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Infinity of subharmonics for Duffing equations with convex and oscillatory nonlinearities

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Abstract
The existence of infinity of subharmonics for Duffing equations with convex and oscillatory nonlinearities is shown. This result is a corollary of two theorems. These theorems, one for a weak sub-quadratic potential and another for a geometric case, roughly speaking, are complementary. The approach of this paper is based on the phase-plane analysis for the time map and using the Poincaré-Birkhoff twist theorem.

Keywords: infinity of subharmonics; convex and oscillatory nonlinearity; time map; Poincaré-Birkhoff twist theorem

1 Introduction and the main results

Many problems in differential equations and dynamical systems are associated with the perturbations of an autonomous system. One usually concerns the existence of special kind of persistent solutions under perturbation. Recently, there are many researches on this subject using the assumption of 'oscillatory nonlinearity' which can be considered as a kind of nondegenerate condition at infinity in view of bifurcation; see [1, 2] and [3]. The approaches of these researches are based on the analysis of the time map for the related autonomous system combining the Poincaré-Birkhoff twist theorem for the planar homeomorphism and topological degree theory.

Consider the Duffing equation

$$x'' + g(x) = p(t), \tag{1.1}$$

where $g, p : \mathbb{R} \to \mathbb{R}$ are continuous and p 2π-least periodic. The related autonomous system of (1.1) is

$$x' = y, \qquad y' = -g(x). \tag{1.2}$$

Its energy integral is

$$\gamma_h : \quad H(x,y) \equiv \frac{1}{2}y^2 + G(x) = h,$$

where h is a parameter and G is the primitive of g.

It is easy to see that, if $\liminf_{|x|\to\infty} \operatorname{sgn}(x)g(x) > 0$, γ_h is a star-shaped closed curve for h large enough, its least period is the so-called time map, or period function,

$$\tau(h) := \int_{x_-}^{x_+} \frac{ds}{\sqrt{h - G(s)}},$$

where $x_- = G_-^{-1}(h) < 0 < x_+ = G_+^{-1}(h)$ with G_-^{-1}, G_+^{-1} the left and right inverse of G, respectively.

The condition called 'oscillatory nonlinearity' can be expressed in terms of the time map as

$$(\tau_0): \quad \Delta\tau = \tau^* - \tau_* := \limsup_{h\to+\infty} \tau(h) - \liminf_{h\to+\infty} \tau(h) > 0.$$

In the study of bifurcations, under a small forced perturbation, the existence of persistent periodic solutions usually needs a setting of nondegenerate conditions which include the nonvanishing of the derivative of the period function (see, for instance, Chicone [4]). Thus, the condition (τ_0) can be interpreted reasonably as a kind of nondegenerate condition at infinity for 'global persistence' under 'large force'. We can understand (τ_0) in a generalized meaning entailing that

$$\limsup_{h\to+\infty} \tau(h) = +\infty \quad \text{or} \quad \liminf_{h\to+\infty} \tau(h) = +\infty.$$

Besides (τ_0), some growth condition on $g(x)$ is needed to guarantee the existence of 2π-periodic solution and infinitely many subharmonics.

In [1], Ding, Iannacci, and Zanolin showed the existence of 2π-periodic solution and infinitely many subharmonics under the assumptions that $g(x)$ is globally Lipschitz and has linear growth at infinity, that is,

$$0 < \liminf_{|x|\to+\infty} \frac{g(x)}{x} \le \limsup_{|x|\to+\infty} \frac{g(x)}{x} < +\infty. \tag{1.3}$$

One of the authors generalized the above result in [2] by dropping the globally Lipschitz property and assuming the following growth conditions on $g(x)$.

$$(g_0): \quad \lim_{|x|\to\infty} \operatorname{sgn}(x)g(x) = +\infty;$$

$$(G_0): \quad \limsup_{|x|\to+\infty} \frac{G(x)}{g^2(x)} < +\infty.$$

Later, Capietto, Mawhin, and Zanolin weakened the condition (G_0) in [3] to the following geometric one

$$(G_0)': \quad \forall c_1 > 0, \exists c_2 > 0 \text{ such that}$$
$$AB > 0 \quad \& \quad \left|\sqrt{G(B)} - \sqrt{G(A)}\right| < c_1 \quad \Longrightarrow \quad |B - A| < c_2.$$

They showed the existence of 2π-periodic solution by developing a new continuation theorem with two functionals. Wang showed in [5] the existence of infinitely many subharmonic solutions under the same assumptions as in [3].

In this paper, we consider the Duffing equation (1.1) with *convex* potential. More precisely, we assume

$$(g_1): \quad \liminf_{|x|\to\infty}\Big(\operatorname{sgn}(x)g(x) - \max_{t\in\mathbb{R}}\big|p(t)\big|\Big) > 0;$$

$$(g_2): \quad g(x) \text{ is nondecreasing for } |x| \geq d, \text{where } d \text{ is a constant.}$$

We will prove that the convexity of the potential and the oscillatory nonlinearity imply that the existence of the infinity of persistent subharmonic solutions, that is, we have the following.

Theorem 1.1 *Assume that* (g_1), (g_2) *and* (τ_0). *Then equation* (1.1) *has at least one* 2π*-periodic solution and infinitely many subharmonic solutions* $x_k(t)$ *with minimal period* $2k\pi$. *Moreover,*

$$\lim_{k\to+\infty}\min_{t\in\mathbb{R}}\big(\big|x_k(t)\big| + \big|x_k'(t)\big|\big) = +\infty.$$

This theorem is a corollary of two theorems. One is in [2] and the other one is the following theorem. It considers the equation with so-called 'weak sub-quadratic potential' (see the condition (G_1) in the following).

Theorem 1.2 *Assume* (g_1) *and*

$$(G_1): \quad \liminf_{x\to+\infty}\frac{G(x)}{x^2} = 0 \quad \textit{or} \quad \liminf_{x\to-\infty}\frac{G(x)}{x^2} = 0.$$

Then equation (1.1) *has at least one* 2π*-periodic solution, and for each* $j\in\mathbb{N}$ *there is* $m_j^*\in\mathbb{N}$, *such that for every* $k\geq m_j^*$ *with k primes with j, there is at least one periodic solution* $x_k(\cdot) = x_{j,k}(\cdot)$ *with minimal period* $2k\pi$. *Moreover,* $x_{j,k}(\cdot)$ *has exactly* $2j$ *zeros in the interval* $[0, 2k\pi)$ *and satisfies*

$$\lim_{k\to+\infty}\min_{t\in\mathbb{R}}\big(\big|x_{j,k}(t)\big| + \big|x_{j,k}'(t)\big|\big) = +\infty.$$

Theorem 1.2 is meaningful itself. We recall that the condition (G_1) is firstly introduced in [6] to guarantee the existence of 2π-periodic solution. Moreover, Ding and Zanolin introduced in [7] the following sub-linear condition on the time map to guarantee the existence of infinitely many subharmonic solutions.

$$(\tau_1): \quad \lim_{h\to+\infty}\tau^+(h) = +\infty \quad \text{or} \quad \lim_{h\to+\infty}\tau^-(h) = +\infty,$$

where $\tau^{\pm}(h) := |\int_0^{x_\pm}\frac{ds}{\sqrt{h-G(s)}}|$.

If we consider the forced equation (1.1), we can prove that $(\tau_1)\Rightarrow(G_1)$.

Actually, if the assumption (G_1) is false, then we have

$$(G_2): \quad \text{There is } a>0 \text{ such that } \quad \liminf_{|x|\to+\infty}\frac{G(x)}{x^2} \geq a > 0,$$

which implies that

$$\lim_{|x|\to+\infty}\left(G(x)-\frac{a}{2}x^2\right)=+\infty,$$

and thus there are sequences $\{x_m^\pm\}$ with $x_m^\pm \to \pm\infty$ as $m\to\infty$, such that

$$G(x_m^+)-G(s)\geq \frac{a}{2}\left((x_m^+)^2-s^2\right),\quad \text{for } 0\leq s\leq x_m^+,$$

and

$$G(x_m^-)-G(s)\geq \frac{a}{2}\left((x_m^-)^2-s^2\right),\quad \text{for } x_m^-\leq s\leq 0.$$

Then

$$\tau^\pm(h_m^\pm)=\left|\int_0^{x_m^\pm}\frac{ds}{\sqrt{2(G(x_m^\pm)-G(s))}}\right|\leq\left|\frac{1}{\sqrt{a}}\int_0^{x_m^\pm}\frac{ds}{\sqrt{(x_m^\pm)^2-s^2}}\right|\leq\frac{\pi}{2a},$$

where $h_m^\pm = G(x_m^\pm)$, which contradicts (τ_1).

Therefore Theorem 1.2 is a generalization of the results both in [6] and in [7].

Now we prove Theorem 1.1. Assume that $G(x)$ is convex, then $G(x)\leq xg(x)$. (G_2) implies that $G(x)\geq\frac{a}{2}x^2$, it follows that $g(x)\geq\frac{a}{2}x$ for $|x|\gg 1$. Therefore $\frac{G(x)}{g^2(x)}\leq\frac{x}{g(x)}\leq\frac{2}{a}$ for $|x|\gg 1$. Thus (G_0) holds. Hence under the assumption (g_1) and (g_2) we see that either (G_1) or (g_0) and (G_0) hold. Then Theorem 1.1 is a corollary of Theorem 1.2 and Theorem A in [2].

The key point of our argument is the estimation of the return time when the solution completes j clock-wise turns around the origin in the phase plane for given j under a forced perturbation.

In the rest of the paper we will give the details of the proof for Theorem 1.2.

2 Duffing equation with weak sub-quadratic potential

Rewrite equation (1.1) as the following equivalent form:

$$x'=y,\qquad y'=-g(x)+p(t). \tag{2.1}$$

We will apply the Poinacré-Birkhoff twist theorem to obtain the existence of the $2k\pi$-periodic solution for equation (2.1) under the assumption of a weak sub-quadratic potential. The existence of a 2π-periodic solution then follows from the Massera theorem.

The key point of applying the Poincaré-Birkhoff twist theorem is constructing an annulus $\mathcal{A}$ in the (x,y) phase plane bounded by two star-shaped curves γ^+ and γ^-, such that the solution starting from γ^+ and γ^- will move more than and less than j clock-wise turns in the time interval $[t_0, t_0+2k\pi]$, respectively.

When the solution passes through the origin at some time, we cannot compute how many clock-wise turns it moves. To avoid this problem, we introduce a modified equation such that if $z(t)=(x(t),y(t))$ is a solution of the modified equation, then $z(t_0)\neq(0,0)$ for some t_0 implies that $z(t)\neq(0,0)$ for all t. Moreover, the twist property of the fixed point obtained by using the Poinacré-Birkhoff twist theorem will help us to guarantee that the $2k\pi$-periodic solution we showed for the modified equation is exactly the $2k\pi$-periodic solution of the original equation (2.1) for sufficiently large k.

Denote by $P = \max\{|p(t)| \mid t \in [0,2\pi]\}$ and $\mathcal{B}_{\mu_0}(g)$ the set of all the continuously differentiable functions $f(x)$ satisfying

$$\bigl|f(x) - g(x)\bigr| \le \mu_0, \quad \text{for all } x \in \mathbb{R},$$

where μ_0 is sufficient small such that $\liminf_{|x|\to\infty} \operatorname{sgn}(x)g(x) > \mu_0 + P$.

We consider the following modified equation:

$$x' = \frac{\partial H(t,x,y)}{\partial y}, \qquad y' = -\frac{\partial H(t,x,y)}{\partial y}, \tag{2.2}$$

where $H(t,x,y) = \frac{y^2}{2} + K(x^2+y^2)[F(x) - xp(t)] + (1 - K(x^2+y^2))x^2$, $F(x) = \int_0^x f(\xi)\,d\xi$ with $f \in \mathcal{B}_{\mu_0}(g)$, and $K \in C^\infty(\mathbb{R})$ satisfies that

$$K\bigl(x^2+y^2\bigr) = \begin{cases} 0, & 0 \le x^2+y^2 \le R_1^2, \\ 1, & x^2+y^2 \ge R_2^2, \end{cases}$$

with some given constants $R_2 > R_1 > 0$.

Let $z(t;t_0,z_0) = (x(t;t_0,z_0), y(t;t_0,z_0))$ be the solution of (2.2) satisfying the initial condition $z(t_0;t_0,z_0) = z_0 = (x_0,y_0)$. Then we have the following fundamental lemma.

Lemma 2.1 *Assume that g satisfies* (g_1) *and* $f \in \mathcal{B}_{\mu_0}(g)$ *for sufficiently small* μ_0. *Then each initial value problem associated with* (2.2) *has a unique solution* $z(t;t_0,z_0) = (x(t;t_0,z_0), y(t;t_0,z_0))$ *which is well defined for* $t \in \mathbb{R}$ *and* $z_0 \ne (0,0)$ *implying that* $z(t;t_0,z_0) \ne (0,0)$ *for all t. Moreover, for any* $L > 0$, *there is* $E = E(L) > 0$ *such that*

$$\bigl|l(t) - l(t_0)\bigr| \le E \quad \text{for } |t-t_0| \le L \text{ and } |z_0| \gg 1,$$

where

$$l(t) = \sqrt{\frac{y^2(t;t_0,z_0)}{2} + G\bigl(x(t;t_0,z_0)\bigr)} \quad \text{for } t > t_0.$$

Proof f is a continuously differentiable function. The existence and uniqueness of the solution associated to the initial condition is ensured by the existence-uniqueness theorem. Moreover, the solution has continuity with respect to initial conditions. It is easy to see that $z(t) \equiv (0,0)$ is the solution of (2.2) satisfying the initial condition $z(t_0;t_0,z_0) = (0,0)$, from which it follows that $z_0 \ne (0,0) \Rightarrow z(t;t_0,z_0) \ne (0,0)$, $\forall t$.

Note that equation (2.2) is the same as the equation

$$x' = y, \qquad y' = -f(x) + p(t)$$

for $x^2 + y^2 \ge R_2^2$. Then we have, for $|z(t;t_0,z_0)| \gg 1$,

$$\begin{aligned} \bigl|l'(t)\bigr| &= \frac{1}{2l(t)}\bigl|y(t)y'(t) + g\bigl(x(t)\bigr)x'(t)\bigr| = \frac{1}{2l(t)}\bigl|y(t)\bigl(g\bigl(x(t)\bigr) - f\bigl(x(t)\bigr) + p(t)\bigr)\bigr| \\ &\le \frac{1}{2l(t)}|P + \mu_0|\bigl|y(t)\bigr| \le |P + \mu_0|. \end{aligned}$$

An obvious induction shows that for any $L > 0$, there is $E = E(L) > 0$ such that

$$\bigl|l(t) - l(t_0)\bigr| \le E \quad \text{for } |t - t_0| \le L \text{ and } |z_0| \gg 1.$$

Moreover, $l(t) \to +\infty \Leftrightarrow |z(t)| \to +\infty$. Then the above estimation implies that the global existence of the solution of (2.2) is a consequence of the continuation theorem. The lemma is thus proved. □

If $z_0 \neq (0,0)$ we can represent the solution $z(t;t_0,z_0)$ by means of polar coordinates as

$$r(t;t_0,r_0,\theta_0) = \bigl|z(t;t_0,z_0)\bigr|, \qquad \theta(t;t_0,r_0,\theta_0) = \operatorname{Arg}\bigl(z(t;t_0,z_0)\bigr),$$

where $|\cdot|$ is Euclidean norm of $\mathbb{R}^2$. Sometimes we write $r(t;t_0,z_0) = |z(t;t_0,z_0)|$ and $\theta(t;t_0,z_0) = \operatorname{Arg}(z(t;t_0,z_0))$ for ease of notation.

Moreover, we have the following.

Lemma 2.2 *Assume that the conditions of Lemma* 2.1 *hold. Then there is $R_0 > R_2 > R_1 > 0$, such that $\theta'(t;t_0,r_0,\theta_0) < 0$ whenever $r(t;t_0,r_0,\theta_0) \ge R_0$.*

Proof From (g_1) and $\liminf_{|x|\to\infty} \operatorname{sgn}(x)g(x) > \mu_0 + P$ we have $d_1 > 0$ such that $x(f(x) - p(t)) > 0$ for $|x| \ge d_1$ and $t \in \mathbb{R}$. For $|z(t)| \ge R_2$, we have

$$\theta'(t) = -\frac{1}{r^2(t)}\bigl(x(t)y'(t) - y(t)x'(t)\bigr) = -\frac{1}{r^2(t)}\bigl(y^2(t) + \bigl(f\bigl(x(t)\bigr) - p(t)\bigr)x(t)\bigr).$$

Let R_0 be a constant such that $R_0 \ge 2\max\{d_1, R_2, \max\{|g(x)| : |x| \le d_1\} + \mu_0\}$. Then for $r(t) \ge R_0$, either $|x(t)| \ge d_1$, and we have $y^2(t) + (f(x(t)) - p(t))x(t) > 0$, or $|x(t)| < d_1$, which implies $|y(t)| > |x(t)|$ and $|y(t)| > 1/2r(t) \ge 1/2R_0$, then $y^2(t) > -(f(x(t)) - p(t))x(t)$. Hence $\theta'(t) < 0$ for $r(t) \ge R_0$.

Lemma 2.2 concludes that $z(t;t_0,z_0)$ moves clock-wise around the origin O if $|z(t;t_0,z_0)| \ge R_0$. □

Lemma 2.3 *Assume that the conditions of Lemma* 2.1 *hold. Then for any solution $z(t;t_0,z_0)$ of* (2.2) *with $z_0 \neq (0,0)$, we have*

$$\theta(t_2;t_0,r_0,\theta_0) - \theta(t_1;t_0,r_0,\theta_0) < \pi, \quad \forall t_2 > t_1.$$

Proof Note that in equation (2.2) $yx' = y^2 > 0$ whenever $x = 0$ and $y \neq 0$. Then for any solution $z(t;t_0,z_0)$ of (2.2) with $z_0 \neq (0,0)$, we have $\theta'(t;t_0,r_0,\theta_0) < 0$ whenever $\theta(t;t_0,r_0,\theta_0) = k\pi + 1/2\pi$, $k \in \mathbb{Z}$. Thus Lemma 2.3 in [7] shows that we have $\theta(t_2;t_0,r_0,\theta_0) - \theta(t_1;t_0,r_0,\theta_0) < \pi$ for all $t_2 > t_1$. This proves the lemma. □

The following lemma is similar to that in [7] with the modified proof as in [7].

Lemma 2.4 *Assume that the conditions of Lemma* 2.1 *hold. Then for any constant $j \in \mathbb{N}$ there is $K_j > R_0$ and there is a continuous increasing function $\zeta = \zeta_j : [K_j, +\infty) \to \mathbb{R}^+$, with*

$\zeta_j(s) > s$, $\forall s \geq K_j$ *such that the following inference holds for each* $z(t)$ *as the solution of* (2.2):

$$\begin{aligned}
&\bigl|z(t_0)\bigr| \leq r, \bigl|z(t_1)\bigr| \geq \zeta_j(r) \text{ or } \bigl|z(t_0)\bigr| \geq \zeta_j(r), \bigl|z(t_1)\bigr| \leq r \text{ whenever } r \geq K_j \\
&\text{and } \bigl|z(t)\bigr| \geq R_0, \forall t \in [t_0, t_1] \quad \Longrightarrow \\
&z(t) \text{ moves at least } j \text{ clock-wise turns on } [t_0, t_1] \text{ around the origin } O.
\end{aligned}$$

Proof Let $z(t)$ be a solution of (2.2) and suppose that $z(t) \geq R_0$ for each $t \in I$, with I an interval. Denote

$$\begin{aligned}
&H_{\pm}(x,y) = \frac{y^2}{2} + G(x) \pm (\mu_0 + P)x, \qquad E_1 = \bigl\{(x,y) : y \geq 0\bigr\}, \\
&H_{\pm}(t) = \frac{y^2(t)}{2} + G\bigl(x(t)\bigr) \pm (\mu_0 + P)x(t), \quad \text{with } z(t) = \bigl(x(t), y(t)\bigr).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{d}{dt}H_+(t) = y(t)\bigl(g\bigl(x(t)\bigr) - f\bigl(x(t)\bigr) + p(t) + \mu_0 + P\bigr) \geq 0; \\
&\frac{d}{dt}H_-(t) = y(t)\bigl(g\bigl(x(t)\bigr) - f\bigl(x(t)\bigr) + p(t) - \mu_0 - P\bigr) \leq 0,
\end{aligned}$$

for $z(t) \in E_1$. Moreover, let $I_1 \subset I$ be a nondegenerate interval such that $z(t) \in E_1$ for all $t \in I_1$. Then

$$H_+(t) \geq H_+(s), \qquad H_-(t) \leq H_-(s), \quad \text{for } \forall t, s \in I_1, t > s. \tag{2.3}$$

From the assumption (g_1), the definition of f and μ_0 sufficiently small, we have for $z = (x, y)$,

$$H_{\pm}(x,y) \to +\infty \quad \Leftrightarrow \quad r = |z| \to +\infty.$$

Then for r sufficiently large, using similar argument as in [7], Lemma 2.7, we have two continuous increasing functions $L_{\pm} : [r_1, +\infty)$ with $r_1 > R_0$ sufficiently large, such that

$$L_-\bigl(|z|\bigr) \leq H_{\pm}(x,y) \leq L_+\bigl(|z|\bigr), \quad \text{for } z = (x,y) \in E_1,$$

and

$$L_{\pm}(r) \to +\infty, \quad \text{for } r \to +\infty.$$

Without loss of generality, we suppose that $L_-(r) < r < L_+(r)$. Using (2.3) we have

$$L_+\bigl(r(t)\bigr) \geq L_-\bigl(r(s)\bigr), \qquad L_-\bigl(r(t)\bigr) \leq L_+\bigl(r(s)\bigr), \quad \text{for } \forall t, s \in I_1, t > s,$$

from which it follows that

$$(\xi_1)^{-1}\bigl(r(s)\bigr) \leq r(t) \leq \xi_1\bigl(r(s)\bigr), \quad \text{for } \forall t, s \in I_1, t > s, \tag{2.4}$$

where $\xi_1(r) = (L_-)^{-1}(L_+(r))$ and $(\xi_1)^{-1}(r) = (L_+)^{-1}(L_-(r))$ with $r < \xi_1(r)$ and $(\xi_1)^{-1}(r) \to +\infty$ for $r \to +\infty$.

Denote by $E_2 = \{(x,y) : y \le 0\}$ and $I_2 \subset I$ be a nondegenerate interval such that $z(t) \in E_2$ for all $t \in I_2$. We can use a similar argument to above to obtain a continuous increasing function $\xi_2 : [r_2, +\infty)$ with $r_2 > R_0$ sufficiently large, such that

$$(\xi_2)^{-1}\bigl(r(s)\bigr) \le r(t) \le \xi_2\bigl(r(s)\bigr), \quad \text{for } \forall t, s \in I_2, t > s. \tag{2.5}$$

Moreover, $r < \xi_2(r)$ and $(\xi_2)^{-1}(r) \to +\infty$ for $r \to +\infty$.

Choose a continuous increasing function $\eta(r) \ge \max\{\xi_i(r), i = 1,2\}$ with $(\eta)^{-1}(r) \to +\infty$ for $r \to +\infty$. Denote $\zeta_1(r) = \eta \circ \eta \circ \eta(r)$. Let K_1 be large enough such that $K_1 \ge \zeta_1(R_0)$. Thus (2.4) and (2.5) show that if $|z(t_0)| \ge K_1$, then $\zeta_1^{-1}(|z(t_0)|) \le |z(t)| \le \zeta_1(|z(t_0)|)$ before $z(t)$ completes 1 clock-wise turn around the origin O. That is, if $|z(t_0)| \ge K_1$, $\theta(t) \ge \theta(t_0) - 2\pi$ and $|z(t)| \ge R_0$, $\forall t \in [t_0, t_1]$ then $\zeta_1^{-1}(|z(t_0)|) \le |z(t_1)| \le \zeta_1(|z(t_0)|)$. In other words, for $r \ge K_1$,

$$\begin{aligned}
&|z(t_0)| \le r, |z(t_1)| \ge \zeta_1(r) \text{ and } |z(t)| \ge R_0, \forall t \in [t_0, t_1] \\
&\text{or } |z(t_0)| \ge \zeta_1(r), |z(t_1)| \le r \text{ and } |z(t)| \ge R_0, \forall t \in [t_0, t_1] \quad \Longrightarrow \\
&z(t) \text{ moves at least one clock-wise turn on } [t_0, t_1] \text{ around the origin } O.
\end{aligned}$$

The lemma is thus proved by induction. □

Let $T_j(t_0, z_0)$ denote the minimum time in which the solution $z(t; t_0, z_0)$ of (2.2) completes j clock-wise turns, $j \in \mathbb{N}$. In the proof of the next lemma we will use Lemma 2.4 to show that $T_j(t_0, z_0)$ is well defined for sufficiently large z_0. Let $T_j^+(t_0, z_0)$ denote the minimum time such that if $T \ge T_j^+(t_0, z_0)$ then the solution $z(t; t_0, z_0)$ of (2.2) completes at least j clock-wise turns in $[t_0, t_0 + T]$. Moreover, denote

$$\begin{aligned}
T_j^-(h) &= \inf\bigl\{T_j(t_0, z_0) \mid \forall z_0 \in \gamma_h, \forall f \in \mathcal{B}_{\mu_0}, \forall t_0\bigr\}; \\
T_j^+(h) &= \sup\bigl\{T_j^+(t_0, z_0) \mid \forall z_0 \in \gamma_h, \forall f \in \mathcal{B}_{\mu_0}, \forall t_0\bigr\}.
\end{aligned}$$

Then we have the following estimations of the twist properties for the solutions of (2.2).

Lemma 2.5 *Assume that the conditions of Lemma* 2.1 *hold. Then there exists a sequence* $\{h_m^-\}$ *with* $\lim_{m\to\infty} h_m^- = +\infty$ *such that* $T_j^+(h_m^-) < +\infty$. *Moreover, if* (G_1) *holds, then there exists another sequence* $\{h_m^+\}$ *with* $\lim_{m\to\infty} h_m^+ = +\infty$ *such that* $\lim_{m\to\infty} T_j^-(h_m^+) = +\infty$.

Proof For sufficiently large h, let

$$\begin{aligned}
r^+(h) &= \max\bigl\{\sqrt{x^2 + y^2} : 1/2y^2 + G(x) = h\bigr\}, \\
r^-(h) &= \min\bigl\{\sqrt{x^2 + y^2} : 1/2y^2 + G(x) = h\bigr\}.
\end{aligned}$$

Then for for any $j \in \mathbb{N}$ and $m \in \mathbb{N}$, let $r^{(m)} \ge \max\{m, K_{j+1}\}$ and choose h_m^- sufficiently large such that $r^-(h_m^-) \ge \zeta_{j+1}(r^{(m)})$. Let

$$a = \inf\bigl\{-\theta'(t; t_0, z_0) : z_0 \in \gamma_{h_m^-}, r^{(m)} \le \bigl|z(t; t_0, z_0)\bigr| \le \zeta_{j+1}\bigl(r^+\bigl(h_m^-\bigr)\bigr)\bigr\}.$$

It is easy to see that

$$a \geq \frac{\inf\{xg(x) - (\mu_0 + P)|x| + y^2 : x^2 + y^2 \geq (r^{(m)})^2\}}{(\zeta_{j+1}(r^+(h_m^-)))^2} > 0.$$

Denote by $L_m = \frac{2(j+1)\pi}{a}$. For any solution $z(t;t_0,z_0)$, $z_0 \in \gamma_{h_m^-}$, there are two cases as follows.

(1) If $r^{(m)} \leq |z(t;t_0,z_0)| \leq \zeta_{j+1}(r^+(h_m^-))$ for $t \in [t_0, t_0 + L_m]$, then

$$\theta(t_0 + L_m; t_0, z_0) - \theta_0 = \int_{t_0}^{t_0+L_m} \theta'(t)\,dt \leq -aL_m \leq -2(j+1)\pi.$$

(2) If there exists a time $t_1' \in [t_0, t_0 + L_m)$ such that $|z(t_1';t_0,z_0)| < r^{(m)}$ or $|z(t_1';t_0,z_0)| > \zeta_{j+1}(r^+(h_m^-))$, then we have $t_1 \in [t_0, t_1')$ such that $|z(t;t_0,z_0)| \geq R_0$ for $t \in [t_0, t_1]$ and

$$\bigl|z(t_1;t_0,z_0)\bigr| < r^{(m)}, \qquad |z_0| \geq r^-\bigl(h_m^-\bigr) \geq \zeta_{j+1}\bigl(r^{(m)}\bigr),$$

or

$$\bigl|z(t_1;t_0,z_0)\bigr| > \zeta_{j+1}\bigl(r^+\bigl(h_m^-\bigr)\bigr), \qquad |z_0| \leq r^+\bigl(h_m^-\bigr).$$

According to Lemma 2.4, it follows that $z(t;t_0,z_0)$ completes at least $j+1$ clock-wise turns in $[t_0, t_1]$. That is,

$$\theta(t_1;t_0,z_0) - \theta_0 \leq -2(j+1)\pi,$$

which implies that

$$\begin{aligned}\theta(t_0 + L_m; t_0, z_0) - \theta_0 &= \theta(t_0 + L_m; t_0, z_0) - \theta(t_1;t_0,z_0) + \theta(t_1;t_0,z_0) - \theta_0 \\ &< \pi - 2(j+1)\pi = -(2j+1)\pi\end{aligned}$$

by using Lemma 2.3.

Hence for both cases, we show that the solution $z(t;t_0,z_0)$, $z_0 \in \gamma_{h_m^-}$ completes at least j clock-wise turns in $[t_0, t_0 + L_m]$. Moreover, the above argument shows that for $T \geq L_m$, the solution $z(t;t_0,z_0)$, $z_0 \in \gamma_{h_m^-}$ completes at least j clock-wise turns in $[t_0, t_0 + T]$, from which it follows that $T_j^+(h_m^-) \leq L_m < +\infty$.

By the way, using the same method as employed above we can prove that if $|z_0| \geq \zeta_{j+1}(K_{j+1})$ then $T_j(t_0,z_0)$ is well defined.

On the other hand, from the assumption (G_1), we have a sequence $\{x_m\}$ with $x_m \to +\infty$ as $m \to \infty$, such that $\lim_{m\to+\infty} \frac{G(x_m)}{x_m^2} = 0$. For any given $L > 0$, choose m_0, such that

$$\frac{G(x_m)}{x_m^2} \leq \frac{1}{9L^2}, \quad \text{and} \quad \sqrt{G(x_m)} \geq 2E, \quad \text{for } m \geq m_0,$$

where $E = E(L)$ is defined as in Lemma 2.1.

Consider the solution $z(t;t_0,z_0)$ starting from $z_0 \in \gamma_{h_m^+}$, where

$$h_m^+ = \left(\sqrt{G(x_m)} + E\right)^2.$$

If $T_j(t_0,z_0) \le L$, then Lemma 2.1 implies that

$$G(x_m) \le \frac{1}{2}y^2(t;t_0,z_0) + G\bigl(x(t;t_0,z_0)\bigr) \le \left(\sqrt{h_m^+} + E\right)^2, \quad \text{for } t_0 \le t \le t_0 + T_j(t_0,z_0),$$

which implies that

$$\frac{|x'(t;t_0,z_0)|}{\sqrt{2((\sqrt{h_m^+}+E)^2 - G(x(t;t_0,z_0)))}} \le 1, \quad \text{for } t_0 \le t \le t_0 + T_j(t_0,z_0).$$

Note that the solution $z(t;t_0,z_0)$, $z_0 \in \gamma_{h_m^+}$, completes at least j clock-wise turns around the origin O in $[t_0, t_0 + T_j(t_0,z_0)]$, then it intersects with the x-axis and y-axis which implies that there exist $t_1, t_2 \in [t_0, t_0 + T_j(t_0,z_0)]$ such that

$$x(t_1) = 0, \qquad y(t_2) = 0 \quad \text{and} \quad x'(t) = y(t) > 0 \quad \text{for } t \in (t_1,t_2)$$

or

$$y(t_1) = 0, \qquad x(t_2) = 0 \quad \text{and} \quad x'(t) = y(t) < 0 \quad \text{for } t \in (t_1,t_2).$$

If $y(t_2) = 0$ then $G(x(t_2)) \ge G(x_m)$, which implies $x(t_2) \ge x_m$. If $y(t_1) = 0$ then $G(x(t_1)) \ge G(x_m)$, which implies $x(t_1) \ge x_m$. In both cases we have

$$\bigl|x(t_2) - x(t_1)\bigr| \ge x_m - 0.$$

Therefore

$$\begin{aligned} T_j(t_0,z_0) &= \int_{t_0}^{t_0+T_j(t_0,z_0)} dt \ge \int_{t_1}^{t_2} dt \ge \left| \int_{x(t_1)}^{x(t_2)} \frac{ds}{\sqrt{2((\sqrt{h_m^+}+E)^2 - G(s))}} \right| \\ &\ge \int_0^{x_m} \frac{ds}{\sqrt{2((\sqrt{h_m^+}+E)^2 - G(s))}} \ge \frac{1}{\sqrt{2}} \frac{x_m}{\sqrt{G(x_m)}} \cdot \frac{\sqrt{G(x_m)}}{\sqrt{h_m^+}+E} \\ &= \frac{1}{\sqrt{2}} \cdot \frac{x_m}{\sqrt{G(x_m)}} \cdot \left(1 + \frac{2E}{\sqrt{G(x_m)}}\right)^{-1} \ge \frac{3L}{2\sqrt{2}} > L. \end{aligned}$$

This is a contradiction. Hence we have proved that $T_j(t_0,z_0) > L$ for $z_0 \in \gamma_{h_m^+}$ and $f \in \mathcal{B}_{\mu_0}$, which implies that $T_j^-(h_m^+) \ge L$. Since L is arbitrary we have

$$T_j^-\bigl(h_m^+\bigr) \to +\infty \quad \text{as } m \to \infty.$$

The lemma is thus proved. □

Now we are in the position to prove Theorem 1.2.

Proof of Theorem 1.2 It follows from Lemma 2.5 that there exist sequences $\{h_l^-\}$ and $\{h_l^+\}$ such that $\lim_{l\to+\infty} h_l^- = +\infty$, $\lim_{l\to+\infty} h_l^+ = +\infty$, $T_j^+(h_l^-) < +\infty$, and $\lim_{l\to+\infty} T_j^-(h_l^+) = +\infty$.

Let h_m^- be large enough such that $r^-(h_m^-) \geq K_{j+1}$, $h_m^- \geq \zeta_{j+1}(R_0)$ and $T_j(0,z_0)$ is well defined for every solution $z(t;z_0) = (x(t;x_0,y_0),y(t;x_0,y_0))$ starting from $z_0 \in \gamma_{h_m^-}$ at $t = 0$. Then there is $m_j^* \in \mathbb{N}$, such that, for every $k \geq m_j^*$, where k primes with j, we can choose $l = l(m,k)$ sufficiently large, such that $h_m^- < h_l^+$ and

$$T_j^+(h_m^-) < 2k\pi < T_j^-(h_l^+). \tag{2.6}$$

Denote by $\mathcal{A}_{m,l}$ the annulus bounded by $\gamma_{h_m^-}$ and $\gamma_{h_l^+}$. Consider the Poincaré map

$$\mathcal{P}_k: \quad (x_0,y_0) \mapsto \bigl(x(2k\pi;x_0,y_0),y(2k\pi;x_0,y_0)\bigr).$$

The uniqueness of the solution to initial value problems for equation (2.2) guarantees that $\mathcal{P}_k$ is an area-preserving homeomorphism such that $\mathcal{P}_k(O) = O$. Moreover, $\mathcal{A}_{m,l}$ are annuli bounded by strictly star-shaped Jordan curves around the origin and the inequality (2.6) implies that $\mathcal{P}_k$ is boundary twisting on the annulus $\mathcal{A}_{m,l}$, that is,

$$\theta(2k\pi;t_0,z_0) - \theta_0 < -2j\pi, \quad \text{for } z_0 \in \gamma_{h_m^-},$$
$$\theta(2k\pi;t_0,z_0) - \theta_0 > -2j\pi, \quad \text{for } z_0 \in \gamma_{h_l^+}.$$

Hence we can use a recent version of the Poincaré-Birkhoff fixed point theorem[a] (Rebelo, [8], Corollary 2) to obtain at least two fixed points of $\mathcal{P}_k$, $z_1,z_2 \in \mathcal{A}_{m,l}$ with $z_1 \neq z_2$. These fixed points are initial points of two $2k\pi$-periodic solutions

$$z_1(t) = z(t;z_1), \qquad z_2(t) = z(t;z_2)$$

of (2.2), such that $z_1(t)$ and $z_2(t)$ satisfy

$$\theta(2k\pi;z_i) - \theta(0;z_i) = -2j\pi, \quad i = 1,2.$$

We assert that

$$\min\bigl\{|z_i(t)| \mid t \in [0,2k\pi)\bigr\} > R_2, \quad i = 1,2.$$

If the assertion would not hold, then there exists $t_1' \in (0,2k\pi]$ such that $|z_1(t_1')| \leq R_2$ (or $t_2' \in (0,2k\pi]$ such that $|z_2(t_2')| \leq R_2$). It follows that there exists $t_1 \in (0,t_1')$ such that $|z_1(0)| = |z_1| \geq r^-(h_m^-) \geq \zeta_{j+1}(K_{j+1})$, $|z_1(t_1)| \leq K_{j+1}$, and $|z_1(t)| \geq R_0 > R_2$ for $t \in [0,t_1]$. According to Lemma 2.4, we have

$$\theta(t_1;z_1) - \theta(0;z_1) \leq -2(j+1)\pi.$$

Then Lemma 2.3 implies that

$$\begin{aligned}\theta(2k\pi;z_1) - \theta(0;z_1) &= \theta(2k\pi;z_1) - \theta(t_1;z_1) + \theta(t_1;z_1) - \theta(0;z_1)\\ &\leq \pi - 2(j+1)\pi = -(2j+1)\pi.\end{aligned}$$

This is a contradiction. Hence $z_1(t) > R_2$, $\forall t \in \mathbb{R}$ (or $z_2(t) > R_2$, $\forall t \in \mathbb{R}$).

Therefore we have proved that $z_i(t)$ is exactly the $2k\pi$-periodic solution of the equation

$$x' = y, \qquad y' = -f(x) + p(t)$$

and $z_i(t)$ completes exactly j clock-wise turns around the origin O in $[0, 2k\pi)$ which implies $x_i(t)$ has exactly $2j$ zeros in $[0, 2k\pi)$, where $x_i(t) = \Pi_1(z_i(t))$, $\Pi_1(\cdot)$ is the projection for the first component, $i = 1, 2$.

Moreover, in the above argument we can choose the same $\mathcal{A}_{m,l}$ for any $f \in \mathcal{B}_{\mu_0}$, μ_0 sufficiently small. Further, denote the two $2k\pi$-periodic solutions obtained above by $z_{f,1}(t) = z_1(t)$ and $z_{f,2}(t) = z_2(t)$. It is easy to find a compact annulus $\mathcal{C}_{m,l}$ such that $z_{f,i}(t)$, $i = 1, 2$, are in $\mathcal{C}_{m,l}$.

Now consider equation (2.1) with $g, p : \mathbb{R} \to \mathbb{R}$ continuous and p 2π-least periodic. We construct the equations

$$x' = y, \qquad y' = -f_q(x) + p(t)$$

with $f_q \in \mathcal{B}_{\mu_0}$, $f_q \to g$ in $\mathcal{C}_{m,l}$ as $q \to \infty$. As showed previously, we have two sequences of $2k\pi$-periodic solutions $z_1^{(q)}(t) = z_{f_q,1}(t)$ and $z_2^{(q)}(t) = z_{f_q,2}(t)$ for the equation

$$x' = y, \qquad y' = -f_q(x) + p(t)$$

in $\mathcal{C}_{m,l}$. Moreover, $x_i^{(q)}(t) = \Pi_1(z_i^{(q)}(t))$ has exactly $2j$ zeros in $[0, 2k\pi)$, $i = 1, 2$. If t_* is the zero time of $x_i^{(q)}(t)$, *i.e.* $x_i^{(q)}(t_*) = 0$, then $|x_i^{(q)'}(t_*)| \geq c$, where c is a positive constant which is independent of q and i. By a standard compactness argument (see, for example, in [9]), we obtain a $2k\pi$-periodic solution $z_{j,k}(t)$ (as the limitation for some subsequence of $\{z_1^{(q)}(t)\}$ and $\{z_2^{(q)}(t)\}$) for equation (2.1) in $\mathcal{C}_{m,l}$. Then we obtain a $2k\pi$-periodic solution $x_{j,k}(t) = \Pi_1(z_{j,k}(t))$ for equation (1.1). Since $p(t)$ is 2π-least periodic, $x_{j,k}(t)$ has exactly $2j$ zeros in $[0, 2k\pi)$ and k is prime with j, and $x_{j,k}(t)$ has a minimal period $2k\pi$. Actually, the solution $z_{j,k}(t) = (r_k(t)\cos\theta_k(t), r_k(t)\sin\theta_k(t))$ satisfies

$$r_k(t) > 0, \qquad \theta_k'(t) < 0, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

Moreover, $z_{j,k}(t)$ moves exactly j clock-wise turns around the origin O in $[0, 2k\pi)$. Suppose, by contradiction, that $z_{j,k}(t)$ has a minimal period $2l\pi$ with $l \in \mathbb{N}$ such that $1 \leq l < k$. Then (2.7) implies that $z_{j,k}(t)$ moves exactly q clock-wise turns around the origin O in $[0, 2l\pi)$ with $q \in \mathbb{N}$ such that $1 \leq q < j$. On the other hand, from the additivity property of the rotation we easily see that $z_{j,k}(t)$ moves exactly kq and lj clock-wise turns around the origin O in $[0, 2lk\pi)$. It follows that $kq = lj$. Then

$$k/j = l/q \quad \text{with } 1 \leq l < k \text{ and } 1 \leq q < j.$$

This contradicts that k is prime with j. The same argument works for the case when $j = 1$. Hence we have proved that $z_{j,k}(t)$ $(x_{j,k}(t))$ has a minimal period $2k\pi$.

Theorem 1.2 is thus proved. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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Endnote

[a] Usually, we can use the generalized version of the Poincaré-Birkhoff fixed point theorem by Ding [10] as in [1, 7, 9]. The result in [10], however, requires an extra assumption, *i.e.* the strictly star-shapedness of the outer boundary of the annular region, as recently pointed out in [11]. Now we refer to [8], Corollary 2, which is a direct reduction to the classical Poincaré-Birkhoff theorem for the standard annulus, already settled in [12]. We also refer to other versions of the Poincaré-Birkhoff theorem due to Franks [13], and Qian-Torres [14], where independent proofs are given.

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