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Existence of positive solutions for nonlinear m -point boundary value problems on time scales

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Abstract

In this article, we study the following m -point boundary value problem on time scales,

$$\begin{cases} (\phi_p(u^\Delta(t)))^\nabla + h(t)f(t, u(t)) = 0, & t \in (0, T)_{\mathbb{T}}, \\ u(0) - \delta u^\Delta(0) = \sum_{i=1}^{m-2} \beta_i u^\Delta(\xi_i), & u^\Delta(T) = 0, \end{cases}$$

where \mathbb{T} is a time scale such that $0, T \in \mathbb{T}, \delta, \beta_i > 0, i = 1, \dots, m-2$,

$\phi_p(s) = |s|^{p-2}s, p > 1, h \in C_{ld}([0, T], (0, +\infty))$, and $f \in C([0, +\infty), (0, +\infty))$,

$0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < T \in \mathbb{T}$. By using several well-known fixed point theorems in a cone, the existence of at least one, two, or three positive solutions are obtained. Examples are also given in this article.

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1 Introduction

The study of dynamic equations on time scales goes back to its founder Hilger [1], and is a new area of still theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics. Further, the study of time scales has led to several important applications, e.g., in the study of insect population models, neural networks, heat transfer, epidemic models, etc. [2].

Multipoint boundary value problems of ordinary differential equations (BVPs for short) arise in a variety of different areas of applied mathematics and physics. For example, the vibrations of a guy wire of a uniform cross section and composed of N parts of different densities can be set up as a multi-point boundary value problem [3]. Many problems in the theory of elastic stability can be handled by the method of multi-point problems [4]. Small size bridges are often designed with two supported points, which leads into a standard two-point boundary value condition and large size bridges are sometimes contrived with multi-point supports, which corresponds to a multi-point boundary value condition [5]. The study of multi-point BVPs for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [6]. Since then many authors have studied more general nonlinear multi-point BVPs, and

the multi-point BVP on time scales can be seen as a generalization of that in ordinary differential equations.

Recently, the existence and multiplicity of positive solutions for nonlinear differential equations on time scales have been studied by some authors [7-11], and there has been some merging of existence of positive solutions to BVPs with p -Laplacian on time scales [12-19].

He [20] studied

$$(\phi_p(u^\Delta(t)))^\nabla + a(t)f(t) = 0, \quad t \in (0, T)_{\mathbb{T}}, \quad (1.1)$$

subject to one of the following boundary conditions

$$\begin{cases} u(0) - B_0(u^\Delta(\eta)) = 0, & u^\Delta(T) = 0, \\ u^\Delta(0) = 0, & u(T) - B_1(u^\Delta(\eta)) = 0, \end{cases} \quad (1.2)$$

where $\eta \in (0, T) \cap \mathbb{T}$. By using a double fixed-point theorem, the authors get the existence of at least two positive solutions to BVP (1.1) and (1.2).

Anderson [21] studied

$$-u^{\Delta^\nabla}(t) = \eta a(t)f(u(t)), \quad t \in (t_1, t_n)_{\mathbb{T}}, \quad (1.3)$$

subject to one of the following boundary conditions

$$u(t_1) = \sum_{i=2}^{n-1} \alpha_i u(t_i), \quad u^\Delta(t_n) = 0, \quad (1.4)$$

$$u^\Delta(t_1) = 0, \quad u(t_n) = \sum_{i=2}^{n-1} \alpha_i u(t_i), \quad (1.5)$$

by using a functional-type cone expansion-compression fixed-point theorem, the author gets the existence of at least one positive solution to BVP (1.3), (1.4) and BVP (1.3), (1.5).

However, to the best of the authors' knowledge, up to now, there are few articles concerned with the existence of m -point boundary value problem with p -Laplacian on time scales. So, in this article, we try to fill this gap. Motivated by the article mentioned above, in this article, we consider the following m -point BVP with one-dimensional p -Laplacian,

$$\begin{cases} (\phi_p(u^\Delta(t)))^\nabla + h(t)f(t, u(t)) = 0, & t \in (0, T)_{\mathbb{T}}, \\ u(0) - \delta u^\Delta(0) = \sum_{i=1}^{m-2} \beta_i u^\Delta(\xi_i), & u^\Delta(T) = 0, \end{cases} \quad (1.6)$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $h \in C_{ld}((0, T), (0, +\infty))$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < T \in \mathbb{T}$, $\delta, \beta_i > 0$, $i = 1, \dots, m-2$.

We will assume throughout

(S1) $h \in C_{ld}((0, T), [0, \infty))$ such that $\int_0^T h(s) \nabla s < \infty$;

(S2) $f \in C([0, \infty), (0, \infty))$, $f \not\equiv 0$ on $f \not\equiv 0$ on $[0, T]_{\mathbb{T}}$;

(S3) By φ_q we denote the inverse to ϕ_p , where $\frac{1}{p} + \frac{1}{q} = 1$;

(S4) By $t \in [a, b]$ we mean that $t \in [a, b] \cap \mathbb{T}$, where $0 \leq a \leq b \leq T$.

2 Preliminaries

In this section, we will give some background materials on time scales.

Definition 2.1. [7,22] For $t < \sup \mathbb{T}$ and $t > \inf \mathbb{T}$, define the forward jump operator σ and the backward jump operator ρ , respectively,

$$\sigma(t) = \inf\{\tau \in \mathbb{T} | \tau > t\} \in \mathbb{T}, \quad \rho(r) = \sup\{\tau \in \mathbb{T} | \tau < r\} \in \mathbb{T}$$

for all $r, t \in \mathbb{T}$. If $\sigma(t) > t$, t is said to be *right scattered*, and if $\rho(r) < r$, r is said to be *left scattered*. If $\sigma(t) = t$, t is said to be *right dense*, and if $\rho(r) = r$, r is said to be *left dense*. If \mathbb{T} has a right scattered minimum m , define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; Otherwise set $\mathbb{T}_\kappa = \mathbb{T}$. The *backward graininess* $\mu_b : \mathbb{T}_\kappa \rightarrow \mathbb{R}_0^+$ is defined by

$$\mu_b(t) = t - \rho(t).$$

If \mathbb{T} has a left scattered maximum M , define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; Otherwise set $\mathbb{T}^\kappa = \mathbb{T}$. The *forward graininess* $\mu_f : \mathbb{T}_\kappa \rightarrow \mathbb{R}_0^+$ is defined by

$$\mu_f(t) = \sigma(t) - t.$$

Definition 2.2. [7,22] For $x : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the “ Δ ” derivative of $x(t)$, $x^\Delta(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is neighborhood U of t such that

$$| [x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s] | < \varepsilon |\sigma(t) - s|$$

for all $s \in U$. For $x : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, we define the “ ∇ ” derivative of $x(t)$, $x^\nabla(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood V of t such that

$$| [x(\rho(t)) - x(s)] - x^\nabla(t)[\rho(t) - s] | < \varepsilon |\rho(t) - s|$$

for all $s \in V$.

Definition 2.3. [22] If $F^\Delta(t) = f(t)$, then we define the “ Δ ” integral by

$$\int_a^t f(s) \Delta s = F(t) - F(a).$$

If $F^\nabla(t) = f(t)$, then we define the “ ∇ ” integral by

$$\int_a^t f(s) \nabla s = F(t) - F(a).$$

Lemma 2.1. [23] *The following formulas hold:*

- (i) $(\int_a^t f(s) \Delta s)^\Delta = f(t)$,
- (ii) $(\int_a^t f(s) \Delta s)^\nabla = f(\rho(t))$,
- (iii) $(\int_a^t f(s) \nabla s)^\Delta = f(\sigma(t))$,
- (iv) $(\int_a^t f(s) \nabla s)^\nabla = f(t)$.

Lemma 2.2. [7, Theorem 1.75 in p. 28] *If $f \in C_{rd}$ and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(\tau) \Delta \tau = \mu_f(t) f(t).$$

According to [23, Theorem 1.30 in p. 9], we have the following lemma, which can be proved easily. Here, we omit it.

Lemma 2.3. *Let $a, b \in \mathbb{T}$ and $f \in C_{ld}$.*

(i) *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_a^b f(t) \nabla t = \int_a^b f(t) dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) *If $[a, b]$ consists of only isolated points, then*

$$\int_a^b f(t) \nabla t = \begin{cases} \sum_{t \in (a, b]} \mu_b(t) f(t), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ - \sum_{t \in (a, b]} \mu_b(t) f(t), & \text{if } a > b. \end{cases}$$

(iii) *If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then*

$$\int_a^b f(t) \nabla t = \begin{cases} \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}} f(kh)h, & \text{if } a < b, \\ 0, & \text{if } a = b, \\ - \sum_{k=\frac{b}{h}+1}^{\frac{a}{h}} f(kh)h, & \text{if } a > b. \end{cases}$$

(iv) *If $\mathbb{T} = \mathbb{Z}$, then*

$$\int_a^b f(t) \nabla t = \begin{cases} \sum_{t=a+1}^b f(t), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ - \sum_{t=b+1}^a f(t), & \text{if } a > b. \end{cases}$$

In what follows, we list the fixed point theorems that will be used in this article.

Theorem 2.4. [24] *Let E be a Banach space and $P \subset E$ be a cone. Suppose $\Omega_1, \Omega_2 \subset E$ open and bounded, $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2$. Assume $A : (\overline{\Omega}_2 \setminus \Omega_1) \cap P \rightarrow P$ is completely continuous. If one of the following conditions holds*

- (i) $\|Ax\| \leq \|x\|, \forall x \in \partial\Omega_1 \cap P, \|Ax\| \geq \|x\|, \forall x \in \partial\Omega_2 \cap P;$
- (ii) $\|Ax\| \geq \|x\|, \forall x \in \partial\Omega_1 \cap P, \|Ax\| \leq \|x\|, \forall x \in \partial\Omega_2 \cap P.$

Then, A has a fixed point in $(\overline{\Omega}_2 \setminus \Omega_1) \cap P$.

Theorem 2.5. [25] *Let P be a cone in the real Banach space E . Set*

$$P(\gamma, r) = \{u \in P, \gamma(u) < r\}.$$

If α and γ are increasing, nonnegative continuous functionals on P , let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that for some positive constants r, M ,

$$\gamma(u) \leq \theta(u) \leq \alpha(u) \quad \text{and} \quad \|u\| \leq M\gamma(u).$$

for all $u \in \overline{P(\gamma, r)}$. Further, suppose there exists positive numbers $a < b < r$ such that

$$\theta(\lambda u) \leq \lambda \theta(u) \quad \text{for all } 0 \leq \lambda \leq 1, \quad u \in \partial P(\theta, b).$$

If $A : \overline{P(\gamma, r)} \rightarrow P$ is completely continuous operator satisfying

- (i) $\gamma(Au) > r$ for all $u \in \partial P(\gamma, r)$;
- (ii) $\theta(Au) < b$ for all $u \in \partial P(\theta, b)$;
- (iii) $P(\alpha, b) \neq \emptyset$ and $\alpha(Au) > a$ for all $u \in \partial P(\alpha, a)$.

Then, A has at least two fixed points u_1 and u_2 such that

$$a < \alpha(u_1), \quad \text{with } \theta(u_1) < b, \quad \text{and } b < \theta(u_2), \quad \text{with } \gamma(u_1) < r,$$

Let a, b, c be constants, $P_r = \{u \in P : \|u\| < r\}$, $P(\psi, b, d) = \{u \in P : a \leq \psi(u), \|u\| \leq b\}$.

Theorem 2.6. [26] Let $A : \overline{P_c} \rightarrow \overline{P_c}$ be a completely continuous map and ψ be a nonnegative continuous concave functional on P such that for $\forall u \in \overline{P_c}$ there holds $\psi(u) \leq \|u\|$. Suppose there exist a, b, d with $0 < a < b < d \leq c$ such that

- (i) $\{u \in P(\psi, b, d) : \psi(u) > b\} \neq \emptyset$ and $\psi(Au) > b$ for all $u \in P(\psi, b, d)$;
- (ii) $\|Au\| < a$ for all $u \in \overline{P_a}$;
- (iii) $\psi(Au) > b$ for all $u \in P(\psi, b, d)$ with $\|Au\| > d$.

Then, A has at least three fixed points u_1, u_2 , and u_3 satisfying

$$\|u_1\| < a, \quad b < \alpha(u_2), \quad \|u_3\| > a, \quad \text{and } \|u_3\| < b.$$

Let the Banach space $E = C_{ld}[0, T]$ be endowed with the norm $\|u\| = \sup_{t \in [0, T]} u(t)$, and cone $P \subset E$ is defined as

$$P = \{u \in E, u(t) \geq 0 \text{ for } t \in [0, T] \text{ and } u^{\Delta \nabla}(t) \leq 0 \text{ for } t \in (0, T), u^\Delta(T) = 0\}.$$

It is obvious that $\|u\| = u(T)$ for $u \in P$. Define $A : P \rightarrow E$ as

$$\begin{aligned} (Au)(t) = & \int_0^t \phi_q \left(\int_s^T h(\tau) f(\tau, u(\tau)) \nabla \tau \right) \Delta s + \delta \phi_q \left(\int_0^T h(s) f(s, u(s)) \nabla s \right) \\ & + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) f(s, u(s)) \nabla s \right) \end{aligned}$$

for $t \in [0, T]$.

In what follows, we give the main lemmas which are important for getting the main results.

Lemma 2.7. $A : P \rightarrow P$ is completely continuous.

Proof. First, we try to prove that $A : P \rightarrow P$.

$$(Au)^\Delta(t) = \phi_q \left(\int_t^T h(s) f(s, u(s)) \nabla s \right).$$

Thus, $(Au)^\Delta(T) = 0$ and by Lemma 2.1 we have $(Au)^{\Delta \nabla}(t) = -h(t)f(t, u(t)) \leq 0$ for $t \in (0, T)$. Consequently, $A : P \rightarrow P$.

By standard argument we can prove that A is completely continuous. For more details, see [27]. The proof is complete.

Lemma 2.8. For $u \in P$, there holds $u(t) \geq \frac{t}{T} \|u\|$ for $t \in [0, T]$ for $t \in [0, T]$.

Proof. For $u \in P$, we have $u^{\Delta \nabla}(t) \leq 0$, it follows that $u^{\Delta}(t)$ is non-increasing. Therefore, for $0 < t < T$,

$$u(t) - u(0) = \int_0^t u^{\Delta}(s) \Delta s \geq tu^{\Delta}(t) \quad (2.1)$$

and

$$u(T) - u(t) = \int_t^T u^{\Delta}(s) \Delta s \leq (T - t)u^{\Delta}(t), \quad (2.2)$$

thus

$$u(T) - u(0) \leq Tu^{\Delta}(t). \quad (2.3)$$

Combining (2.1) and (2.3) we have

$$T(u(t) - u(0)) \geq Ttu^{\Delta}(t) \geq t(u(T) - u(0)),$$

as $u(0) \geq 0$, it is immediate that

$$u(t) \geq \frac{tu(T) + (T - t)u(0)}{T} \geq \frac{t}{T}u(T) = \frac{t}{T} \|u\|.$$

The proof is complete.

3 Existence of at least one positive solution

First, we give some notations. Set

$$\begin{aligned} \Lambda &= \left(\delta + \sum_{i=1}^{m-2} \beta_i + T \right) \phi_q \left(\int_0^T h(s) \nabla s \right), \\ B &= \frac{\xi_1}{T} \left(\delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^T \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s \right). \end{aligned}$$

Theorem 3.1. Assume in addition to (S1) and (S2), the following conditions are satisfied, there exists $0 < r < \frac{\xi_1 \rho}{T} < \rho < \infty$ such that

(H1) $f(t, u) \leq \phi_p\left(\frac{u}{\Lambda}\right)$, for $t \in [0, T]$, $u \in [0, r]$;

(H2) $f(t, u) \geq \phi_p\left(\frac{u}{B}\right)$, for $t \in [\xi_1, T]$, $u \in [\frac{\xi_1 \rho}{T}, \rho]$.

Then, BVP (1.6) has at least one positive solution.

Proof. Cone P is defined as above. By Lemma 2.7 we know that $A : P \rightarrow P$ is completely continuous. Set $\Omega_r = \{u \in E, \|u\| < r\}$. In view of (H1), for $u \in \partial \Omega_r \cap P$,

$$\begin{aligned} \|Au\| &= (Au)(T) = \delta \phi_q \left(\int_0^T h(s) f(s, u(s)) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) f(s, u(s)) \nabla s \right) \\ &\quad + \int_0^T \phi_q \left(\int_s^T h(\tau) f(\tau, u(\tau)) \nabla \tau \right) \Delta s \\ &\leq \left(\delta + \sum_{i=1}^{m-2} \beta_i + T \right) \phi_q \left(\int_0^T \phi_p \left(\frac{u(s)}{\Lambda} \right) h(s) \nabla s \right) \\ &\leq \frac{\|u\|}{\Lambda} \left(\delta + \sum_{i=1}^{m-2} \beta_i + T \right) \phi_q \left(\int_0^T h(s) \nabla s \right) \leq \|u\|, \end{aligned}$$

which means that for $u \in \partial \Omega_r \cap P$, $\|Au\| \leq \|u\|$.

On the other hand, for $u \in P$, in view of Lemma 2.8, there holds $u(t) \geq \frac{\xi_1}{T} \|u\|$, for $t \in [\xi_1, T]$. Denote $\Omega_\rho = \{u \in E, \|u\| < \rho\}$. Then for $u \in \partial \Omega_\rho \cap P$, considering (H2), we have

$$\begin{aligned}
\|Au\| &= (Au)(T) = \delta \phi_q \left(\int_0^T h(s) f(s, u(s)) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) f(s, u(s)) \nabla s \right) \\
&\quad + \int_0^T \phi_q \left(\int_s^T h(\tau) f(\tau, u(\tau)) \nabla \tau \right) \Delta s \\
&\geq \delta \phi_q \left(\int_0^T \phi_p \left(\frac{u(s)}{B} \right) h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T \phi_p \left(\frac{u(s)}{B} \right) h(s) \nabla s \right) \\
&\quad + \int_0^T \phi_q \left(\int_s^T \phi_p \left(\frac{u(\tau)}{B} \right) h(\tau) \nabla \tau \right) \Delta s \\
&\geq \frac{\xi_1 \|u\|}{TB} \left(\delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^T \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s \right) \\
&= \|u\|.
\end{aligned}$$

which implies that for $u \in \partial\Omega_\rho \cap P$, $\|Au\| \geq \|u\|$. Therefore, the immediate result of Theorem 2.4 is that A has at least one fixed point $u \in (\Omega_\rho \setminus \Omega_r) \cap P$. Also, it is obvious that the fixed point of A in cone P is equivalent to the positive solution of BVP (1.6), this yields that BVP (1.6) has at least one positive solution u satisfies $r \leq \|u\| \leq \rho$. The proof is complete.

Here is an example.

Example 3.2. Let $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$. Consider the following four point BVP on time scale $\mathbb{P}_{1,1}$.

$$\begin{cases} x^{\Delta \nabla}(t) + f(t, u(t)) = 0, & t \in [0, T]_{\mathbb{T}}, \\ x(0) - 2x^{\Delta}(0) = x^{\Delta}(2) + x^{\Delta}(3), & x^{\Delta}(4) = 0, \end{cases} \quad (3.1)$$

where

$$f(t, u) = \begin{cases} \frac{tu}{128}, & 0 \leq u \leq 100, \\ \frac{128}{39t}(u - 100) + \frac{25t}{32}, & 100 \leq u \leq 500, \\ \frac{512}{16t}, & u \geq 500, \end{cases}$$

and $h(t) = 1$, $T = 4$, $\zeta_1 = 2$, $\zeta_2 = 3$, $\delta = 2$, $\beta_1 = \beta_2 = 1$, $p = q = 2$. In what follows, we try to calculate Λ , B . By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
\Lambda &= \left(\delta + \sum_{i=1}^{m-2} \beta_i + T \right) \phi_q \left(\int_0^T h(s) \nabla s \right) \\
&= (2 + 1 + 1 + 4) \int_0^4 \nabla s \\
&= 8 \times \left(\int_0^1 ds + \int_2^3 ds + \int_1^2 \nabla s + \int_3^4 \nabla s \right) \\
&= 8 \times \left(\int_0^1 ds + \int_2^3 ds + \nu(2) \times 1 + \nu(4) \times 1 \right) \\
&= 8 \times (1 + 1 + 1 + 1) = 32. \\
B &= \frac{\xi_1}{T} \left(\delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^T \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s \right) \\
&= \frac{2}{4} \left(2 \int_0^4 \nabla s + \int_2^4 \nabla s + \int_3^4 \nabla s + \int_0^4 \int_s^4 \nabla \tau \Delta s \right) \\
&= \frac{1}{2} \left(8 + 2 + 1 + \int_0^1 \int_s^4 \nabla \tau \Delta s + \int_1^2 \int_2^4 \nabla \tau \Delta s + \int_2^3 \int_s^4 \nabla \tau \Delta s + \int_3^4 \int_s^4 \nabla \tau \Delta s \right) \\
&= \frac{1}{2} \left(11 + \int_0^1 \int_s^4 \nabla \tau \Delta s + \int_1^2 \int_s^4 \nabla \tau \Delta s + \int_2^3 \int_s^4 \nabla \tau \Delta s + \int_3^4 \int_s^4 \nabla \tau \Delta s \right),
\end{aligned}$$

where

$$\begin{aligned}\int_0^1 \int_s^4 \nabla \tau \Delta s &= \int_0^1 \left(\int_s^1 \nabla \tau + \int_1^4 \nabla \tau \right) \Delta s \\ &= \int_0^1 \int_s^1 d\tau ds + \int_0^1 \left(\int_1^2 \nabla \tau + \int_2^3 \nabla \tau + \int_3^4 \nabla \tau \right) \Delta s \\ &= \int_0^1 \left(\int_s^1 d\tau + \int_2^3 d\tau \right) ds + \int_0^1 \left(\int_1^2 \nabla \tau + \int_3^4 \nabla \tau \right) \Delta s \\ &= \frac{1}{2} + 1 + 1 + 1 = \frac{7}{2}, \\ \int_2^3 \int_s^4 \nabla \tau \Delta s &= \int_2^3 \left(\int_s^3 \nabla \tau + \int_3^4 \nabla \tau \right) \Delta s \\ &= \int_2^3 \int_s^3 d\tau ds + \int_2^3 \int_3^4 \nabla \tau \Delta s \\ &= \frac{1}{2} + 1 = \frac{3}{2}, \\ \int_1^2 \int_s^4 \nabla \tau \Delta s &= \sigma(1) \times \int_1^4 \nabla \tau = 3, \\ \int_3^4 \int_s^4 \nabla \tau \Delta s &= \sigma(3) \times \int_3^4 \nabla \tau = 1.\end{aligned}$$

Thus, $B = \frac{1}{2} \left(11 + \frac{7}{2} + \frac{3}{2} + 3 + 1 \right) = 10$. Let $r = 100 < \frac{2}{4}\rho < \rho = 1000$. Then, we have

- (i) $f(t, u) \leq \frac{4u}{128} = \frac{u}{32} = \phi_p\left(\frac{u}{\lambda}\right)$, for $t \in [0, 4]$, $u \in [0, 100]$;
- (ii) $f(t, u) \geq \frac{2u}{16} = \frac{u}{8} > \phi_p\left(\frac{u}{B}\right)$, for $t \in [2, 4]$, $u \in [500, 1000]$.

Thus, if all the conditions in Theorem 3.1 satisfied, then BVP (3.1) has at least one positive solution lies between 100 and 1000.

4 Existence of at least two positive solutions

In this section, we will apply fixed point Theorem 2.5 to prove the existence of at least two positive solutions to the nonlinear BVP (1.6).

Fix $\eta \in \mathbb{T}$ such that

$$0 < \xi_{m-2} \leq \eta < T,$$

and define the increasing, nonnegative, continuous functionals γ, θ, α on P by

$$\begin{aligned}\gamma(u) &= \min_{t \in [\xi_1, \eta]} u(t) = u(\xi_1), \\ \theta(u) &= \max_{t \in [0, \xi_{m-2}]} u(t) = u(\xi_{m-2}), \\ \alpha(u) &= \min_{t \in [\eta, T]} u(t) = u(\eta).\end{aligned}$$

We can see that, for $u \in P$, there holds

$$\gamma(u) \leq \theta(u) \leq \alpha(u).$$

In addition, Lemma 2.8 implies that $\gamma(u) = u(\xi_1) \geq \frac{\xi_1}{T} \|u\|$ which means that

$$\|u\| \leq \frac{T}{\xi_1} \gamma(u) \text{ for } u \in P.$$

We also see that

$$\theta(\lambda u) = \lambda \theta(u) \quad \text{for } \lambda \in [0, 1], \quad u \in \partial P(\theta, b).$$

For convenience, we give some notations,

$$\begin{aligned} K &= \left(\delta + \sum_{i=1}^{m-2} \beta_i + \xi_{m-2} \right) \phi_q \left(\int_0^T h(s) \nabla s \right), \\ M &= \delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^{\xi_1} \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s, \\ L &= \delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^\eta \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s. \end{aligned}$$

Theorem 4.1. Assume in addition to (S1), (S2) there exist positive constants $a < \frac{T}{\eta} a < b < \frac{T}{\xi_{m-2}} b < c$ such that the following conditions hold

(H3) $f(t, u) > \phi_p(c/M)$ for $t \in [\xi_1, T]$, $u \in [c, Tc/\xi_1]$;

(H4) $f(t, u) < \phi_p(b/K)$ for $t \in [0, \xi_{m-2}]$, $u \in [b, Tb/\xi_{m-2}]$;

(H5) $f(t, u) > \phi_p(a/L)$ for $t \in [\eta, T]$, $u \in [a, Ta/\eta]$.

Then BVP (1.6) has at least two positive solutions u_1 and u_2 such that

$$\alpha(u_1) > a, \quad \text{with } \theta(u_1) < b, \quad \text{and } b < \theta(u_2), \quad \text{with } \gamma(u_2) < c. \quad (4.1)$$

Proof. From Lemma 2.7 we know that $A : P(\gamma, c) \rightarrow P$ is completely continuous. In what follows, we will prove the result step by step.

Step one: To verify (i) of theorem 2.5 holds.

We choose $u \in \partial P(\gamma, c)$, then $\gamma(u) = \min_{t \in [\xi_1, \eta]} u(t) = u(\xi_1) = c$. This implies that $u(t) \geq c$ for $t \in [\xi_1, T]$, considering that $\|u\| \leq \frac{T}{\xi_1} \gamma(u) = \frac{T}{\xi_1} c$, we have

$$c \leq u(t) \leq \frac{T}{\xi_1} c \text{ for } t \in [\xi_1, T].$$

As a consequence of (H3),

$$f(t, u(t)) > \phi_p\left(\frac{c}{M}\right) \text{ for } t \in [\xi_1, T].$$

Since $Au \in P$, we have

$$\begin{aligned} \gamma(Au) &= (Au)(\xi_1) = \delta \phi_q \left(\int_0^T h(s) f(s, u(s)) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) f(s, u(s)) \nabla s \right) \\ &\quad + \int_0^{\xi_1} \phi_q \left(\int_s^T h(\tau) f(\tau, u(\tau)) \nabla \tau \right) \Delta s \\ &> \frac{c}{M} \left(\delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^{\xi_1} \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s \right) \\ &= c. \end{aligned}$$

Thus, (i) of Theorem 2.5 is satisfied.

Step two: To verify (ii) of Theorem 2.5 holds.

Let $u \in \partial P(\theta, b)$, then $\theta(u) = \max_{t \in [0, \xi_{m-2}]} u(t) = u(\xi_{m-2}) = b$, this implies that $0 \leq u(t) \leq b$, $t \in [0, \xi_{m-2}]$ and since $u \in P$, we have $\|u\| = u(T)$, note that

$$\|u\| \leq \frac{T}{\xi_{m-2}} u(\xi_{m-2}) = \frac{T}{\xi_{m-2}} \theta(u) = \frac{T}{\xi_{m-2}} b. \text{ So,}$$

$$0 \leq u(t) \leq \frac{T}{\xi_{m-2}} b \text{ for } t \in [0, T].$$

From (H4) we know that $f(t, u(t)) < \phi_p(\frac{b}{K})$ for $t \in [0, \xi_{m-2}]$ and so

$$\begin{aligned} \theta(Au) &= (Au)(\xi_{m-2}) = \delta \phi_q \left(\int_0^T h(s) f(s, u(s)) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) f(s, u(s)) \nabla s \right) \\ &\quad + \int_0^{\xi_{m-2}} \phi_q \left(\int_s^T h(\tau) f(\tau, u(\tau)) \nabla \tau \right) \Delta s \\ &< \frac{b}{K} \left(\delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) \right. \\ &\quad \left. + \int_0^{\xi_{m-2}} \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s \right) \\ &< \frac{b}{K} \left(\delta + \sum_{i=1}^{m-2} \beta_i + \xi_{m-2} \right) \phi_q \left(\int_0^T h(s) \nabla s \right) = b. \end{aligned}$$

Thus, (ii) of Theorem 2.5 holds.

Step three: To verify (iii) of Theorem 2.5 holds.

Choose $u_0(t) = \frac{a}{2}$, $t \in [0, T]$, obviously, $u_0(t) \in P(\alpha, a)$ and $\alpha(u_0) = \frac{a}{2} < a$, thus $P(\alpha, a) \neq \emptyset$.

Now, let $u \in \partial P(\alpha, a)$, then, $\alpha(u) = \min_{t \in [\eta, T]} u(t) = u(\eta) = a$. Recalling that

$$\|u\| \leq \frac{T}{\eta} u(\eta) = \frac{T}{\eta} \alpha(u) = \frac{T}{\eta} a. \text{ Thus, we have}$$

$$a \leq u(t) \leq \frac{T}{\eta} a \text{ for } t \in [\eta, T].$$

From assumption (H5) we know that

$$f(t, u(t)) > \phi_p\left(\frac{a}{L}\right) \text{ for } t \in [\eta, T],$$

and so

$$\begin{aligned} \alpha(Au) &= (Au)(\eta) = \delta \phi_q \left(\int_0^T h(s) f(s, u(s)) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) f(s, u(s)) \nabla s \right) \\ &\quad + \int_0^{\eta} \phi_q \left(\int_s^T h(\tau) f(\tau, u(\tau)) \nabla \tau \right) \Delta s \\ &> \frac{a}{L} \left(\delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^{\eta} \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s \right) \\ &= a. \end{aligned}$$

Therefore, all the conditions of Theorem 2.5 are satisfied, thus A has at least two fixed points in $P(\gamma c)$, which implies that BVP (1.6) has at least two positive solutions u_1, u_2 which satisfies (4.1). The proof is complete.

Example 4.2. Let $\mathbb{T} = \{2^n, n \in \mathbb{Z}\} \cup \{0\}$. Consider the following four point boundary value problem on time scale \mathbb{T} .

$$\begin{cases} (\phi_p(x^\Delta))^\nabla(t) + tf(t, u(t)) = 0, & t \in [0, 8]_{\mathbb{T}}, \\ x(0) - x^\Delta(0) = x^\Delta(1) + 2x^\Delta(2), x^\Delta(8) = 0, \end{cases} \quad (4.2)$$

where

$$f(t, u) = \begin{cases} |\sin t| + \frac{u}{10^5}, & 0 \leq u \leq 9.3 \times 10^6, \\ |\sin t| + 93, & 9.3 \times 10^6 \leq u \leq 4 \times 10^8, \\ |\sin t| + \frac{247u}{6 \times 10^8} - \frac{215}{3}, & u \geq 4 \times 10^8, \end{cases}$$

and $h(t) = t$, $T = 8$, $\zeta_1 = 1$, $\zeta_2 = 2$, $\delta = 1$, $\beta_1 = 1$, $\beta_2 = 2$, $p = 3/2$, $q = 3$. In what follows, we try to calculate K , M , L . By Lemmas 2.2 and 2.3, we have

$$\begin{aligned} K &= \left(\delta + \sum_{i=1}^{m-2} \beta_i + \xi_{m-2} \right) \phi_q \left(\int_0^T h(s) \nabla s \right) \\ &= (1 + 1 + 2 + 2) \phi_q \left(\int_0^8 s \nabla s \right) \\ &= 6 \times \left(\int_0^1 s \nabla s + \int_1^2 s \nabla s + \int_2^4 s \nabla s + \int_4^8 s \nabla s \right)^2 \\ &= 6 \times (\nu(1) \times 1 + \nu(2) \times 2 + \nu(4) \times 4 + \nu(8) \times 8)^2 \\ &= 6 \times (1 + 2 + 8 + 32)^2 = 6 \times 1849 = 11094. \\ M &= \delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^{\xi_i} \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s \\ &= \left(\int_0^8 s \nabla s \right)^2 + \left(\int_1^8 s \nabla s \right)^2 + 2 \left(\int_2^8 s \nabla s \right)^2 + \int_0^1 \phi_q \left(\int_s^8 \tau \nabla \tau \right) \Delta s \\ &= (1 + 2 + 8 + 32)^2 + (2 + 8 + 32)^2 + 2 \times (8 + 32)^2 + \left(\int_0^8 s \nabla s \right)^2 \\ &= 2 * (1 + 2 + 8 + 32)^2 + (2 + 8 + 32)^2 + 2 \times (8 + 32)^2 = 8662. \\ L &= \delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^\eta \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s \\ &= \left(\int_0^8 s \nabla s \right)^2 + \left(\int_1^8 s \nabla s \right)^2 + 2 \left(\int_2^8 s \nabla s \right)^2 + \int_0^4 \left(\int_s^8 h(\tau) \nabla \tau \right)^2 \Delta s \\ &= (1 + 2 + 8 + 32)^2 + (2 + 8 + 32)^2 + 2 \times (8 + 32)^2 \\ &\quad + \int_0^1 \left(\int_s^8 h(\tau) \nabla \tau \right)^2 \Delta s + \int_1^2 \left(\int_s^8 h(\tau) \nabla \tau \right)^2 \Delta s + \int_2^4 \left(\int_s^8 h(\tau) \nabla \tau \right)^2 \Delta s \\ &= 6813 + \mu(0) \times \left(\int_0^8 s \nabla s \right)^2 + \mu(1) \times \left(\int_0^8 s \nabla s \right)^2 + \mu(2) \times \left(\int_0^8 s \nabla s \right)^2 \\ &= 6813 + (1 + 2 + 8 + 32)^2 + (2 + 8 + 32)^2 + 2 \times (8 + 32)^2 = 13626. \end{aligned}$$

Let $a = 10^6$, $b = 10^8$, $c = 10^9$, then we have

- (i) $f(t, u) \geq 340 > \left(\frac{10^9}{8662} \right)^{1/2} = \phi_p \left(\frac{c}{M} \right)$, for $t \in [1, 8]$, $u \in [10^9, 8 \times 10^9]$;
- (ii) $f(t, u) \leq 94 < \left(\frac{10^8}{11094} \right)^{1/2} = \phi_p \left(\frac{b}{K} \right)$, for $t \in [0, 2]$, $u \in [10^8, 4 \times 10^8]$;

$$(iii) \ f(t, u) > 9 > \left(\frac{10^6}{13326} \right)^{1/2} = \phi_p \left(\frac{a}{L} \right), \text{ for } t \in [4, 8], \ u \in [10^6, 2 \times 10^6].$$

Thus, if all the conditions in Theorem 4.1 are satisfied, then BVP (4.2) has at least two positive solutions satisfying (4.1).

5 Existence of at least three positive solutions

Let $\psi(u) = \min_{t \in [\xi_1, T]} u(t)$, then $0 < \psi(u) \leq \|u\|$. Denote

$$D = \delta \phi_q \left(\int_0^T h(s) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) \nabla s \right) + \int_0^{\xi_1} \phi_q \left(\int_s^T h(\tau) \nabla \tau \right) \Delta s,$$

$$R = \left(\delta + \sum_{i=1}^{m-2} \beta_i + T \right) \phi_q \left(\int_0^T h(s) \nabla s \right).$$

In this section, we will use fixed point Theorem 2.6 to get the existence of at least three positive solutions.

Theorem 5.1. Assume that there exists positive number d, v, g satisfying $d < v < \min \left\{ \frac{\xi_1}{T}, \frac{D}{R} \right\} g < g$, such that the following conditions hold.

(H6) $f(t, u) < \phi_p(d/R)$, $t \in [0, T], u \in [0, d]$;

(H7) $f(t, u) > \phi_p(v/D)$, $t \in [\xi_1, T], u \in [v, Tv/\xi_1]$;

(H8) $f(t, u) \leq \phi_p(g/R)$, $t \in [0, T], u \in [0, g]$,

then BVP (1.6) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\begin{aligned} \|u_1\| &< d, \quad \psi(u_2) > v, \\ \|u_3\| &> d, \quad \text{with } \psi(u_3) < v. \end{aligned} \tag{5.1}$$

Proof. From Lemma 2.8 we know that $A : P \rightarrow P$ is completely continuous. Now we only need to show that all the conditions in Theorem 2.6 are satisfied.

For $u \in \bar{P}_g, \|u\| \leq g$. By (H8), one has

$$\begin{aligned} \|Au\| &= (Au)(T) = \delta \phi_q \left(\int_0^T h(s) f(s, u(s)) \nabla s \right) + \sum_{i=1}^{m-2} \beta_i \phi_q \left(\int_{\xi_i}^T h(s) f(s, u(s)) \nabla s \right) \\ &\quad + \int_0^T \phi_q \left(\int_s^T h(\tau) f(\tau, u(\tau)) \nabla \tau \right) \Delta s \\ &\leq \frac{g}{R} \left(\delta + \sum_{i=1}^{m-2} \beta_i + T \right) \phi_q \left(\int_0^T h(s) \nabla s \right) \\ &= g. \end{aligned}$$

Thus, $A : \bar{P}_g \rightarrow \bar{P}_g$. Similarly, by (H6), we can prove (ii) of Theorem 2.6 is satisfied.

In what follows, we try to prove that (i) of theorem 2.6 holds. Choose

$u_1(t) = \frac{T}{\xi_1} v$, $t \in [0, T]$, obviously, $\psi(u_1) > v$, thus $\{u \in P(\psi, v, Tv/\xi_1) : \psi(u) > v\} \neq \emptyset$. For $u \in P(\psi, v, Tv/\xi_1)$,

$$\begin{aligned}\psi(Au) &= (Au)(\xi_1) = \delta\phi_q\left(\int_0^T h(s)f(s, u(s))\nabla s\right) + \sum_{i=1}^{m-2} \beta_i\phi_q\left(\int_{\xi_i}^T h(s)f(s, u(s))\nabla s\right) \\ &\quad + \int_0^{\xi_1} \phi_q\left(\int_s^T h(\tau)f(\tau, u(\tau))\nabla\tau\right)\Delta s \\ &> \frac{\nu}{D}\left(\delta\phi_q\left(\int_0^T h(s)\nabla s\right) + \sum_{i=1}^{m-2} \beta_i\phi_q\left(\int_{\xi_i}^T h(s)\nabla s\right) + \int_0^{\xi_1} \phi_q\left(\int_s^T h(\tau)\nabla\tau\right)\Delta s\right) \\ &= \nu.\end{aligned}$$

It remains to prove (iii) of Theorem 2.6 holds. For $u \in P(\psi, \nu, Tv/\zeta_1)$, with $\|Au\| > Tv/\zeta_1$, in view of Lemma 2.8, there holds $\psi(Au) - (Au)(\xi_1) \geq \frac{\xi_1}{T}\|Au\| > \nu$, which implies that (iii) of Theorem 2.6 holds.

Therefore, all the conditions in Theorem 2.6 are satisfied. Thus, BVP (1.6) has at least three positive solutions satisfying (5.1). The proof is complete.

Example 5.2. Let $\mathbb{T} = [0, 1] \cup \mathbb{N}$. Consider the following four point boundary value problem on time scale \mathbb{T} .

$$\begin{cases} (\phi_p(x^\Delta))^\nabla(t) + e^t(t, u(t)) = 0, & t \in [0, T]_{\mathbb{T}}, \\ x(0) - 3x^\Delta(0) = 2x^\Delta(1/2) + 3x^\Delta(1), & x^\Delta(8) = 0, \end{cases} \quad (5.2)$$

where

$$f(t, u) = \begin{cases} \frac{t}{20} + \left(\frac{u^2}{840}\right)^3, & 0 \leq u \leq 126, \\ \frac{t}{20} + 18.9^3, & u \geq 126, \end{cases}$$

and $h(t) = e^t$, $T = 2$, $\zeta_1 = 1/2$, $\zeta_2 = 1$, $\delta = 3$, $\beta_1 = 2$, $\beta_2 = 3$, $p = 4$, $q = 4/3$. In what follows, we try to calculate D , R . By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}D &= \delta\phi_q\left(\int_0^T h(s)\nabla s\right) + \sum_{i=1}^{m-2} \beta_i\phi_q\left(\int_{\xi_i}^T h(s)\nabla s\right) + \int_0^{\xi_1} \phi_q\left(\int_s^T h(\tau)\nabla\tau\right)\Delta s \\ &= 3\left(\int_0^2 e^s\nabla s\right)^{1/3} + 2\left(\int_{1/2}^2 e^s\nabla s\right)^{1/3} + 3\left(\int_1^2 e^s\nabla s\right)^{1/3} + \int_0^{1/2} \left(\int_s^2 e^\tau\nabla\tau\right)^{1/3}\Delta s \\ &= 3\left(\int_0^1 e^s ds + \int_1^2 e^s\nabla s\right)^{1/3} + 2\left(\int_{1/2}^1 e^s ds + \int_1^2 e^s\nabla s\right)^{1/3} + 3\left(\int_1^2 e^s\nabla s\right)^{1/3} \\ &\quad + \int_0^{1/2} \left(\int_s^1 e^\tau d\tau + \int_1^2 e^\tau\nabla\tau\right)^{1/2}\Delta s \\ &= 3(e + e^2 - 1)^{1/3} + 2(e + e^2 - e^{1/2})^{1/3} + 3e^{2/3} + \frac{3}{4}(e + e^2 - 1)^{4/3} - \frac{3}{4}(e + e^2 - e^{1/2})^{4/3} \\ &\approx 17.5216.\end{aligned}$$

$$\begin{aligned}R &= \left(\delta + \sum_{i=1}^{m-2} \beta_i + T\right)\phi_q\left(\int_0^T h(s)\nabla s\right) = (3 + 2 + 3 + 2)\left(\int_0^2 e^s\nabla s\right)^{1/3} \\ &= 10(e + e^2 - 1)^{1/3} = 20.8832.\end{aligned}$$

Let $d = 40$, $\nu = 50$, $g = 400$, then we have

- (i) $f(t, u) < 7.027 = (40/20.8832)^3 = \phi_p(d/R)$, for $t \in [0, 2]$, $u \in [0, 40]$;
- (ii) $f(t, u) > 23.2375 = (50/17.5216)^3 = \phi_p(\nu/D)$, for $t \in [1/2, 2]$, $u \in [50, 200]$;
- (iii) $f(t, u) < 7027.305 = (400/20.8832)^3 = \phi_p(g/R)$, for $t \in [0, 2]$, $u \in [0, 400]$.

Thus, if all the conditions in Theorem 5.1 are satisfied, then BVP (5.2) has at least three positive solutions satisfying (5.1).

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Authors' contributions

WG and HL conceived of the study, and participated in its coordination. JZ drafted the manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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