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Fixed point and common fixed point results in cone metric space and application to invariant approximation

Anil Kumar[1,2*] and Savita Rathee[1]

*Correspondence: anilkshk84@gmail.com
[1]Department of Mathematics, Maharshi Dayanand University, Rohtak, Haryana 124001, India
[2]Department of Mathematics, Govt. College Bahu, Jhajjar, Haryana 124142, India

Abstract

In this work, the concept of almost contraction for multi-valued mappings in the setting of cone metric spaces is defined and then we establish some fixed point and common fixed point results in the set-up of cone metric spaces. As an application, some invariant approximation results are obtained. The results of this paper extend and improve the corresponding results of multi-valued mapping from metric space theory to cone metric spaces. Further our results improve the recent result of Arshad and Ahmad (Sci. World J. 2013:481601, 2013).

MSC: 46T99; 47H10; 54H25

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1 Introduction

Fixed point theory has many applications in different branches of science. This theory itself is a beautiful mixture of analysis, topology, and geometry. Since the appearance of the Banach contraction mapping principle, there has been a lot of activity in this area and several well-known fixed point theorems came into existence as a generalization of that principle. Many authors generalized and extended the notion of metric spaces such as b-metric spaces, partial metric spaces, generalized metric spaces, complex-valued metric space *etc.* For a useful discussion of these generalizations of metric spaces, one may refer to [1].

In 2007, Huang and Zhang [2] introduced the concept of cone metric space as a generalization of metric space, in which they replace the set of real numbers with a real Banach space. Although they proved several fixed point theorems for contractive type mappings on a cone metric space when the underlying cone is normal. Rezapour and Hamlbarani [3] proved such fixed point theorems omitting the assumptions of normality of cone. After that, the study of fixed point theorems in cone metric spaces was followed by many others (*e.g.*, see [4–17] and the references therein).

On the other side, Nadler [18] and Markin [19] initiated the study of fixed point theorems for multi-valued mappings and established the multi-valued version of the Banach contraction mapping principle. Since the theory of multi-valued mappings has many applications, it became a focus of research over the years. Recently, many authors worked out results on multi-valued mappings defined on a cone metric space when the underlying cone is normal or regular (see [20–23]). In 2011, Janković *et al.* [24] showed that most of the fixed point results in the set-up of normal cone metric space can be obtained as a consequence of the corresponding results in metric spaces. In the light of this, Arshad and Ahmad [25] improved Wardowski's results by proving the same without the assumption of the normality of the cones.

Here, the concept of almost contraction for multi-valued mappings in the setting of cone metric spaces is defined and then we establish some fixed point and common fixed point results in the set-up of cone metric spaces. In this way our results extend the results of Arshad and Ahmad [25] and also improve the corresponding results of both single-valued and multi-valued mappings existing in the literature. Before starting our work we need the following well-known definitions and results.

Definition 1 Let E be a real Banach space with norm $\|\cdot\|$ and P be a subset of E. Then P is called a cone if

(1) P is nonempty, closed, and $P \neq \{\theta\}$, where θ is the zero element of E;
(2) for any non-negative real numbers a, b and for any $x, y \in P$, one has $ax + by \in P$;
(3) $x \in P$ and $-x \in P$ implies $x = \theta$.

Given a cone $P \subseteq E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$ while $x \ll y$ if and only if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ is the interior of P. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P. In the following we suppose that E is a real Banach space and P is a cone in E with $\operatorname{int} P \neq \phi$ and $\preceq$ is a partial ordering with respect to P.

Definition 2 [2] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies the following:

(d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$;
(d_2) $d(x, y) = \theta$ if and only if $x = y$;
(d_3) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_4) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 3 [2] Let (X, d) be a cone metric space and let $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ obeys the following.

(1) $\{x_n\}$ converges to x, if for every $c \in E$ with $\theta \ll c$ there exists a positive integer N such that $d(x_n, x) \ll c$, for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$.
(2) $\{x_n\}$ is said to be Cauchy if for every $c \in E$ with $\theta \ll c$ there exists a positive integer N such that $d(x_n, x_m) \ll c$, for all $n, m \geq N$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 4 [24] *Let P be a cone in Banach space E. Then the following properties hold*:
(1) *If $c \in \operatorname{int} P$ and $a_n \to \theta$, then there exists a positive integer N such that for all $n > N$, we have $a_n \ll c$.*
(2) *If $a \preceq ka$, where $a \in P$ and $0 \leq k < 1$, then $a = \theta$.*

Definition 5 [25] Let (X, d) be a cone metric space and let $C(X)$ be the family of all nonempty and closed subsets of X. A map $H : C(X) \times C(X) \to E$ is called an H-cone metric on $C(X)$ induced by d if the following conditions hold:

(H_1) $\theta \preceq H(A, B)$ for all $A, B \in C(X)$.
(H_2) $H(A, B) = \theta$ if and only if $A = B$.
(H_3) $H(A, B) = H(B, A)$ for all $A, B \in C(X)$.
(H_4) $H(A, B) \preceq H(A, C) + H(C, B)$ for all $A, B, C \in C(X)$.
(H_5) If $A, B \in C(X)$, $\theta \prec \epsilon \in E$ with $H(A, B) \prec \epsilon$, then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \prec \epsilon$.

Example 6 Let (X, d) be a metric space. Then the mapping $H_u : C(X) \times C(X) \to \mathbb{R}$ defined by

$$H_u(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\} \tag{1.1}$$

is an H-cone metric induced by d. It is also known as the usual Hausdorff metric induced by d.

It is to be noted that $(C(X), H)$ is a complete metric space whenever (X, d) is a complete metric space.

Definition 7 Let X be a nonempty set, $T : X \to C(X)$ be a multi-valued mapping, and $f : X \to X$. Then an element $x \in X$ is said to be
(i) a fixed point of T, if $x \in Tx$;
(ii) a common fixed point of T and f, if $x = fx \in Tx$;
(iii) a coincidence point of T and f, if $w = fx \in Tx$, and w is called the point of coincidence of T and f.

We denote $C(f, T) = \{x \in X : fx \in Tx\}$, the set of coincidence point of f and T. The set of fixed point of T and the set of common fixed point of f and T is denoted by $F(T)$ and $F(f, T)$, respectively.

Definition 8 [26] Let X be a nonempty set, $T : X \to C(X)$ be a multi-valued mapping, and $f : X \to X$. Then f is called T-weakly commuting at $x \in X$ if $ffx \in Tfx$.

2 Main result

We start this section with the following definition.

Definition 9 Let (X, d) be a cone metric space and let there exist an H-cone metric on $C(X)$ induced by d. A map $T : X \to C(X)$ is said to be a multi-valued almost contraction if there exist two constants $\lambda \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \preceq \lambda d(x,y) + Ld(y,u) \tag{2.1}$$

for all $x, y \in X$ and $u \in Tx$.

Theorem 10 *Let (X,d) be a complete cone metric space and let there exist an H-cone metric on $C(X)$ induced by d. Suppose $T : X \to C(X)$ is a multi-valued almost contraction. Then T has a fixed point in X.*

Proof Let x_0 be an arbitrary fixed element and $x_1 \in Tx_0$, if $x_0 = x_1$, then x_0 is fixed point of T; if $x_0 \neq x_1$, then $\theta \prec d(x_0,x_1)$. As $\lambda > 0$, we have $H(Tx_0, Tx_1) \prec \epsilon$, where $\epsilon = H(Tx_0, Tx_1) + \lambda d(x_0,x_1)$. Then, by the definition of H-cone metric there exists $x_2 \in Tx_1$ such that

$$d(x_1,x_2) \prec \epsilon = H(Tx_0, Tx_1) + \lambda d(x_0,x_1).$$

Clearly, $H(Tx_1, Tx_2) \prec H(Tx_1, Tx_2) + \lambda^2 d(x_0,x_1)$. Since $x_2 \in Tx_1$, for $\epsilon = H(Tx_1, Tx_2) + \lambda^2 d(x_0,x_1)$, there exists $x_3 \in Tx_2$ such that

$$d(x_2,x_3) \prec H(Tx_1, Tx_2) + \lambda^2 d(x_0,x_1).$$

In the same way, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, for each $n \in \mathbb{N} \cup \{0\}$ and

$$d(x_n,x_{n+1}) \prec H(Tx_{n-1}, Tx_n) + \lambda^n d(x_0,x_1). \tag{2.2}$$

Since T is a multi-valued almost contraction, in view of (2.2) we have

$$d(x_n,x_{n+1}) \prec \lambda d(x_{n-1},x_n) + Ld(x_n,u) + \lambda^n d(x_0,x_1) \tag{2.3}$$

for each $u \in Tx_{n-1}$. Also, as $x_n \in Tx_{n-1}$, for each $n \geq 1$ we have

$$d(x_n,x_{n+1}) \prec \lambda d(x_{n-1},x_n) + \lambda^n d(x_0,x_1). \tag{2.4}$$

By repeated use of (2.4), we get

$$\begin{aligned} d(x_n,x_{n+1}) &\prec \lambda^n d(x_0,x_1) + n\lambda^n d(x_0,x_1) \\ &= (n+1)\lambda^n d(x_0,x_1). \end{aligned} \tag{2.5}$$

Now, for any $m > n$,

$$\begin{aligned} d(x_n,x_m) &\preceq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m) \\ &\preceq (n+1)\lambda^n d(x_0,x_1) + (n+2)\lambda^{n+1} d(x_0,x_1) + \cdots + m\lambda^{m-1} d(x_0,x_1) \\ &\preceq d(x_0,x_1) \sum_{i=n}^{\infty} (i+1)\lambda^i. \end{aligned} \tag{2.6}$$

Let $c \in E$ be any with $\theta \ll c$. Choose $\delta > 0$ such that $c + N_\delta(\theta) \subset \operatorname{int} P$, where $N_\delta(\theta) = \{x \in E : \|x\| < \delta\}$. Also, since the series $\sum_{n=1}^{\infty}(n+1)\lambda^n$ is convergent, there exists a natural number N such that $d(x_0, x_1)\sum_{i=n}^{\infty}(i+1)\lambda^i \in N_\delta(\theta)$, for all $n \geq N$. Thus $d(x_0, x_1)\sum_{i=n}^{\infty}(i+1)\lambda^i \ll c$, for all $n \geq N$. Hence, (2.6) implies

$$d(x_n, x_m) \ll c$$

for all $m > n \geq N$. Thus, the sequence $\{x_n\}$ is Cauchy. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = \theta$. Now we shall show that z is a fixed point of T, that is, $z \in Tz$. As $x_{n+1} \in Tx_n$ and

$$H(Tx_n, Tz) \prec H(Tx_n, Tz) + \lambda^n d(x_0, x_1),$$

using the definition of the H-cone metric there exists $y_n \in Tz$ such that

$$d(x_{n+1}, y_n) \preceq H(Tx_n, Tz) + \lambda^n d(x_0, x_1). \tag{2.7}$$

Since T is a multi-valued almost contraction and $x_{n+1} \in Tx_n$, it follows from (2.7) that

$$d(x_{n+1}, y_n) \preceq \lambda d(x_n, z) + Ld(z, x_{n+1}) + \lambda^n d(x_0, x_1).$$

Then, by the triangle inequality, we get

$$\begin{aligned} d(z, y_n) &\preceq d(z, x_{n+1}) + d(x_{n+1}, y_n) \\ &\preceq d(z, x_{n+1}) + \lambda d(x_n, z) + Ld(z, x_{n+1}) + \lambda^n d(x_0, x_1). \end{aligned} \tag{2.8}$$

Further, since $d(x_n, z) \to \theta$ and $\lambda^n \to 0$ as $n \to \infty$, the right-hand side of the inequality (2.8) tends to θ as $n \to \infty$. Now, by Lemma 4, for any $c \in E$ with $\theta \ll c$ there exists a positive integer N_1 such that $d(z, y_n) \ll c$ for all $n \geq N_1$. Thus, the sequence $\{y_n\}$ converges to z. As $y_n \in Tz$ and Tz is closed, we get $z \in Tz$. □

Now we present an example in support of the proved result.

Example 11 Let $X = [0,1]$, $E = C^1_{\mathbb{R}}([0,1])$ with the norm $\|\varphi\| = \sup_{x\in X}|\varphi(x)| + \sup_{x\in X}|\varphi'(x)|$ and consider the cone $P = \{\varphi \in E : \varphi(t) \geq 0\}$. Suppose $\varphi, \phi \in E$ are defined as

$$\varphi(x) = x \quad \text{and} \quad \phi(x) = x^{2n} \quad \text{for each } n \geq 1$$

Then $\theta \preceq \phi \preceq \varphi$ and $\|\varphi\| = 2$, $\|\phi\| = 2n+1$. Given any $K > 0$ we can find a positive integer n such that $2n + 1 > 2K$. So, $\|\phi\| \npreceq K\|\varphi\|$ for any $K > 0$. Thus, P is non-normal cone. Now, define $d : X \times X \to E$ by

$$d(x, y) = |x - y|\varphi,$$

where $\varphi : [0,1] \to R$ with $\varphi(t) = e^t$. Then (X, d) be a complete cone metric space. Let $C(X)$ be the family of all nonempty and closed subsets of X and define a mapping $H : C(X) \times$

$C(X) \to E$ as

$$H(A,B) = H_u(A,B)\varphi \quad \text{for all } A,B \in C(X),$$

where H_u is the usual Hausdorff metric induced by $d(x,y) = |x-y|$. Also define $T : X \to C(X)$ by

$$T(x) = \begin{cases} [0, \frac{x}{2}] & \text{for } x \in [0, \frac{1}{2}], \\ [\frac{2}{3}, \frac{x}{3} + \frac{1}{2}] & \text{for } x \in (\frac{1}{2}, 1]. \end{cases}$$

Now we shall show that T is a multi-valued almost contraction, that is, we show that T will satisfy condition (2.1). For this, we consider the following possible cases:

Case (1). If $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$, then condition (2.1) can be written as

$$\left|\frac{x}{2} - \left(\frac{y}{3} + \frac{1}{2}\right)\right| e^t \leq \lambda |x-y| e^t + L|y-u|e^t \tag{2.9}$$

for all $u \in Tx = [0, \frac{x}{2}]$. Here, we observe that $|\frac{x}{2} - (\frac{y}{3} + \frac{1}{2})| \leq \frac{5}{6}$, $|x-y| \in (0,1]$, and $|y-u| > \frac{1}{4}$ for all $u \in [0, \frac{x}{2}]$. Thus, the inequality (2.9) is true for any $\lambda \in (0,1)$ and $L \geq \frac{10}{3}$.

Case (2). If $x \in (\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2}]$, then condition (2.1) takes the form

$$\left|\left(\frac{x}{3} + \frac{1}{2}\right) - \frac{y}{2}\right| e^t \leq \lambda |x-y| e^t + L|y-u|e^t \tag{2.10}$$

for all $u \in Tx = [\frac{2}{3}, \frac{x}{3} + \frac{1}{2}]$. In this case $|(\frac{x}{3} + \frac{1}{2}) - \frac{y}{2}| \leq \frac{5}{6}$, $|x-y| \in (0,1]$ and $|y-u| \geq \frac{1}{6}$ for all $u \in [\frac{2}{3}, \frac{x}{3} + \frac{1}{2}]$. Thus, the inequality (2.10) is true for any $\lambda \in (0,1)$ and $L \geq 5$.

Case (3). If $x, y \in [0, \frac{1}{2}]$, then

$$\begin{aligned} H(Tx, Ty) &= H_u(Tx, Ty)\varphi \\ &= H_u\left(\left[0, \frac{x}{2}\right], \left[0, \frac{y}{2}\right]\right) e^t \\ &= \frac{1}{2}|x-y|e^t \\ &\leq \lambda d(x,y) + Ld(y,u) \end{aligned}$$

for any $\lambda \in [\frac{1}{2}, 1)$ and $L \geq 0$, where u is arbitrary element of Tx.

Case (4). If $x, y \in (\frac{1}{2}, 1]$, then

$$\begin{aligned} H(Tx, Ty) &= H_u(Tx, Ty)\varphi \\ &= H_u\left(\left[\frac{2}{3}, \frac{x}{3} + \frac{1}{2}\right], \left[\frac{2}{3}, \frac{y}{3} + \frac{1}{2}\right]\right) e^t \\ &= \frac{1}{3}|x-y|e^t \\ &\leq \lambda d(x,y) + Ld(y,u) \end{aligned}$$

for any $\lambda \in [\frac{1}{3}, 1)$ and $L \geq 0$, where u is arbitrary element of Tx.

Now, from all the cases, it is concluded that the multi-valued mapping T satisfies the inequality (2.1) for $\lambda = \frac{1}{2}$ and $L = 5$. Hence, T is an almost multi-valued contraction that satisfies all the hypotheses of Theorem 10. Thus, the mapping T has a fixed point. Here $x = 0$ is such a fixed point.

Remark 12

(i) Theorem 3.1 of Arshad and Ahmad [25], Theorem 2.4 of Dorić [27], and Theorem 3.1 of Wardowski [20] are direct consequences of Theorem 10.

(ii) In Example 11, for $x = \frac{1}{2}$ and $y = \frac{2}{3}$, we get $Tx = [0, \frac{1}{4}]$, $Ty = [\frac{2}{3}, \frac{13}{18}]$, therefore $H(Tx, Ty) = \frac{17}{36}e^t$. Then it can easily be checked that there does not exist any $\lambda \in (0,1)$ such that the mapping T satisfies the conditions (D1), (D2), (D3), (D4) given in Definition 2.1 of Dorić [27]. Hence, Theorem 2.4 of Dorić [27] cannot be applied to Example 11. It is also to be noted that Theorem 3.1 of Arshad and Ahmad [25] and Theorem 3.1 of Wardowski [20] are not applicable to Example 11.

In [28] Haghi *et al.* proved the following lemma.

Lemma 13 *Let X be a nonempty set and $f : X \to X$ be a function. Then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one to one.*

Theorem 14 *Let (X, d) be a cone metric space and let there exist an H-cone metric on $C(X)$ induced by d. Suppose $f : X \to X$ is a self map such that $f(X)$ is a complete subspace of X and $T : X \to C(X)$ is a multi-valued mapping with $Tx \subseteq f(X)$ for each $x \in X$. If there exist two constants $\lambda \in (0,1)$ and $L \geq 0$ such that*

$$H(Tx, Ty) \preceq \lambda d(fx, fy) + Ld(fy, u) \tag{2.11}$$

for all $x, y \in X$ and $u \in Tx$. Then T and f have a coincidence point in X. Moreover, if $ffx = fx$ for each $x \in C(f, T)$, then T and f have a common fixed point in X.

Proof By Lemma 13, there exists $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one to one. Now, we define a map $g : f(E) \to C(f(E))$ by $g(f(x)) = Tx$. Clearly g is well defined as f is one to one. Also

$$\begin{aligned} H\bigl(g(fx), g(fy)\bigr) &= H(Tx, Ty) \\ &\preceq \lambda d(fx, fy) + Ld(fy, u) \end{aligned} \tag{2.12}$$

for all $fx, fy \in f(E)$ and $u \in Tx = g(fx)$. Then, by Theorem 10, there exists $fx_0 \in f(E)$ such that $fx_0 \in g(fx_0) = Tx_0$. Thus x_0 is a coincidence point of f and T and hence $ffx_0 = fx_0$. Let $w = fx_0$, therefore $fw = ffx_0 = fx_0 \in Tx_0$. By (2.11), we have

$$\begin{aligned} H(Tx_0, Tw) &\preceq \lambda d(fx_0, fw) + Ld(fw, fw) \\ &= \theta, \end{aligned}$$

which gives $Tx_0 = Tw$. Therefore, $w = fw \in Tw$, that is, w is a common fixed point of f and T. □

Theorem 15 *Let (X, d) be a cone metric space and let there exist an H-cone metric on $C(X)$ induced by d. Assume K is a nonempty closed subset of X such that for each $x \in K$ and $y \notin K$ there exists $z \in \delta K$ such that*

$$d(x,z) + d(x,y) = d(x,y).$$

Suppose $T : K \to C(X)$ and $f : K \to X$ are two non-self maps satisfying

$$H(Tx, Ty) \preceq \lambda d(fx, fy) + Ld(fy, u) \tag{2.13}$$

for all $x, y \in K$ and $u \in Tx$ with some $\lambda \in (0,1)$ and $L \geq 0$ such that $\lambda(1 + L) < 1$. Further assume

(i) $\delta K \subseteq fK$;
(ii) $(\bigcup_{x\in K} Tx) \cap K \subseteq fK$;
(iii) $fx \in \delta K \Rightarrow Tx \subseteq K$;
(iv) *fK is closed in X.*

Then T and f have a coincidence point in X. Moreover, if $ffx = fx$ for each $x \in C(f, T)$, then there exists a common fixed point of f and T.

Proof Let $x \in \delta K$. We construct two sequences $\{x_n\}$ in K and $\{y_n\}$ in fK in the following way. Since $\delta K \subseteq fK$, there exists $x_0 \in K$ such that $fx_0 = x \in \delta K$. So, by (iii) we get $Tx_0 \subseteq K$. Since $(\bigcup_{x\in K} Tx) \cap K \subseteq fK$, we have $Tx_0 \subseteq fK$. Let $y_1 \in Tx_0$, then there exists $x_1 \in K$ such that $y_1 = fx_1$. Consider the element $H(Tx_0, Tx_1) \in E$. If the right-hand side of (2.13) is θ at $x = x_0$ and $y = x_1$, then, as $fx_1 \in Tx_0$, we have $d(fx_0, fx_1) = \theta$ and hence $fx_1 = fx_0$. This and $fx_1 \in Tx_0$ imply $fx_0 \in Tx_0$. Thus, x_0 is coincidence point of f and T.

Assume the right-hand side of (2.13) is not θ. Let $e \in P$ be a fixed element such that $e \neq \theta$. Since $\lambda > 0$, we have $H(Tx_0, Tx_1) \prec \epsilon$, where $\epsilon = H(Tx_0, Tx_1) + \lambda e$. Then, as $y_1 \in Tx_0$, by the definition of an H-cone metric there exists $y_2 \in Tx_1$ such that

$$d(y_1, y_2) \prec \epsilon = H(Tx_0, Tx_1) + \lambda e.$$

If $y_2 \in K$, then from (ii), we have $y_2 \in fK$. Therefore, there exists $x_2 \in K$ such that $y_2 = fx_2$. If $y_2 \notin K$, then, as $fx_1 \in K$, there exists a point $p \in \delta K$ such that

$$d(fx_1, p) + d(p, y_2) = d(fx_1, y_2). \tag{2.14}$$

Since $p \in \delta K \subseteq fK$, there exists a point $x_2 \in K$ such that $p = fx_2$. Then, by (2.14)

$$d(fx_1, fx_2) + d(fx_2, y_2) = d(fx_1, y_2). \tag{2.15}$$

Clearly, $H(Tx_1, Tx_2) \prec H(Tx_1, Tx_2) + \lambda^2 e$. Then, again using the definition of the H-cone metric, there exists $y_3 \in Tx_2$ such that

$$d(y_2, y_3) \prec H(Tx_1, Tx_2) + \lambda^2 e.$$

If $y_3 \in K$, then, again using (ii), we have $y_3 \in fK$. So, there is a point $x_3 \in K$ such that $y_3 = fx_3$. If $y_3 \notin K$, then there exists a point $q \in \delta K$ such that

$$d(fx_2, q) + d(q, y_3) = d(fx_2, y_3). \tag{2.16}$$

Again, since $q \in \delta K \subseteq fK$, there exists a point $x_3 \in K$ such that $q = fx_3$. Then, by (2.16),

$$d(fx_2, fx_3) + d(fx_3, y_3) = d(fx_2, y_3). \tag{2.17}$$

Repeating the foregoing procedure we construct two sequences $\{x_n\}$ in K and $\{y_n\}$ in fK such that

(a) $y_{n+1} \in Tx_n$, for each $n \in \mathbb{N} \cup \{0\}$;
(b) $d(y_n, y_{n+1}) \preceq H(Tx_{n-1}, Tx_n) + \lambda^n e$;
(c) if $y_n \in K$, then $y_n = fx_n$;
(d) if $y_n \notin K$, then $fx_n \in \delta K$ with

$$d(fx_{n-1}, fx_n) + d(fx_n, y_n) = d(fx_{n-1}, y_n). \tag{2.18}$$

Now we show that the sequence $\{fx_n\}$ is Cauchy and for this we define two sets P and Q as follows:

$$P = \bigl\{fx_i \in \{fx_n\} : fx_i = y_i\bigr\}, \qquad Q = \bigl\{fx_i \in \{fx_n\} : fx_i \neq y_i\bigr\}.$$

Clearly, if $fx_n \in Q$, then fx_{n-1} and fx_{n+1} lies in P. Now, it can be concluded that there are three possibilities.

Case 1. If $fx_n \in P$ and $fx_{n+1} \in P$, then $fx_n = y_n$ and $fx_{n+1} = y_{n+1}$. Therefore, by using (b)

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(y_n, y_{n+1}) \\ &\preceq H(Tx_{n-1}, Tx_n) + \lambda^n e \\ &\preceq \lambda d(fx_{n-1}, fx_n) + Ld(fx_n, u) + \lambda^n e \quad \text{for each } u \in Tx_{n-1}. \end{aligned} \tag{2.19}$$

Thus, in view of (a) and (2.19), we have

$$d(fx_n, fx_{n+1}) \preceq \lambda d(fx_{n-1}, fx_n) + \lambda^n e. \tag{2.20}$$

Case 2. If $fx_n \in P$ and $fx_{n+1} \in Q$, then $fx_n = y_n$ and $y_{n+1} \notin K$. Thus, (d) implies $fx_{n+1} \in \delta K$ with

$$d(y_n, fx_{n+1}) + d(fx_{n+1}, y_{n+1}) = d(y_n, y_{n+1}). \tag{2.21}$$

Regarding (2.21) and (b), we get

$$\begin{aligned} d(fx_n, fx_{n+1}) &\prec d(y_n, y_{n+1}) \\ &\preceq H(Tx_{n-1}, Tx_n) + \lambda^n e \\ &\preceq \lambda d(fx_{n-1}, fx_n) + Ld(fx_n, u) + \lambda^n e \quad \text{for each } u \in Tx_{n-1}. \end{aligned} \tag{2.22}$$

Since $fx_n = y_n \in Tx_{n-1}$, (2.21) gives

$$d(fx_n, fx_{n+1}) \prec \lambda d(fx_{n-1}, fx_n) + \lambda^n e. \tag{2.23}$$

Case 3. If $fx_n \in Q$ and $fx_{n+1} \in P$, then $fx_{n-1} \in P$, $y_n \notin K$ and $fx_{n+1} = y_{n+1}$. Thus, from (d) $fx_n \in \delta K$ such that

$$d(y_{n-1}, fx_n) + d(fx_n, y_n) = d(y_{n-1}, y_n). \tag{2.24}$$

By the triangle inequality, (2.24), and (b), we obtain

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(fx_n, y_{n+1}) \\ &\preceq d(fx_n, y_n) + d(y_n, y_{n+1}) \\ &\preceq d(y_{n-1}, y_n) - d(y_{n-1}, fx_n) + d(y_n, y_{n+1}) \\ &\preceq d(y_{n-1}, y_n) - d(y_{n-1}, fx_n) + H(Tx_{n-1}, Tx_n) + \lambda^n e. \end{aligned} \tag{2.25}$$

□

Since $y_n \in Tx_{n-1}$, on account of (2.13), we derive from (2.25) that

$$\begin{aligned} d(fx_n, fx_{n+1}) &\preceq d(y_{n-1}, y_n) - d(y_{n-1}, fx_n) + \lambda d(fx_{n-1}, fx_n) + Ld(fx_n, y_n) + \lambda^n e \\ &\preceq d(y_{n-1}, y_n) + Ld(fx_n, y_n) + \lambda^n e, \quad \text{as } \lambda \in (0,1) \\ &\preceq d(y_{n-1}, y_n) + Ld(y_{n-1}, y_n) - Ld(y_{n-1}, fx_n) + \lambda^n e \\ &\preceq (1+L)d(y_{n-1}, y_n) + \lambda^n e. \end{aligned} \tag{2.26}$$

Therefore, by using (b) and (2.13), we conclude that

$$\begin{aligned} d(fx_n, fx_{n+1}) &\preceq (1+L)H(Tx_{n-2}, Tx_{n-1}) + (1+L)\lambda^{n-1}e + \lambda^n e \\ &\preceq (1+L)\lambda d(fx_{n-2}, fx_{n-1}) + (1+L)Ld(fx_{n-1}, y_n) + (1+L)\lambda^{n-1}e + \lambda^n e \\ &\preceq (1+L)\lambda d(fx_{n-2}, fx_{n-1}) + (1+L)\lambda^{n-1}e + \lambda^n e. \end{aligned} \tag{2.27}$$

Now, we define

$$\mu = \max\{\lambda, (1+L)\lambda\}.$$

Thus, from Case 1-Case 3, it follows that

$$d(fx_n, fx_{n+1}) \preceq \mu\zeta_n + (\mu^{n-1} + \mu^n)e, \tag{2.28}$$

where

$$\zeta_n \in \{d(fx_{n-1}, fx_n), d(fx_{n-2}, fx_{n-1})\}.$$

Now we claim that for each $n > 1$,

$$d(fx_n, fx_{n+1}) \preceq \mu^{\frac{n-1}{2}}\zeta_2 + 2n\mu^{\frac{n}{2}}e, \tag{2.29}$$

where

$$\zeta_2 \in \{d(fx_0, fx_1), d(fx_1, fx_2)\}.$$

We shall prove it by mathematical induction. If $n = 2$, then (2.28) gives

$$\begin{aligned} d(fx_2,fx_3) &\preceq \mu\zeta_2 + \left(\mu + \mu^2\right)e \\ &= \mu\zeta_2 + \mu(1+\mu)e \\ &\preceq \mu^{\frac{1}{2}}\zeta_2 + 2\cdot(2\mu e), \quad \text{as } \mu < 1. \end{aligned}$$

Thus, (2.29) holds for $n = 2$. Let (2.29) be true for $2 \leq n \leq m$, then we have to show that it is also true for $n = m + 1$. From (2.28), for $n = m + 1$, we have

$$d(fx_{m+1},fx_{m+2}) \preceq \mu\zeta_{m+1} + \left(\mu^m + \mu^{m+1}\right)e, \tag{2.30}$$

where

$$\zeta_{m+1} \in \left\{d(fx_m,fx_{m+1}), d(fx_{m-1},fx_m)\right\}.$$

Case 1. If $\zeta_{m+1} = d(fx_m,fx_{m+1})$, then (2.30) implies

$$\begin{aligned} d(fx_{m+1},fx_{m+2}) &\preceq \mu d(fx_m,fx_{m+1}) + \left(\mu^m + \mu^{m+1}\right)e \\ &\preceq \mu\left(\mu^{\frac{m-1}{2}}\zeta_2 + 2m\mu^{\frac{m}{2}}e\right) + \left(\mu^m + \mu^{m+1}\right)e \\ &\preceq \mu^{\frac{1}{2}}\left(\mu^{\frac{m-1}{2}}\zeta_2 + 2m\mu^{\frac{m}{2}}e\right) + 2\mu^{\frac{m+1}{2}}e \\ &= \mu^{\frac{(m+1)-1}{2}}\zeta_2 + 2(m+1)\mu^{\frac{m+1}{2}}e. \end{aligned}$$

Case 2. If $\zeta_{m+1} = d(fx_{m-1},fx_m)$, then it follows from (2.30) that

$$\begin{aligned} d(fx_{m+1},fx_{m+2}) &\preceq \mu d(fx_{m-1},fx_m) + \left(\mu^m + \mu^{m+1}\right)e \\ &\preceq \mu\left(\mu^{\frac{(m-1)-1}{2}}\zeta_2 + (m-1)\mu^{\frac{m-1}{2}}e\right) + \left(\mu^m + \mu^{m+1}\right)e \\ &\preceq \mu^{\frac{m}{2}}\zeta_2 + (m-1)\mu^{\frac{m+1}{2}}e + \mu^{\frac{m+1}{2}}(1+\mu)e \\ &\preceq \mu^{\frac{m}{2}}\zeta_2 + (m-1)\mu^{\frac{m+1}{2}}e + \mu^{\frac{m+1}{2}}(1+\mu)e \\ &\preceq \mu^{\frac{m}{2}}\zeta_2 + 2(m-1)\mu^{\frac{m+1}{2}}e + 2\cdot 2\mu^{\frac{m+1}{2}}e \\ &= \mu^{\frac{(m+1)-1}{2}}\zeta_2 + 2(m+1)\mu^{\frac{m+1}{2}}e. \end{aligned}$$

Therefore, in both cases, (2.29) is true for $n = m+1$. Thus, by the principle of mathematical induction the inequality (2.29) holds for each $n > 1$. Now, by the triangle inequality and (2.29), for any $m > n$, we have

$$\begin{aligned} d(fx_n,fx_m) &\preceq d(fx_n,fx_{n+1}) + d(fx_{n+1},fx_{n+2}) + \cdots + d(fx_{m-1},fx_m) \\ &\preceq \left[\mu^{\frac{n-1}{2}}\zeta_2 + 2n\mu^{\frac{n}{2}}e\right] \\ &\quad + \left[\mu^{\frac{n}{2}}\zeta_2 + 2(n+1)\mu^{\frac{n+1}{2}}e\right] \\ &\quad + \cdots \\ &\quad + \left[\mu^{\frac{m-2}{2}}\zeta_2 + 2(m-1)\mu^{\frac{m-1}{2}}e\right] \\ &\preceq \left[\mu^{\frac{n-1}{2}} + \mu^{\frac{n}{2}} + \cdots + \mu^{\frac{m-2}{2}}\right]\zeta_2 + R_n(\mu)2e, \end{aligned} \tag{2.31}$$

where $R_n(\mu)$ is the remainder of the convergent series $\sum_{n=1}^{\infty} n\mu^{\frac{n}{2}}$. As $\mu < 1$, by (2.31) we get

$$d(fx_n, fx_m) \preceq \frac{\mu^{\frac{n-1}{2}}}{1-\mu^{\frac{1}{2}}}\zeta_2 + R_n(\mu)2e \to \theta, \quad \text{as } n \to \infty. \tag{2.32}$$

Thus, by Lemma 4, for any $c \in E$ with $\theta \ll c$ there exists a positive integer N_1 such that $d(fx_n, fx_m) \ll c$ for all $n \geq N_1$. Hence, the sequence $\{fx_n\}$ is Cauchy. Also, $\{fx_n\}$ is a sequence in $K \cap f(K)$ and, being a closed subset of complete space X, $K \cap f(K)$ is complete. Therefore, there exists $z \in K \cap f(K)$ such that $fx_n \to z$ as $n \to \infty$. Further, as $z \in f(K)$, there exists $w \in K$ such that $z = f(w)$. By the construction of $\{fx_n\}$ there is a subsequence fx_m such that

$$fx_m = y_m \in Tx_{m-1}.$$

Now we shall show that w is a coincidence point of T and f, that is, $fw \in Tw$. As $fx_m \in Tx_{m-1}$ and

$$H(Tx_{m-1}, Tw) \prec H(Tx_{m-1}, Tw) + \lambda^{m-1}e,$$

using the definition of H-cone metric, there exists $z_m \in Tw$ such that

$$d(fx_m, z_m) \prec H(Tx_{m-1}, Tw) + \lambda^{m-1}e. \tag{2.33}$$

Regarding (2.13) and $fx_m \in Tx_{m-1}$, we obtain from (2.33)

$$d(fx_m, z_m) \preceq \lambda d(fx_{m-1}, fw) + Ld(fw, fx_m) + \lambda^{m-1}e.$$

Then, by the triangle inequality, we get

$$\begin{aligned} d(fw, z_m) &\preceq d(fw, fx_m) + d(fx_m, z_m) \\ &\preceq d(fw, fx_m) + \lambda d(fx_{m-1}, fw) + Ld(fw, fx_m) + \lambda^{m-1}e. \end{aligned} \tag{2.34}$$

Since the subsequence $\{fx_m\}$ converges to $z = f(w)$ and $\lambda < 1$, the right-hand side of the inequality (2.34) converges to θ as $m \to \infty$. Therefore, in view of Lemma 4, for any $c \in E$ with $\theta \ll c$, we can choose a positive integer N_2 such that $d(fw, z_m) \ll c$ for all $m \geq N_2$. Thus the sequence z_m converges to $f(w)$. As $z_m \in Tw$ and Tw is closed, we get $fw \in Tw$. Since $w \in C(f, T)$, it follows that $ffw = fw$. Let $z = fw$ and so $fz = ffw = fw \in Tw$. In view of (2.13) we have

$$\begin{aligned} H(Tw, Tz) &\preceq \lambda d(fw, fz) + Ld(fz, fz) \\ &= \theta, \end{aligned}$$

which gives $Tw = Tz$. Therefore, $z = fz \in Tz$, that is, z is a common fixed point of f and T.

If we let $f = I$ (identity map) in Theorem 15, we obtain the following result as an extension of Theorem 9 of [29] to a cone metric space.

Corollary 16 *Let (X, d) be a cone metric space and let there exist an H-cone metric on $C(X)$ induced by d. Assume K is a nonempty closed subset of X such that for each $x \in K$ and $y \notin K$ there exists $z \in \delta K$ such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Suppose that $T : K \to C(X)$ is a non-self map satisfying

$$H(Tx, Ty) \preceq \lambda d(x,y) + Ld(y,u) \tag{2.35}$$

for all $x, y \in K$ and $u \in Tx$ with some $\lambda \in (0,1)$ and $L \geq 0$ such that $\lambda(1 + L) < 1$. Further assume $x \in \delta K \Rightarrow Tx \subseteq K$, then there exists $z \in K$ such that $z \in Tz$.

Now, we present a non-trivial example which shows the generality of Corollary 16 over the corresponding existing theorems.

Example 17 Let $X = [-1, \infty)$, $E = C^1_{\mathbb{R}}([0,1])$ with supremum norm, $P = \{\varphi \in E : \varphi(t) \geq 0\}$ and $d : X \times X \to E$ defined by $d(x,y) = |x - y|\varphi$, where $\varphi : [0,1] \to \mathbb{R}$ with $\varphi(t) = e^t$. Then (X, d) is a cone metric space with a non-normal cone P. Let $C(X)$ be the family of all nonempty and closed subsets of X and define a mapping $H : C(X) \times C(X) \to E$ as

$$H(A,B) = H_u(A,B)\varphi \quad \text{for all } A, B \in C(X),$$

where H_u is the usual Hausdorff metric induced by $d(x,y) = |x - y|$. Take $K = [0,1]$ and define $T : K \to C(X)$ as given in Example 11 of [29]:

$$T(x) = \begin{cases} \{\frac{x}{9}\} & \text{for } x \in [0, \frac{1}{2}), \\ \{-1\} & \text{for } x = \frac{1}{2}, \\ [\frac{17}{18}, \frac{x}{9} + \frac{8}{9}] & \text{for } x \in (\frac{1}{2}, 1]. \end{cases}$$

Here $\delta K = \{0, 1\}$. Clearly, for each $x \in K$ and $y \notin K$ there exists a point $z = 0$ or $z = 1 \in \delta K$ such that $d(x,z) + d(z,y) = d(x,y)$. Further, as $0 \in \delta K \Rightarrow T0 = \{0\} \subseteq [0,1] = K$ and $1 \in \delta K \Rightarrow T1 = [\frac{17}{18}, 1] \subseteq K$, so $x \in \delta K \Rightarrow Tx \subseteq K$. Now, using a routine calculation as done in Example 11, it can easily be shown that the inequality (2.35) holds for $\delta = \frac{1}{9}$ and $L = \frac{9}{2}$. Thus, all the conditions of Corollary 16 are satisfied and hence T has a fixed point in K. Here $x = 1$ is such a point.

Remark 18

(i) Theorem 1 of Assad and Kirk [30] is a direct consequence of Corollary 16.

(ii) In Example 17, for $x = 1$ and $y = \frac{1}{2}$, it can be checked that the inequality

$$H(Tx, Ty) \preceq \delta d(x,y)$$

is not satisfied for any $\delta \in (0,1)$. Therefore, Theorem 1 of Assad and Kirk [30] is not applicable to Example 17.

3 Application to invariant approximation

Since the appearance of Meinardus' result in best approximation theory, several authors have obtained best approximation results for single-valued maps as an application of fixed point and common fixed point results. The best approximation results for multi-valued mappings was obtained by Kamran [26], Al-Thagafi and Shahzad [31], Beg *et al.* [32], O'Regan and Shahzad [33], and Markin and Shahzad [34]. Further, best approximation results in the setting of cone metric space were for the first time considered by Rezapour [35] (see also [36]).

In this section the best approximation results for a multi-valued mapping in the setting of cone metric spaces are obtained.

Definition 19 Let M be a nonempty subset of a cone metric space X. A point $y \in M$ is said to be a best approximation to $p \in X$, if $d(y,p) \preceq d(z,p)$ for all $z \in M$. The set of best approximations to p in M is denoted by $B_M(p)$.

As an application of Theorem 14, we obtain the following theorem, which ensures the existence of a best approximation.

Theorem 20 *Let M be subset of a cone metric space X, $p \in X$, and let there exist an H-cone metric on $C(M)$ induced by d. Suppose $f : M \to M$ is a single-valued mapping and $T : M \to C(M)$ is a multi-valued mapping such that for all $x, y \in B_M(p)$ and $u \in Tx$ we have*

$$H(Tx,Ty) \preceq \lambda d(fx,fy) + Ld(fy,u), \tag{3.1}$$

where $\lambda \in (0,1)$ and $L \geq 0$. Also the following conditions hold:

(i) $f(B_M(p)) = B_M(p)$.

(ii) *$ffv = fv$ for $v \in C(f,T) \cap B_M(p)$.*

(iii) *$d(y,p) \preceq d(fx,p)$ for all $x \in B_M(p)$ and $y \in Tx$.*

(iv) *$f(B_M(p))$ is complete.*

Then $F(f,T) \cap B_M(P) \neq \phi$.

Proof First we show that $T|_{B_M(p)} : B_M(p) \to C(B_M(p))$ is a multi-valued mapping. For this let $x \in B_M(p)$ and $u \in Tx$. Then, as $f(B_M(p)) = B_M(p)$, we get $fx \in B_M(p)$ and hence $d(fx,p) \preceq d(z,p)$ for all $z \in M$.

Since $u \in Tx$, by (iii) we obtain

$$d(u,p) \preceq d(fx,p) \preceq d(z,p) \quad \text{for all } z \in M.$$

Thus, $u \in B_M(p)$ and hence $Tx \subseteq B_M(p)$ for all $x \in B_M(p)$. Since Tx is closed for all $x \in M$, therefore also Tx is closed for all $x \in B_M(p)$. So, $T|_{B_M(p)}$ is a multi-valued mapping from $B_M(p)$ to $C(B_M(p))$. Moreover, $Tx \subseteq B_M(p) = f(B_M(p))$ for each $x \in B_M(p)$. Further, as $f(B_M(p)) = B_M(p)$ and $f : M \to M$, the mapping $f|_{B_M(p)} : B_M(p) \to B_M(p)$ is single-valued. Clearly

$$F(f|_{B_M(p)}, T|_{B_M(p)}) = F(f,T) \cap B_M(p).$$

Therefore, by applying Theorem 14 for $X = B_M(p)$, $F(f,T) \cap B_M(P) \neq \phi$. □

Corollary 21 *Let M be subset of a cone metric space X, $p \in X$, and let there exist an H-cone metric on $C(M)$ induced by d. Suppose $T : M \to C(M)$ is a multi-valued mapping such that for all $x, y \in B_M(p)$ and $u \in Tx$ we have*

$$H(Tx, Ty) \preceq \lambda d(x, y) + Ld(y, u), \tag{3.2}$$

where $\lambda \in (0, 1)$ and $L \geq 0$. Also the following conditions hold:

(i) *$d(y, p) \preceq d(x, p)$ for all $x \in B_M(p)$ and $y \in Tx$.*

(ii) *$B_M(p)$ is complete.*

Then $F(T) \cap B_M(P) \neq \phi$.

Let M be a subset of a cone metric space (X, d) and let there exist an H-cone metric on $C(M)$ induced by d. A family $F = \{h_A : A \in C(M)\}$ of functions from $[0, 1]$ into $C(M)$ with the property $h_A(1) = A$ for each $A \in C(M)$ is said to be contractive if there exists a mapping $\varphi : (0, 1) \to (0, 1)$ such that for all $A, B \in C(M)$ and $t \in (0, 1)$, we have

$$H\bigl(h_A(t), h_B(t)\bigr) \preceq \varphi(t)H(A, B).$$

Such a family F is said to be jointly continuous if $A \to A_0$ in $C(M)$ and $t \to t_0$ in $(0, 1)$ imply $h_A(t) \to h_{A_0}(t_0)$.

Theorem 22 *Let M be a subset of a cone metric space (X, d), and let there exist an H-cone metric on $C(M)$ induced by d. Suppose $F = \{h_A : A \in C(M)\}$ is a contractive and joint continuous family, $T : M \to C(M)$ is a multi-valued mapping and there exists $L \geq 0$ such that*

$$H(Tx, Ty) \preceq d(x, y) + Ld(y, u) \tag{3.3}$$

for all $x, y \in M$ and $u \in h_{Tx}(k)$, where $k \in (0, 1)$ is any fixed element. If M is compact and T is continuous, then T has a fixed point in M.

Proof For each $n \geq 1$, if we put $k_n = \frac{n}{n+1}$, then k_n is a real sequence which lies in $(0, 1)$ with $k_n \to 1$ as $n \to \infty$. Now, we define $T_n : M \to C(M)$ by

$$T_n x = h_{Tx}(k_n) \quad \text{for all } x \in M.$$

Firstly, we show that T_n is a multi-valued almost contraction on M. For this let $x, y \in M$, as F is a contractive family, we have

$$\begin{aligned} H(T_n x, T_n y) &= H\bigl(h_{Tx}(k_n), h_{Ty}(k_n)\bigr) \\ &\preceq \varphi(k_n)H(Tx, Ty) \\ &\preceq \varphi(k_n)d(x, y) + L\varphi(k_n)d(y, u) \end{aligned} \tag{3.4}$$

for all $u \in h_{Tx}(k_n) = T_n x$. For each $n \geq 1$, define $\lambda_n = \varphi(k_n)$ and $L_n = L\varphi(k_n)$. Clearly, $\lambda_n \in (0, 1)$ and $L_n \geq 0$ for each $n \geq 1$. Therefore, for each $n \geq 1$, (3.4) implies

$$H(T_n x, T_n y) \preceq \lambda_n d(x, y) + L_n d(y, u)$$

for all $x, y \in M$ and $u \in T_n x$. Hence, T is a multi-valued almost contraction. Since M is compact, by Theorem 10 for each $n \geq 1$ there exists $x_n \in M$ such that $x_n \in T_n x_n$. Again compactness of M implies that there exists a convergent subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to z \in M$ as $m \to \infty$. Since T is continuous, the family F is jointly continuous and $x_m \in T_m x_m = h_{Tx_m}(k_m)$, we have $z \in Tz$, as $k_m \to 1$ as $m \to \infty$. Thus T has a fixed point in M. □

The following theorem ensures the existence of a fixed point from the set of best approximations.

Theorem 23 *Let M be a subset of a cone metric space (X, d) and let there exist an H-cone metric on $C(X)$ induced by d. Suppose $T : X \to C(X)$, $p \in X$, and $B_M(p)$ is nonempty, compact, and it has a joint contractive family $F = \{h_A : A \in C(B_M(p))\}$. If T is continuous on $B_M(p)$, (3.3) holds for all $x, y \in B_M(p)$ and also $d(y, p) \preceq d(x, p)$ for all $x \in B_M(p)$ and $y \in Tx$. Then $B_M(p) \cap F(T) \neq \phi$.*

Proof We claim $Tx \subseteq B_M(p)$ for each $x \in B_M(p)$. To prove this let $x \in B_M(p)$, then $d(x, p) \preceq d(z, p)$ for all $z \in M$. If $u \in Tx$, then by the given hypothesis

$$d(u, p) \preceq d(x, p) \preceq d(z, p) \quad \text{for all } z \in M.$$

Thus, $u \in B_M(p)$. So $T : B_M(p) \to C(B_M(p))$ is a multi-valued mapping and hence by applying Theorem 22 for $B_M(p)$, it follows that $B_M(p) \cap F(T) \neq \phi$. □

Remark 24 All results in this paper hold as well in the frame of tvs-cone metric spaces (see [16]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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References

1. Van An, T, Van Dung, N, Kadelburg, Z, Radenović, S: Various generalizations of metric spaces and fixed point theorems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. (2015). doi:10.1007/s13398-014-0173-7
2. Huang, G, Zhang, X: Cone metric spaces and fixed point theorems of contractive maps. J. Math. Anal. Appl. **332**, 1467-1475 (2007)
3. Rezapour, S, Hamlbarani, R: Some note on the paper 'Cone metric spaces and fixed point theorems of contractive mappings'. J. Math. Anal. Appl. **345**(2), 719-724 (2008)
4. Kadelburg, Z, Radenović, S, Rakočević, S: A note on the equivalence of some metric and cone metric fixed point results. Appl. Math. Lett. **24**, 370-374 (2011)
5. Kumar, A, Rathee, S, Kumar, N: The point of coincidence and common fixed point for three mappings in cone metric spaces. J. Appl. Math. **2013**, Article ID 146794 (2013)
6. Radenović, S, Rakočević, V, Resapour, S: Common fixed points for (g, f) type maps in cone metric spaces. Appl. Math. Comput. **218**, 480-491 (2011)
7. Pathak, HK, Shahzad, N: Fixed point results for generalized quasicontraction mappings in abstract metric spaces. Nonlinear Anal. TMA **71**(12), 6068-6076 (2009)
8. Radenović, S: A pair of nonself mappings in cone metric space. Kragujev. J. Math. **36**(2), 189-198 (2012)
9. Karapinar, E: Some non unique fixed point theorems of Ćirić type on cone metric spaces. Abstr. Appl. Anal. **2010**, Article ID 123094 (2010)

10. Abdeljawad, T, Karapinar, E: A gap in the paper 'A note on cone metric fixed point theory and its equivalence' [Nonlinear Analysis 72(5), (2010), 2259-2261]. Gazi Univ. J. Sci. **24**(2), 233-234 (2011)
11. Karapinar, E: Fixed point theorems in cone Banach spaces. Fixed Point Theory Appl. **2009**, Article ID 609281 (2009)
12. Haghi, RH, Rakočević, V, Rezapour, S, Shahzad, N: Best proximity results in regular cone metric spaces. Rend. Circ. Mat. Palermo **60**, 323-327 (2011)
13. Alnafei, SH, Radenović, S, Shahzad, N: Fixed point theorems for mappings with convex diminishing diameters on cone metric spaces. Appl. Math. Lett. **24**, 2162-2166 (2011)
14. Rezapour, S, Haghi, RH, Shahzad, N: Some notes on fixed points of quasi-contraction maps. Appl. Math. Lett. **23**, 498-502 (2010)
15. Khamsi, MA: Remarks on cone metric spaces and fixed point theorems of contractive mappings. Fixed Point Theory Appl. **2010**, Article ID 315398 (2010)
16. Radenović, S, Simić, S, Cakić, N, Golubović, Z: A note on tvs-cone metric fixed point theory. Math. Comput. Model. **54**, 2418-2422 (2011)
17. Kadelburg, Z, Radenović, S: A note on various types of cones and fixed point results in cone metric spaces. Asian J. Math. Appl. **2013**, Article ID ama0104 (2013)
18. Nadler, SB Jr: Multivalued contraction mappings. Pac. J. Math. **30**, 475-488 (1969)
19. Markin, JT: Continuous dependence of fixed point sets. Proc. Am. Math. Soc. **38**, 545-547 (1973)
20. Wardowski, D: On set-valued contractions of Nadler type in cone metric spaces. Appl. Math. Lett. **24**(3), 275-278 (2011)
21. Latif, A, Shaddad, FY: Fixed point results for multivalued maps in cone metric spaces. Fixed Point Theory Appl. **2010**, Article ID 941371 (2010)
22. Radenović, S, Kadelburg, Z: Some results on fixed points of multifunctions on abstract metric spaces. Math. Comput. Model. **53**, 746-753 (2011)
23. Rezopour, S, Haghi, RH: Fixed point of multifunctions on cone metric spaces. Numer. Funct. Anal. Optim. **30**(7-8), 1-8 (2009)
24. Janković, S, Kadelburg, K, Radenović, S: On cone metric spaces: a survey. Nonlinear Anal. TMA **74**, 2591-2601 (2011)
25. Arshad, M, Ahmad, J: On multivalued contractions in cone metric spaces without normality. Sci. World J. **2013**, Article ID 481601 (2013)
26. Kamran, T: Coincidence and fixed points for hybrid strict contractions. J. Math. Anal. Appl. **299**, 235-241 (2004)
27. Dorić, D: Common fixed point theorems for generalized multivalued contractions on cone metric spaces over a non-normal solid cone. Fixed Point Theory Appl. **2014**, 159 (2014)
28. Haghi, RH, Rezapour, S, Shahzad, N: Some fixed point generalizations are not real generalization. Nonlinear Anal. **74**, 1799-1803 (2011)
29. Alghamdi, MA, Berinde, V, Shahzad, N: Fixed points of multivalued nonself almost contractions. J. Appl. Math. **2013**, Article ID 621614 (2013)
30. Assad, NA, Kirk, WA: Fixed point theorems for set-valued mappings of contractive type. Pac. J. Math. **43**, 553-562 (1972)
31. Al-Thagafi, MA, Shahzad, N: Coincidence points, generalized *I*-nonexpansive multimaps, and applications. Nonlinear Anal. TMA **67**(7), 2180-2188 (2007)
32. Beg, I, Khan, AR, Hussain, H: Approximation of $*$-nonexpansive random multivalued operators on Banach spaces. J. Aust. Math. Soc. **76**, 51-66 (2004)
33. O'Regan, D, Shahzad, N: Coincidence points and invariant approximations results for multimaps. Acta Math. Sin. Engl. Ser. **23**, 1601-1610 (2007)
34. Markin, J, Shahzad, N: Best approximation theorem for nonexpansive and condensing mappings in hyperconvex spaces. Nonlinear Anal. TMA **70**(16), 2435-2441 (2009)
35. Rezapour, S: Best approximations in cone metric spaces. Math. Morav. **11**, 85-88 (2007)
36. Abbas, M, Vetro, P: Invariant approximation results in cone metric spaces. Ann. Funct. Anal. **2**(2), 101-113 (2011)