{A_{k-1}\cdots A_1} z_n \to 0, \quad n \to \infty. \tag{3.20}$$

Again, using a similar technique to (3.18), we have

$$\begin{aligned}
\|w_n - p\|^2 &\le (1-\beta_n)\|z_n - p\|^2 + \beta_n \big\|S_{r_n}^{A_{k-1}\cdots A_1} z_n - p\big\|^2 \\
&\le (1-\beta_n)\|z_n - p\|^2 + \beta_n \big(\big\|S_{r_n}^{A_{k-2}A_{k-3}\cdots A_1} z_n - p\big\|^2 \\
&\quad - \big\|S_{r_n}^{A_{k-1}\cdots A_1} z_n - S_{r_n}^{A_{k-2}A_{k-3}\cdots A_1} z_n\big\|^2\big) \\
&\le \|z_n - p\|^2 - \beta_n \big\|S_{r_n}^{A_{k-1}A_{k-2}\cdots A_1} z_n - S_{r_n}^{A_{k-2}A_{k-3}\cdots A_1} z_n\big\|^2 \\
&\le \|x_n - p\|^2 - \beta_n \big\|S_{r_n}^{A_{k-1}A_{k-2}\cdots A_1} z_n - S_{r_n}^{A_{k-2}A_{k-3}\cdots A_1} z_n\big\|^2.
\end{aligned}$$

Using the same method in (3.19) and (3.20), we have

$$S_{r_n}^{A_{k-1}A_{k-2}\cdots A_1} z_n - S_{r_n}^{A_{k-2}\cdots A_1} z_n \to 0, \quad n \to \infty. \tag{3.21}$$

By induction, we have the following results:

$$S_{r_n}^{A_{k-2}A_{k-3}\cdots A_1} z_n - S_{r_n}^{A_{k-3}\cdots A_1} z_n \to 0, \quad n \to \infty, \tag{3.22}$$

$$\cdots,$$

$$S_{r_n}^{A_2A_1}z_n - S_{r_n}^{A_1}z_n \to 0, \quad n \to \infty, \tag{3.23}$$

$$S_{r_n}^{A_1}z_n - z_n \to 0, \quad n \to \infty. \tag{3.24}$$

From (3.20)-(3.24), we claim

$$S_{r_n}^{A_kA_{k-1}\cdots A_1}z_n - z_n \to 0, \quad n \to \infty. \tag{3.25}$$

By Lemma 2.2, Lemma 2.3, Lemma 2.4, (3.4), and the convexity of $\|\cdot\|^2$, $\forall p \in D$, we have

$$\begin{aligned} \|w_n - p\|^2 &\le (1-\beta_n-\gamma_n)\|z_n - p\|^2 + \beta_n\left\|S_{r_n}^{A_kA_{k-1}\cdots A_1}z_n - p\right\|^2 + \gamma_n\|T_{r_n}z_n - p\|^2 \\ &\le (1-\gamma_n)\|z_n - p\|^2 + \gamma_n\|T_{r_n}z_n - p\|^2 \\ &\le (1-\gamma_n)\|z_n - p\|^2 + \gamma_n\left(a_0\|z_n - p\|^2 + \sum_{j=1}^{l} a_j\left\|J_{r_n}^{B_j}z_n - p\right\|^2\right) \\ &\le (1-\gamma_n)\|z_n - p\|^2 + \gamma_n\left(a_0\|z_n - p\|^2 + \sum_{j=1}^{l} a_j\left(\|z_n - p\|^2 - \left\|J_{r_n}^{B_j}z_n - z_n\right\|^2\right)\right) \\ &\le \|z_n - p\|^2 - \gamma_n\sum_{j=1}^{l} a_j\left\|J_{r_n}^{B_j}z_n - z_n\right\|^2 \\ &\le \|x_n - p\|^2 - \gamma_n\sum_{j=1}^{l} a_j\left\|J_{r_n}^{B_j}z_n - z_n\right\|^2. \end{aligned} \tag{3.26}$$

Then (3.26) implies that

$$\begin{aligned} \left\|J_{r_n}^{B_j}z_n - z_n\right\|^2 &\le \frac{1}{\gamma_n\sum_{j=1}^{l} a_j}\left(\|x_n - p\|^2 - \|w_n - p\|^2\right) \\ &\le \frac{1}{\gamma_n\sum_{j=1}^{l} a_j}\left(\|x_n - p\| + \|w_n - p\|\right)\|x_n - w_n\|. \end{aligned} \tag{3.27}$$

From the boundedness of $\{x_n\}$ and $\{w_n\}$, condition (iii), (3.10) and (3.27), we have

$$J_{r_n}^{B_j}z_n - z_n \to 0, \quad n \to \infty. \tag{3.28}$$

For $j = 1, 2, \ldots, l$. From (3.28), we have

$$\lim_{n\to\infty}\|T_{r_n}z_n - z_n\| = 0. \tag{3.29}$$

By Algorithm 1.1, we have

$$\|w_n - z_n\| = \beta_n\left\|S_{r_n}^{A_kA_{k-1}\cdots A_1}z_n - z_n\right\| + \gamma_n\|T_{r_n}z_n - z_n\|^2.$$

By (3.25) and (3.29), we have

$$\lim_{n\to\infty}\|w_n - z_n\| = 0. \tag{3.30}$$

By (3.10) and (3.30), we get

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.31}$$

By (3.17) and (3.31), we get

$$\lim_{n\to\infty} \|z_n - t_n\| = 0. \tag{3.32}$$

Step 4. $W(x_n) \subset D$, where $W(x_n)$ denotes the set of all the weak limit points of $\{x_n\}$.

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$; for simplicity, we still denote it by $\{x_n\}$, such that $x_n \xrightarrow{w} u$ as $n \to \infty$. In the following, we will prove $u \in D$.

First, we show $u \in F(S)$. Assume $u \notin F(S)$, *i.e.*, $u \neq Su$. Since $z_n = \alpha_n x_n + (1-\alpha_n)St_n$, we have

$$\begin{aligned}(1-\alpha_n)\|St_n - t_n\| &= \big\|\alpha_n(t_n - x_n) + (z_n - t_n)\big\| \\ &\leq \alpha_n\|t_n - x_n\| + \|z_n - t_n\|. \end{aligned}\tag{3.33}$$

By virtue of condition (i), (3.17), (3.32) and (3.33), we have

$$\lim_{n\to\infty} \|St_n - t_n\| = 0. \tag{3.34}$$

By (3.17) and $x_n \xrightarrow{w} u$, we have $t_n \xrightarrow{w} u$, where $\{t_n\}$ is a subsequence of $\{t_n\}$ for simplicity. From (3.34) and Opial's condition [16], we have

$$\begin{aligned}\liminf_{n\to\infty} \|t_n - u\| &< \liminf_{n\to\infty} \|t_n - Su\| \\ &= \liminf_{n\to\infty} \|t_n - St_n + St_n - Su\| \\ &\leq \liminf_{n\to\infty} \|t_n - St_n\| + \liminf_{n\to\infty} \|St_n - Su\| \\ &\leq \liminf_{n\to\infty} \|t_n - u\|.\end{aligned}$$

This is a contradiction. So, we obtain $u \in F(S)$.

Second, we show $u \in \bigcap_{i=1}^k A_i^{-1}0$. From $x_n - z_n \to 0$ and $x_n \xrightarrow{w} u$, we have $z_n \xrightarrow{w} u$, where $\{z_n\}$ is a subsequence of $\{z_n\}$ for simplicity. From (3.20)-(3.24), we have $S_{r_n}^{A_1} z_n \xrightarrow{w} u$, $S_{r_n}^{A_2A_1} z_n \xrightarrow{w} u$, $S_{r_n}^{A_{k-1}A_{k-2}\cdots A_1} z_n \xrightarrow{w} u$ and $S_{r_n}^{A_kA_{k-1}\cdots A_1} z_n \xrightarrow{w} u$.

Since $(I + r_nA_1)S_{r_n}^{A_1} z_n = z_n$, by (3.24) and condition (v), we have

$$A_1S_{r_n}^{A_1} z_n = \frac{1}{r_n}\big(z_n - S_{r_n}^{A_1} z_n\big) \to 0, \quad n \to \infty.$$

So, $A_1u = 0$ and then $u \in A_1^{-1}0$.

Since $(I + r_nA_2)S_{r_n}^{A_2A_1} z_n = S_{r_n}^{A_1} z_n$, and by (3.23), we have

$$A_2S_{r_n}^{A_2A_1} z_n = \frac{1}{r_n}\big(S_{r_n}^{A_1} z_n - S_{r_n}^{A_2A_1} z_n\big) \to 0, \quad n \to \infty.$$

So, $A_2u = 0$ and then $u \in A_2^{-1}0$.

By induction, we have

$$A_k S_{r_n}^{A_k A_{k-1}\cdots A_2 A_1} z_n = \frac{1}{r_n}\big(S_{r_n}^{A_{k-1}\cdots A_2 A_1} z_n - S_{r_n}^{A_k A_{k-1}\cdots A_2 A_1} z_n\big) \to 0, \quad n \to \infty.$$

So, $A_k u = 0$ and then $u \in A_k^{-1}0$. Thus, $u \in \bigcap_{i=1}^{k} A_i^{-1}0$.

Third, we show $u \in \bigcap_{j=1}^{l} B_j^{-1}0$. In fact, by virtue of (3.28) and condition (v), we have

$$B_j J_{r_n}^{B_j} z_n = \frac{1}{r_n}\big(z_n - J_{r_n}^{B_j} z_n\big) \to 0, \quad n \to \infty.$$

So, $B_j u = 0$, for $j = 1, 2, \ldots, l$. We can easily see that $u \in \bigcap_{j=1}^{l} B_j^{-1}0$.

Finally, we show $u \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \phi, & \text{if } v \notin C, \end{cases}$$

where $N_C v$ is the normal cone to C at $v \in C$. Let $G(T)$ be the graph of T and $(v, \omega) \in G(T)$. So, we have $\omega \in Tv = Av + N_C v$ and hence $\omega - Av \in N_C v$. From the definition of the normal cone and $t_n = P_C(x_n - \lambda_n A y_n) \in C$, we have

$$\langle v - t_n, \omega - Av \rangle \geq 0. \tag{3.35}$$

From Lemma 2.1(i), we have

$$\langle x_n - \lambda_n A y_n - t_n, t_n - v \rangle \geq 0, \quad \forall v \in C,$$

and hence

$$\left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + A y_n \right\rangle \geq 0. \tag{3.36}$$

For simplicity, we assume that $\{y_n\}$ and $\{t_n\}$ are also subsequences of $\{y_n\}$ and $\{t_n\}$ respectively. Because of $x_n - y_n \to 0$, $x_n - t_n \to 0$ and $x_n \stackrel{w}{\longrightarrow} u$, we have $y_n \stackrel{w}{\longrightarrow} u$ and $xt_n \stackrel{w}{\longrightarrow} u$. By the monotonicity of A, (3.35), and (3.36), we obtain

$$\begin{aligned}
\langle v - t_n, \omega \rangle &\geq \langle v - t_n, Av \rangle \\
&\geq \langle v - t_n, Av \rangle - \left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + A y_n \right\rangle \\
&= \langle v - t_n, Av - At_n \rangle + \langle v - t_n, At_n - Ay_n \rangle - \left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} \right\rangle \\
&\geq \langle v - t_n, At_n - Ay_n \rangle - \left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} \right\rangle.
\end{aligned}$$

Hence, we obtain $\langle v - u, \omega - 0 \rangle \geq 0$ as $n \to \infty$. Since T is maximal monotone, we have $0 \in Tu$ and so $u \in VI(C, A)$.

Simply stated, $u \in D$. We get $W(x_n) \subset D$.

Step 5. $x_n \to P_D x$ as $n \to \infty$.

Let $p_0 = P_D x$, $\forall u \in W(x_n)$, from step 4, we have $u \in D$. Suppose $x_n \xrightarrow{w} u$ as $n \to \infty$, where $\{x_n\}$ is looked as a subsequence of $\{x_n\}$ for simplicity. From the definition of the metric projection and the weak lower semi-continuity of $\|\cdot\|$, we have

$$\|p_0 - x\| \le \|u - x\| \le \liminf_{n\to\infty} \|x_n - x\| \le \limsup_{n\to\infty} \|x_n - x\| \le \|p_0 - x\|.$$

So, we obtain

$$\lim_{n\to\infty} \|x_n - x\| = \|u - x\|.$$

From $x_n - x \xrightarrow{w} u - x$ and the Kadec-Klee property, we have $x_n - x \to u - x$ and hence $x_n \to u$.

Since $x_n = P_{Q_n} x$, $p_0 \in D \subset C_n \cap Q_n \subset Q_n$ and Lemma 2.1(i), we have

$$-\|p_0 - x_n\|^2 = \langle p_0 - x_n, x_n - x\rangle + \langle p_0 - x_n, x - p_0\rangle \ge \langle p_0 - x_n, x - p_0\rangle.$$

As $n \to \infty$, we get $-\|p_0 - u\|^2 \ge \langle p_0 - u, x - p_0\rangle \ge 0$. Hence $u = p_0$. This implies that $x_n \to p_0 = P_D x$ as $n \to \infty$. From step 3, it is easy to see $y_n \to P_D x$, $z_n \to P_D x$ and $w_n \to P_D x$ as $n \to \infty$.

This completes the proof. □

From Theorem 3.1, we can get some strong convergence theorems.

Theorem 3.2 *Let H be a real Hilbert space, C be a nonempty, closed, and convex subset of H, $A : C \to H$ be a monotone and ρ-Lipschitz-continuous mapping, A_i, B_j ($i = 1, 2, \dots, k$; $j = 1, 2, \dots, l$) $: C \to C$ be two families of finite maximal monotone mappings such that $D = (\bigcap_{i=1}^k A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0) \cap VI(C, A)$ is nonempty. Suppose*

$$S_{r_n}^{A_k A_{k-1} \cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l},$$

with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $\sum_{m=0}^l a_m = 1$ and $a_m \in (0, 1)$.

The sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = P_C(x_n - \lambda_n A y_n), \\ w_n = (1 - \beta_n - \gamma_n) z_n + \beta_n S_{r_n}^{A_k A_{k-1} \cdots A_1} z_n + \gamma_n T_{r_n} z_n, \\ C_n = \{z \in C : \|w_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

if $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $0 < c \le \beta_n \le 1$;

(ii) $0 < d \le \gamma_n \le 1$, $\beta_n + \gamma_n \le 1$;

(iii) $0 < p \le \lambda_n \le q < \frac{1}{\rho}$;

(iv) $0 < \eta \le r_n < +\infty$,

for every $n = 0, 1, 2, \ldots$, *where* c, d, p, q *and* η *are constants, then the sequences* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{w_n\}$ *converge strongly to* $P_D x$.

Proof Put $S = I$ and $\alpha_n = 0$ for all $n = 0, 1, 2, \ldots$. By Theorem 3.1, we get the desired results. □

Theorem 3.3 *Let* H *be a real Hilbert space,* C *be a nonempty, closed, and convex subset of* H, $S : C \to C$ *be a nonexpansive mapping,* A_i, B_j $(i = 1, 2, \ldots, k; j = 1, 2, \ldots, l) : C \to C$ *be two families of finite maximal monotone mappings such that* $D = (\bigcap_{i=1}^{k} A_i^{-1}0) \cap (\bigcap_{j=1}^{l} B_j^{-1}0) \cap F(S)$ *is nonempty. Suppose*

$$S_{r_n}^{A_k A_{k-1} \cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l},$$

with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $\sum_{m=0}^{l} a_m = 1$ *and* $a_m \in (0, 1)$.

The sequences $\{x_n\}$, $\{z_n\}$, *and* $\{w_n\}$ *are generated by*

$$\begin{cases} x_0 = x \in C, \\ z_n = \alpha_n x_n + (1 - \alpha_n) S P_C x_n, \\ w_n = (1 - \beta_n - \gamma_n) z_n + \beta_n S_{r_n}^{A_k A_{k-1} \cdots A_1} z_n + \gamma_n T_{r_n} z_n, \\ C_n = \{z \in C : \|w_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

if $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{r_n\}$ *satisfy the following conditions*:

(i) $0 \le \alpha_n \le b < 1$;
(ii) $0 < c \le \beta_n \le 1$;
(iii) $0 < d \le \gamma_n \le 1$, $\beta_n + \gamma_n \le 1$;
(iv) $0 < \eta \le r_n < +\infty$,

for every $n = 0, 1, 2, \ldots$, *where* b, c, d *and* η *are constants, then the sequences* $\{x_n\}$, $\{z_n\}$ *and* $\{w_n\}$ *converge strongly to* $P_D x$.

Proof Let $A = 0$ in Theorem 3.1, we obtain the result. □

Theorem 3.4 *Let* H *be a real Hilbert space,* C *be a nonempty, closed, and convex subset of* H, A_i, B_j $(i = 1, 2, \ldots, k; j = 1, 2, \ldots, l) : C \to C$ *be two families of finite maximal monotone mappings such that* $D = (\bigcap_{i=1}^{k} A_i^{-1}0) \cap (\bigcap_{j=1}^{l} B_j^{-1}0)$ *is nonempty. Suppose*

$$S_{r_n}^{A_k A_{k-1} \cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l},$$

with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $\sum_{m=0}^{l} a_m = 1$ *and* $a_m \in (0, 1)$. *The sequences* $\{x_n\}$ *and* $\{y_n\}$ *are generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = (1 - \beta_n - \gamma_n) x_n + \beta_n S_{r_n}^{A_k A_{k-1} \cdots A_1} x_n + \gamma_n T_{r_n} x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

if $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{r_n\}$ *satisfy the following conditions*:

(i) $0 < c \le \beta_n \le 1$;

(ii) $0 < d \le \gamma_n \le 1$, $\beta_n + \gamma_n \le 1$;

(iii) $0 < \eta \le r_n < +\infty$,

for every $n = 0,1,2,\ldots$, *where* c, d, *and* η *are constants, then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $P_D x$.

Proof Let $A = 0$, $S = I$ and $\alpha_n = 0$ for all $n = 0,1,2,\ldots$ in Theorem 3.1, we obtain the result.

If A_i, B_j $(i = 1,2,\ldots,k; j = 1,2,\ldots,l) : H \to 2^H$ are set-valued mappings, the resolvent operator J_r^T $(r > 0)$ of T, is defined by $J_r^T x = \{z \in H : x \in z + rTz\} = (I + rT)^{-1}x$, $\forall x \in H$, where I denote the identity mapping on H. As is well known, $J_r^T : H \to H$ is a single valued mapping. We have the following strong convergence theorem. □

Theorem 3.5 *Let* H *be a real Hilbert space*, $A : H \to H$ *be a monotone and* ρ*-Lipschitz-continuous mapping*, $S : H \to H$ *be a nonexpansive mapping*, A_i, B_j $(i = 1,2,\ldots,k; j = 1,2,\ldots,l) : H \to 2^H$ *be two families of finite set-valued maximal monotone mappings such that* $D = (\bigcap_{i=1}^k A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0) \cap F(S) \cap A^{-1}0$ *is nonempty. Suppose*

$$S_{r_n}^{A_k A_{k-1}\cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l},$$

with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $\sum_{m=0}^l a_m = 1$ *and* $a_m \in (0,1)$.

The sequences $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *are generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S(x_n - \lambda_n A(x_n - \lambda_n A x_n)), \\ z_n = (1 - \beta_n - \gamma_n) y_n + \beta_n S_{r_n}^{A_k A_{k-1}\cdots A_1} y_n + \gamma_n T_{r_n} y_n, \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

if $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, *and* $\{r_n\}$ *satisfy the following conditions in Theorem* 3.1, *then the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $P_D x$.

Proof As is well known, $P_H = I$ and $VI(H,A) = A^{-1}0$. Using the similar arguments to those in the proof of Theorem 3.1, we get the desired result immediately. □

Remark 3.1 Theorems 3.1-3.5 greatly improve and extend the previous work in the following respects:

(1) We study the problem of finding a common element of the set of common zeros of two families of finite maximal monotone mappings, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping, *i.e.*, $(\bigcap_{i=1}^k A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0) \cap F(S) \cap VI(C,A)$. The problems of finding common elements of [7], Theorem 2.2 and [17], Theorems 3.1 and 4.1 are all special cases of our problem.

(2) The hybrid iterative Algorithm 1.1 greatly generalizes and extends some corresponding algorithms in [4, 6, 10, 14, 19]. It is first used for finding common

zeros of two families of finite maximal monotone mappings. The method of proof is also different from the earlier ones.

(3) All parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy weaker restrictions in our theorems than those in the theorems of [3, 4]. For example, $0 \le \alpha_n \le b < 1$, neither $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$, nor $\alpha_n \to 0$ as $n \to \infty$.

4 Applications

In this section, we will give several examples from the practice with numerical analysis with their new algorithms.

Let H be a real Hilbert space, $\varphi : H \to R \cup \{+\infty\}$ be a proper, convex, lower semicontinuous functional. The subdifferentiable operator of φ, denoted by $\partial\varphi : H \to 2^H$, is defined at $x \in H$ by

$$\partial\varphi(x) = \bigl\{u \in H : \varphi(y) \ge \varphi(x) + \langle y - x, u\rangle, \forall y \in H\bigr\}.$$

For each $x \in H$, $\partial\varphi(x)$ is called the subgradient of φ at x. Using different methods, Rockafellar [20] and Alves and Svaiter [21] proved that the subdifferentiable operator is a maximal monotone mapping, respectively. Thus, from Theorem 3.5, we get the following result immediately.

Theorem 4.1 *Let H be a real Hilbert space, $A : H \to H$ be a monotone and ρ-Lipschitz-continuous mapping, $S : H \to H$ be a nonexpansive mapping, $\varphi_{1i}, \varphi_{2j}$ $(i = 1, 2, \ldots, k;$ $j = 1, 2, \ldots, l) : H \to R \cup \{+\infty\}$ be two finite families of proper, convex, lower semicontinuous functionals, $\partial\varphi_{1i}$ and $\partial\varphi_{2j}$ be their subdifferentiable operators, respectively, such that $D = (\bigcap_{i=1}^{k} \partial\varphi_{1i}^{-1}0) \cap (\bigcap_{j=1}^{l} \partial\varphi_{2j}^{-1}0) \cap A^{-1}0$ is nonempty. Suppose*

$$S_{r_n}^{\partial\varphi_{1k}\partial\varphi_{1,k-1}\cdots\partial\varphi_{11}} = J_{r_n}^{\partial\varphi_{1k}} J_{r_n}^{\partial\varphi_{1,k-1}} \cdots J_{r_n}^{\partial\varphi_{11}}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{\partial\varphi_{21}} + a_2 J_{r_n}^{\partial\varphi_{22}} + \cdots + a_l J_{r_n}^{\partial\varphi_{2l}},$$

with $J_{r_n}^{\partial\varphi_i} = (I + r_n \partial\varphi_i)^{-1}$, $\sum_{m=0}^{l} a_m = 1$ and $a_m \in (0, 1)$. The sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated by

$$\begin{cases} x_0 = x \in H, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S(x_n - \lambda_n A(x_n - \lambda_n A x_n)), \\ z_n = (1 - \beta_n - \gamma_n) y_n + \beta_n S_{r_n}^{\partial\varphi_{1k}\partial\varphi_{1,k-1}\cdots\partial\varphi_{11}} y_n + \gamma_n T_{r_n} y_n, \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

if $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, and $\{r_n\}$ satisfy the conditions in Theorem 3.1, *then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $P_D x$.*

We also know a mapping $T : C \to C$ is called pseudocontractive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \bigl\|(I - T)x - (I - T)y\bigr\|^2, \quad \forall x, y \in C.$$

It is equivalent to the following definition:

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2, \quad \forall x, y \in C.$$

If T is a pseudocontractive, ρ-Lipschitz-continuous mapping, then $A = I - T$ is monotone and $(\rho + 1)$-Lipschitz-continuous mapping, and $F(T) = VI(C,A)$, more details see [14, 19]. So, by Theorem 3.1, we have the following result immediately.

Theorem 4.2 *Let H be a real Hilbert space, C be a nonempty, closed, and convex subset of H, $T : C \to C$ be a pseudocontractive and ρ-Lipschitz-continuous mapping, $S : C \to C$ be a nonexpansive mapping, A_i, B_j $(i = 1,2,\ldots,k; j = 1,2,\ldots,l) : C \to C$ be two families of finite maximal monotone mappings such that $D = (\bigcap_{i=1}^k A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0) \cap F(S) \cap F(T)$ is nonempty. Suppose*

$$S_{r_n}^{A_k A_{k-1} \cdots A_1} = J_{r_n}^{A_k} J_{r_n}^{A_{k-1}} \cdots J_{r_n}^{A_1}, \qquad T_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l},$$

with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$, $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, $\sum_{m=0}^l a_m = 1$ and $a_m \in (0,1)$.

The sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are generated by

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n(x_n - Tx_n)), \\ z_n = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n(y_n - Ty_n)), \\ w_n = (1 - \beta_n - \gamma_n) z_n + \beta_n S_{r_n}^{A_k A_{k-1} \cdots A_1} z_n + \gamma_n T_{r_n} z_n, \\ C_n = \{z \in C : \|w_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases}$$

for every $n = 0,1,2,\ldots$. If $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, and $\{r_n\}$ satisfy the following conditions:

(i) $0 \le \alpha_n \le b < 1$;

(ii) $0 < c \le \beta_n \le 1$;

(iii) $0 < d \le \gamma_n \le 1$, $\beta_n + \gamma_n \le 1$;

(iv) $0 < p \le \lambda_n \le q < \frac{1}{\rho+1}$;

(v) $0 < \eta \le r_n < +\infty$,

where b, c, d, p, q and η are constants, then the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ converge strongly to $P_D x$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgements
This work was supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), the Ph.D. Program Foundation of Ministry of Education of China (20123127110002) and the Program for Shanghai Outstanding Academic Leaders in Shanghai City (15XD1503100).

Received: 9 June 2015 Accepted: 24 September 2015 Published online: 06 October 2015

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