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Research Article

Coefficient Estimates for Certain Subclasses of Biunivalent Functions Defined by Convolution

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We introduce two new subclasses of the function class Σ of biunivalent functions in the open disc defined by convolution. Estimates on the coefficients $|a_2|$ and $|a_3|$ for the two subclasses are obtained. Moreover, we verify Brannan and Clunie's conjecture $|a_2| \le \sqrt{2}$ for our subclasses.

1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (a_j \ge 0) \tag{1}$$

which are analytic in the open disc $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$. Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of univalent functions $f(z)$ of form (1).

For $f(z)$ defined by (1) and $h(z)$ defined by

$$h(z) = z + \sum_{j=2}^{\infty} h_j z^j, \quad (h_j \ge 0), \tag{2}$$

the Hadamard product (or convolution) of f and h is defined by

$$(f * h)(z) = z + \sum_{j=2}^{\infty} a_j h_j z^j = (h * f)(z). \tag{3}$$

It is well known that every function $f \in S$ has an inverse f^{-1} defined by

$$\begin{aligned} f^{-1}(f(z)) &= z \quad (z \in \Delta), \\ f(f^{-1}(w)) &= w \quad \left(|w| < r_0(f) : r_0(f) \ge \frac{1}{4}\right). \end{aligned} \tag{4}$$

Indeed, the inverse function may have an analytic continuation to Δ, with

$$\begin{aligned} f^{-1}(w) = {} & w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ & - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \end{aligned} \tag{5}$$

A function $f \in \mathscr{A}$ is said to be biunivalent in Δ if $f(z)$ and $f^{-1}(z)$ are univalent in Δ. Let Σ denote the class of biunivalent functions in Δ given by (1). In 1967, Lewin [1] investigated the biunivalent function class Σ and showed that $|a_2| < 1.51$. Brannan and Clunie [2] conjectured that $|a_2| \le \sqrt{2}$. Netanyahu [3] introduced certain subclasses of biunivalent function class Σ similar to the familiar subclasses $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ of starlike and convex functions of order α $(0 < \alpha \le 1)$. Brannan and Taha [4] defined $f \in \mathscr{A}$ in the class $\mathscr{S}_\Sigma(\alpha)$ of strongly bistarlike functions of order α $(0 < \alpha \le 1)$ if each of the following conditions is satisfied:

$$\begin{aligned} & f \in \Sigma, \\ & \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1,\ z \in \Delta), \\ & \left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1,\ w \in \Delta), \end{aligned} \tag{6}$$

where g is as defined by (5). They also introduced the class of all bistarlike functions of order β defined as a function $f \in \mathscr{A}$, which is said to be in the class $\mathcal{S}^*_\Sigma(\beta)$ if the following conditions are satisfied:

$$\begin{gathered} f \in \Sigma, \\ \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad (z \in \Delta;\ 0 \le \beta < 1), \\ \Re\left(\frac{wg'(w)}{g(w)}\right) > \beta, \quad (w \in \Delta;\ 0 \le \beta < 1), \end{gathered} \tag{7}$$

where the function g is as defined in (5). The classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bistarlike functions of order β and biconvex functions of order β, corresponding to the function classes $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$, were introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\beta)$ and $\mathcal{K}^*_\Sigma(\beta)$, they found nonsharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (see [2, 5]). Some examples of biunivalent functions are $z/(1-z)$, $(1/2)\log((1+z)/(1-z))$, and $-\log(1-z)$ (see [6]). The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients, $|a_n|$ ($n \in \mathbb{N}$, $n \ge 3$), is still open ([6]). Various subclasses of biunivalent function class Σ were introduced and nonsharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series (1) were found in several investigations (see [7–11]).

In this present investigation, motivated by the works of Brannan and Taha [2] and Srivastava et al. [6], we introduce two new subclasses of biunivalent functions involving convolution. The first two initial coefficients of each of these two new subclasses are obtained. Further, we prove that Brannan and Clunie's conjecture is true for our subclasses.

In order to derive our main results, we have to recall the following lemma.

Lemma 1 (see [12]). *If $p \in \mathscr{P}$, then $|p_k| \le 2$ for each k, where $\mathscr{P}$ is the family of all functions $p(z)$ analytic in Δ for which* $\mathrm{Re}\{p(z)\} > 0$*;*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad \forall z \in \Delta. \tag{8}$$

2. Coefficient Bounds for the Classes $S_\Sigma(h,\alpha,\lambda)$ and $S^*_\Sigma(h,\beta,\lambda)$

Definition 2. A function $f(z)$ given by (1) is said to be in the class $S_\Sigma(h,\alpha,\lambda)$, if the following conditions are satisfied:

$$\begin{gathered} f \in \Sigma, \\ \left|\arg\left(\frac{z(f*h)'(z) + \lambda z^2 (f*h)''(z)}{(f*h)(z)}\right)\right| < \frac{\alpha\pi}{2}, \\ 0 < \alpha \le 1,\ \lambda \ge 0,\ z \in \Delta, \\ \left|\arg\left(\frac{w\left((f*h)^{-1}\right)'(w) + \lambda w^2 \left((f*h)^{-1}\right)''(z)}{\left((f*h)^{-1}\right)(w)}\right)\right| \\ < \frac{\alpha\pi}{2}, \quad 0 < \alpha \le 1,\ \lambda \ge 0,\ w \in \Delta, \end{gathered} \tag{9}$$

where the function $h(z)$ is defined by (2) and $(f*h)^{-1}(w)$ is defined by

$$\begin{aligned} (f*h)^{-1}(w) = {} & w - a_2 h_2 w^2 + \left(2a_2^2 h_2^2 - a_3 h_3\right) w^3 \\ & - \left(5a_2^3 h_2^3 - 5a_2 h_2 a_3 h_3 + a_4 h_4\right) w^4 \\ & + \cdots \end{aligned} \tag{10}$$

$$\begin{aligned} \left((f*h)^{-1}(w)\right)' = {} & 1 - 2a_2 h_2 w + 3\left(2a_2^2 h_2^2 - a_3 h_3\right) w^2 \\ & - \cdots . \end{aligned} \tag{11}$$

Clearly, $S_\Sigma(z/(1-z),\alpha,0) \equiv S_\Sigma(\alpha)$, the class of all strong bistarlike functions of order α introduced by Brannan and Taha [2].

Definition 3. A function $f(z)$ given by (1) is said to be in the class $S^*_\Sigma(h,\beta,\lambda)$, if the following conditions are satisfied:

$$\begin{gathered} f \in \Sigma, \\ \Re\left[\frac{z(f*h)'(z) + \lambda z^2 (f*h)''(z)}{(f*h)(z)}\right] > \beta \\ 0 \le \beta < 1,\ \lambda \ge 0,\ z \in \Delta, \\ \Re\left[\frac{w\left((f*h)^{-1}\right)'(w) + \lambda w^2 \left((f*h)^{-1}\right)''(w)}{\left((f*h)^{-1}\right)(w)}\right] \\ > \beta, \quad 0 \le \beta < 1,\ \lambda \ge 0,\ w \in \Delta, \end{gathered} \tag{12}$$

where $h(z)$ and $(f*h)^{-1}(w)$ are defined, respectively, as in (2) and (10).

Clearly, $S_\Sigma(z/(1-z),\beta,0) \equiv S^*_\Sigma(\beta)$, the class of all strong bistarlike functions of order β introduced by Brannan and Taha [2].

Theorem 4. *Let $f(z)$ given by (1) be in the class $S_\Sigma(h,\alpha,\lambda)$, $0 < \alpha \le 1$ and $\lambda \ge 0$. Then*

$$\begin{aligned} |a_2| &\le \frac{2\alpha}{h_2\sqrt{(\alpha+1)(4\lambda+1) + 4\lambda^2(\alpha-1)}}, \\ |a_3| &\le \frac{4\alpha^2}{h_3(1+2\lambda)^2} + \frac{\alpha}{h_3(1+3\lambda)}. \end{aligned} \tag{13}$$

Further, for the choice of $h(z) = z/(1-z)^2 = z + \sum_{n=2}^{\infty} nz^n$, one gets

$$\begin{aligned} |a_2| &\le \frac{\alpha}{\sqrt{(\alpha+1)(4\lambda+1) + 4\lambda^2(\alpha-1)}}, \\ |a_3| &\le \frac{4\alpha^2}{3(1+2\lambda)^2} + \frac{\alpha}{3(1+3\lambda)}. \end{aligned} \tag{14}$$

Proof. It follows from (9) that

$$\frac{z(f*h)'(z)+\lambda z^2(f*h)''(z)}{(f*h)(z)}=[p(z)]^{\alpha},$$
$$\frac{w\left(\left(f*h\right)^{-1}\right)'(w)+\lambda w^2\left(\left(f*h\right)^{-1}\right)''(w)}{\left(\left(f*h\right)^{-1}\right)(w)}=[q(w)]^{\alpha}, \quad (15)$$

where $p(z)$ and $q(w)$ satisfy the following inequalities:

$$\operatorname{Re}\{p(z)\}>0 \quad (z\in\Delta),$$
$$\operatorname{Re}\{q(w)\}>0 \quad (w\in\Delta). \quad (16)$$

Furthermore, the functions $p(z)$ and $q(w)$ have the forms

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots, \quad (17)$$

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots. \quad (18)$$

Now, equating the coefficients in (15), we get

$$(1+2\lambda)a_2h_2=\alpha p_1 \quad (19)$$

$$2(1+3\lambda)a_3h_3=p_2\alpha+\frac{\alpha(\alpha-1)}{2}p_1^2+\frac{\alpha^2p_1^2}{(1+2\lambda)} \quad (20)$$

$$-(1+2\lambda)a_2h_2=\alpha q_1, \quad (21)$$

$$2(1+3\lambda)\left(2a_2^2h_2^2-a_3h_3\right)=q_2\alpha+\frac{\alpha(\alpha-1)}{2}q_1^2+\frac{\alpha^2q_1^2}{(1+2\lambda)}. \quad (22)$$

From (19) and (21), we get

$$p_1=-q_1, \quad (23)$$

$$2(1+2\lambda)^2a_2^2h_2^2=\alpha^2\left(p_1^2+q_1^2\right). \quad (24)$$

Now, from (20), (22), and (24), we obtain

$$4(1+3\lambda)a_2^2h_2^2=\alpha\left(p_2+q_2\right)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right)+\frac{\alpha^2\left(p_1^2+q_1^2\right)}{(1+2\lambda)}. \quad (25)$$

Applying Lemma 1 for the coefficients p_2 and q_2, we immediately have

$$|a_2|\le\frac{2\alpha}{h_2\sqrt{(\alpha+1)(4\lambda+1)+4\lambda^2(1-\alpha)}}. \quad (26)$$

This gives the bound on $|a_2|$.

Next, in order to find the bound on $|a_3|$, by subtracting (20) from (22), we get

$$4(1+3\lambda)\left(a_3h_3-a_2^2h_2^2\right)=\alpha\left(p_2-q_2\right)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2-q_1^2\right)+\frac{\alpha^2\left(p_1^2-q_1^2\right)}{(1+2\lambda)}. \quad (27)$$

Upon substituting the value of a_2^2 from (24) and observing that $p_1^2=q_1^2$, it follows that

$$a_3=a_2^2+\frac{\alpha\left(p_2-q_2\right)}{4h_3(1+3\lambda)}=\frac{\alpha^2\left(p_1^2+q_1^2\right)}{2h_3(1+2\lambda)^2}+\frac{\alpha\left(p_2-q_2\right)}{4h_3(1+3\lambda)}. \quad (28)$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1, and q_2, we get

$$|a_3|\le\frac{4\alpha^2}{h_3(1+2\lambda)^2}+\frac{\alpha}{h_3(1+3\lambda)}. \quad (29)$$

This completes the proof. □

Remark 5. When $h(z)=z/(1-z)$ and $\lambda=0$, in (13), we get the results obtained due to [4].

Remark 6. When $\lambda=0$, $\alpha=1$, and $h_2=1$, we obtain Brannan and Clunie's [2] conjecture $|a_2|\le\sqrt{2}$.

Theorem 7. *Let $f(z)$ given by (1) be in the class $S_{\Sigma}^{*}(h,\beta,\lambda)$, $0\le\beta<1$ and $\lambda\ge0$. Then*

$$|a_2|\le\frac{1}{h_2}\sqrt{\frac{2(1-\beta)}{(1+4\lambda)}},$$
$$|a_3|\le\frac{1}{h_3}\left[\frac{4(1-\beta)^2}{(1+2\lambda)^2}+\frac{(1-\beta)}{(1+3\lambda)}\right]. \quad (30)$$

Further, for the choice of $h(z)=z/(1-z)^2=z+\sum_{n=2}^{\infty}nz^n$, we get

$$|a_2|\le\sqrt{\frac{(1-\beta)}{(1+4\lambda)}}, \quad (31)$$

$$|a_3|\le\frac{1}{3}\left[\frac{4(1-\beta)^2}{(1+2\lambda)^2}+\frac{(1-\beta)}{(1+3\lambda)}\right]. \quad (32)$$

Proof. It follows from (12) that there exist $p(z)$ and $q(w)$, such that

$$\begin{aligned}&\frac{z(f*h)'(z)+\lambda z^2(f*h)''(z)}{(f*h)(z)}\\&\quad=\beta+(1-\beta)p(z),\\&\frac{w\left((f*h)^{-1}\right)'(w)+\lambda w^2\left((f*h)^{-1}\right)''(w)}{\left((f*h)^{-1}\right)(w)}\\&\quad=\beta+(1-q)q(w)\end{aligned}\tag{33}$$

where $p(z)$ and $q(w)$ have forms (17) and (18), respectively.

Equating coefficients in (33) we obtain

$$(1+2\lambda)a_2h_2=p_1(1-\beta)\tag{34}$$

$$2(1+3\lambda)a_3h_3=p_2(1-\beta)+\frac{p_1^2(1-\beta)^2}{(1+2\lambda)}\tag{35}$$

$$-(1+2\lambda)a_2h_2=q_1(1-\beta),\tag{36}$$

$$2(1+3\lambda)\left(2a_2^2h_2^2-a_3h_3\right)=q_2(1-\beta)+\frac{q_1^2(1-\beta)^2}{(1+2\lambda)}.\tag{37}$$

From (34) and (36), we get

$$p_1=-q_1,\tag{38}$$

$$2(1+2\lambda)^2a_2^2h_2^2=(1-\beta)^2\left(p_1^2+q_1^2\right).\tag{39}$$

Now from (35), (37), and (39), we obtain

$$\begin{aligned}4(1+3\lambda)a_2^2h_2^2&=(1-\beta)(p_2+q_2)\\&\quad+\frac{(1-\beta)^2\left(p_1^2+q_1^2\right)}{(1+2\lambda)}.\end{aligned}\tag{40}$$

Therefore, we have

$$a_2^2=\frac{(1-\beta)(p_2+q_2)}{2h_2^2(1+4\lambda)}.\tag{41}$$

Applying Lemma 1, for the coefficients p_2 and q_2, we immediately have

$$|a_2|\le\frac{1}{h_2}\sqrt{\frac{2(1-\beta)}{(1+4\lambda)}}.\tag{42}$$

Next, in order to find the bound on $|a_3|$, by subtracting (35) from (37), we get

$$\begin{aligned}&4(1+3\lambda)\left(a_3h_3-a_2^2h_2^2\right)=(1-\beta)(p_2-q_2)\\&a_3=\frac{(1-\beta)^2(p_2-q_2)}{4(1+3\lambda)h_3}+\frac{(1-\beta)^2\left(p_1^2+q_1^2\right)}{2h_3(1+2\lambda)^2}.\end{aligned}\tag{43}$$

Applying Lemma 1 for the coefficients p_1, p_2, q_1, and q_2, we readily get

$$|a_3|\le\frac{1}{h_3}\left[\frac{4(1-\beta)^2}{(1+2\lambda)^2}+\frac{(1-\beta)}{(1+3\lambda)}\right].\tag{44}$$

□

Remark 8. When $h(z)=z/(1-z)$ and $\lambda=0$ in (30), we have the following result due to [4]. The bounds are

$$\begin{aligned}|a_2|&\le\sqrt{2(1-\beta)},\\|a_3|&\le(1-\beta)+4(1-\beta)^2.\end{aligned}\tag{45}$$

Remark 9. When $h(z)=z/(1-z)$ and $\lambda=0$ in (30), we have the following result due to [4]. The bounds are

$$\begin{aligned}|a_2|&\le\sqrt{2(1-\beta)},\\|a_3|&\le(1-\beta)+4(1-\beta)^2.\end{aligned}\tag{46}$$

Remark 10. When $\lambda=0$, $\beta=0$, and $h_2=1$ we obtain Brannan and Clunie's [2] conjecture $|a_2|\le\sqrt{2}$.

Competing Interests

The authors declare that they have no competing interests.

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