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A Novel Approach to Stability of Interval Delayed Systems with Nonlinear Perturbations

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Abstract: This paper focuses on the stability problem for a class of linear systems with interval time-varying delays and nonlinear perturbations. By developing a delay decomposition approach, the information of the delayed plant states can be taken into full consideration in a new Lyapunov-Krasovskii (LK) functional, and a delay-fractional-dependent sufficient stability criterion is obtained in terms of linear matrix inequalities without involving any direct approximation in the time-derivative of the LK functional. The merits of the proposed results lie in their less conservatism, which are realized by choosing different Lyapunov matrices in the decomposed integral intervals and utilizing some suitable integral inequalities to estimate some tighter upper bounds in some cross terms more exactly. This development leads to a less conservative LMI criterion as seen through numerical examples.

Keywords: Interval time-varying delay, Nonlinear perturbations, Lyapunov-Krasovskii (LK) functional, Linear matrix inequality (LMI), Maximum allowable delay bound (MADB).

1. INTRODUCTION

Stability is a central issue in dynamical system and control theory. A dynamical system is called stable (in the sense of Lyapunov) if starting the system somewhere near its desired operating point implies that it will stay around that point ever after (Gu et al., 2003). In control systems, time delay is always one of the sources of instability and poor performance, especially, in some practical systems time delay may be time-varying and the delay may vary in a range for which the lower bound is not restricted to being zero, such systems are referred to as interval time-varying delay systems (Gu et al., 2003). In recently years, the stability analysis and control synthesis with interval time-varying delay have received considerable attention (He et al., 2007a; Shao, 2009; Gao et al., 2006; Sun et al., 2010; Briat, 2011; Park et al., 2011; Shao and Han, 2012; Zhang et al., 2005; Qian et al., 2012; Fridman et al., 2009; Zhu and Yang, 2008; Liu et al., 2012; Zhu et al., 2010; Tang et al., 2012; Zhang and Han, 2013; Lee and Park, 2014),and the references therein.

Generally speaking, the delay-dependent stability criterion is less conservative than delay-independent stability when the time-delay is small. To derive the delay-dependent stability conditions, many methods have been proposed based on linear matrix inequality (LMI) approach, such as descriptor system approach, bounding techniques, and free weighting matrix approach. However, it is also known that the bounding technology and the model transformation technique are the main source of conservation. Therefore, the free-weighting matrix method was proposed in (He et al., 2007) to investigate the delay-dependent stability of continuous time systems with time-varying delay. In (Shao,2009; Gao et al., 2006; Sun et al., 2010; Briat, 2011; Park et al., 2011; Shao and Han, 2012; Zhang et al., 2005; Qian et al., 2012; Fridman et al., 2009; Zhu and Yang, 2008; Liu et al., 2012; Zhu et al., 2010),Jensen's integral inequality approach was employed to utilize different integral inequality for dealing the cross-terms that emerge from the time derivative of the L-K functional. From this it is seen that the integral inequality method may have some potential in the study of delay-dependent stability. Meanwhile, the delay decomposition approach is also attracted much researchers (Gao et al., 2006; Zhu and Yang, 2008; Liu et al., 2012; Zhu et al., 2010; Zhang and Han, 2013; Lee and Park, 2014). (Shao and Han, 2012) derived some less conservative results by constructing a new LK functional and introducing few free matrices. When the upper bound of delay derivative may be larger than or equal to 1, (Zhu and Yang, 2008; Zhu et al., 2010) used a delay decomposition approach, and new stability results were derived. Compared with (He et al., 2007a), the stability results in (Zhu and Yang, 2008; Zhu et al., 2010) are simpler and less conservative. Most recently, some less conservative results in (Zhang and Han, 2013) were derived by constructing a delay-dependent LK functional and using new bounding techniques; stability conditions in (Lee and Park, 2014) were improved by developing a new second-order reciprocally convex approach based on the reciprocally convex approach in (Park et al., 2011). In all these works, the state equation being analyzed for stability does not incorporate any perturbations in the current as well as in the delayed states.

In practice, it is very difficult to obtain an exact mathematical model due to environment noise, uncertain or slowly varying parameters, etc, and then the systems almost contain some uncertainties. Therefore, the stability problem of time-delay systems with nonlinear perturbations has received increasing attention in (Zuo and Wang, 2006; He et al., 2007b; Wang et al., 2010; Ramakrishnan and Ray, 2011; Han, 2004; Zeng et al., 2012; Hui et al., 2013; Cao and Lam, 2000). Generally, an important issue in this field is to enlarge the feasible region of stability criteria, so how to reduce the conservativeness is still the key problem. (Cao and Lam, 2000) introduced a model transformation method, while (Han, 2004) employed a descriptor model transformation together with decomposition technique using the delay term matrix. Similarly, the free-weighting approach was introduced in (Zhang et al., 2010) to derive a less conservative delay-dependent stability criterion by using a candidate L-K functional, and bounding the cross terms using free-weighting matrices. Recently, in (Ramakrishnan and Ray, 2011) authors provided a less conservative delay-dependent stability criterion by partitioning the delay-interval into two segments of equal length, while in (Zeng et al., 2012) authors introduced a general N delay partitioning technique by using improved free-weighting method. Most recently, in (Hui et al., 2013) authors derived a less delay-range-dependent stability criterion by using the delay-central point approach and introducing some free-weighting matrices. Nevertheless, there is further scope for reduction in conservatism in the delay-range bound for systems with nonlinear perturbations, and the delay interval may be divided into two unequal subintervals or more subintervals (Zhu and Yang, 2008; Liu et al., 2012; Zhu et al., 2010; Tang et al., 2012). This motivates the present research to develop a novel method for stability problem of the concerned systems with less conservatism by making full use of the information of time-delays and constructing a novel LK functional via variable delay dividing technique.

Motivated by the above discussions, this paper will focus on the stability problem for systems with interval time-varying delay and nonlinear perturbations. Firstly, a new delay-fractional-dependent LK functional is constructed by developing a variable delay decomposition approach. Secondly, a stability criterion for system (1) is derived by suitably using the integral inequalities and estimating more exactly the cross team in the time derivative of LK functional without any direct approximation of the delay terms in the derivation process. Since a tuning parameter α and different delay partitioning method are introduced, the derived LMIs also may be different in the stability conditions, and thus the variable and different Lyapunov matrices-based method may lead to less conservatism. Finally, numerical examples are included to show that the proposed method is effective and can provide less conservative results.

Notation: Throughout this paper, $P > 0$ means that P is symmetric positive definite; I is the identity matrix of appropriate dimensions; $col\{\cdots\}$ denotes a column vector; " := " denotes the definition or denotation; the symmetric term in a symmetric matrix is denoted by *, e.g., $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$.

2. PROBLEM FORMULATION

Consider the following system with a time-varying state delay and nonlinear perturbations:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_\tau x(t-\tau(t)) + f(x(t),t) + g(x(t-\tau(t)),t) \\ x(t) = \phi(t), \forall t \in [-h_b, 0], \end{cases} \tag{1}$$

where $x(t)$ is the state vector. $\phi(t)$ is the continuous initial vector function defined on $[-h_b, 0]$; A, A_τ are constant matrices with appropriate dimensions; the function $f(x(t),t)$ and $g(x(t-\tau(t)),t)$ are unknown nonlinear perturbations with respect to the current state $x(t)$ and the delay state $x(t-\tau(t))$,respectively, which satisfy that $f(0,t) = 0$, $g(0,t) = 0$ and

$$f^T(x(t),t) f(x(t),t) \le \lambda^2 x^T(t) F^T F x(t) \tag{2}$$

$$g^T(x(t-\tau(t)),t) g(x(t-\tau(t)),t) \le \mu^2 x^T(t-\tau(t)) G^T G x(t-\tau(t)) \tag{3}$$

where $\lambda \ge 0$, $\mu \ge 0$ are known scalars, and F, G is known constant matrix. For simplicity, the marks are denoted $\bar{\tau} = h_b - h_a$, $f = f(x(t),t)$,and $g = g(x(t-\tau(t)),t)$.

In this paper, the delay $\tau(t)$ is assumed to be time-varying delay as the following two cases:

Case 1. $\tau(t)$ is a differentiable function, satisfying for all $t \ge 0$:

$$0 < h_a \le \tau(t) \le h_b , \dot{\tau}(t) \le h_d \tag{4}$$

Case 2. $\tau(t)$ is not differentiable or the upper bound of the derivative of $\tau(t)$ is unknown, and $\tau(t)$ satisfies:

$$0 < h_a \le \tau(t) \le h_b \tag{5}$$

where h_a, h_b and h_d are some given positive values.

The purpose of this paper is to find new stability criteria, which are less conservative than the existing results. One usually uses a maximum allowable delay bound (MADB) on h_b of the time-varying delay as a performance index to judge the conservatism of the derived criterion. For given values of h_a , d_1 and d_2 , the larger is the upper bound h_b of the time-varying delay, the less conservatism yields the stability criterion.

To end this section, the following lemma is introduced which has an important role in the derivation of the stability results for the systems.

Lemma 1 (*Integral inequalities (Gu et al., 2003; Sun et al., 2010; Zhang et al., 2005)*)

Let $x(t) \in \Re^n$ be a vector-valued function with first-order continuous-derivative entries. Then, for any matrices $M, N \in \Re^{n\times n}$, $Z \in \Re^{2n\times 2n}$, $X = X^T \in \Re^{n\times n}$, and some given scalars $0 \le \tau_1 < \tau_2$, the following integral inequalities hold:

1). When $X > 0$ and τ_1, τ_2 are constant values,

$$(\tau_2 - \tau_1)\int_{t-\tau_2}^{t-\tau_1} x^T(s)Xx(s)ds \ge \int_{t-\tau_2}^{t-\tau_1} x^T(s)dsX\int_{t-\tau_2}^{t-\tau_1} x(s)ds \quad (6)$$

$$-(\tau_2 - \tau_1)\int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)X\dot{x}(s)ds \le \begin{bmatrix} x(t-\tau_1) \\ x(t-\tau_2) \end{bmatrix}^T \begin{bmatrix} -X & X \\ * & -X \end{bmatrix} \begin{bmatrix} x(t-\tau_1) \\ x(t-\tau_2) \end{bmatrix} \quad (7)$$

2). When τ_1, τ_2 are time-varying and $\tau_2 - \tau_1 := h(t) \ge 0$, and $X > 0$,

$$-\int_{t-\tau_2}^{t-\tau_1} x^T(s)Xx(s)ds \le \begin{bmatrix} x(t-\tau_1) \\ x(t-\tau_2) \end{bmatrix}^T \Xi \begin{bmatrix} x(t-\tau_1) \\ x(t-\tau_2) \end{bmatrix} \quad (8)$$

where $\Xi = \begin{bmatrix} M+M^T & -M+N^T \\ * & -N-N^T \end{bmatrix} + h(t)\begin{bmatrix} M \\ N \end{bmatrix} X^{-1}\begin{bmatrix} M^T & N^T \end{bmatrix}$

3). When τ_1, τ_2 are time-varying and $\tau_2 - \tau_1 := h(t) \ge 0$, and X is any symmetric matrix,

$$-\int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)X\dot{x}(s)ds \le \begin{bmatrix} x(t-\tau_1) \\ x(t-\tau_2) \end{bmatrix}^T \Psi \begin{bmatrix} x(t-\tau_1) \\ x(t-\tau_2) \end{bmatrix} \quad (9)$$

where $\Psi = \begin{bmatrix} M+M^T & -M+N^T \\ * & -N-N^T \end{bmatrix} + h(t)Z$

with $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \ge 0$, and $Y = \begin{bmatrix} M & N \end{bmatrix}$.

3. MAIN RESULTS

Firstly, the delay interval $[0, h_a]$ and $[h_a, h_b]$ are divided into four segments: $[h_{i-1}, h_i], i = 1,2,3,4$ where $h_0 = 0$, $h_1 = \frac{h_a}{2}$, $h_2 = h_a$, $h_3 = h_a + \alpha\bar{\tau}$, $h_4 = h_b$, $(0 < \alpha < 1)$. For the sake of convenience, the following marks are denoted $\bar{\tau}_0 = h_a - 0$, $\tau_i = h_i - h_{i-1}$, $(i = 1,2,3,4)$, and

$$\begin{aligned}\xi(t) := col\{&x(t), x(t-\tau(t)), x(t-h_1), x(t-h_2), \\ &x(t-h_3), x(t-h_4), f, g, x(t-\delta\tau(t)), \dot{x}(t-\delta\tau(t))\}\end{aligned},$$

and $e_i, (i = 1,2,\cdots,10)$ are block entry matrices, for example,

$$e_1^T = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the system (1),based on the Lyapunov stability theorem, a stability condition will be given by using an above variable delay decomposition method and utilizing suitably some integral inequalities for cross terms in the time derivative of the LK functional.

Theorem 1

(i) In Case 1,for given scalars $0 < h_a \le h_b$, $0 < \alpha < 1$, and $0 < \delta < 1$, h_d satisfying $\delta h_d < 1$, the system (1) is asymptotically stable if there exist real symmetric matrices $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \ge 0$ with $P_1 > 0$, $P_\delta > 0$, $Q_i > 0$, $R_i > 0$, $\begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} > 0$, $Q_\tau \ge 0$, $Q_\delta \ge 0$, $R_\tau \ge 0$, and two non-negative scalar $\varepsilon_1 \ge 0, \varepsilon_2 \ge 0$ and any matrices $Z_j = \begin{bmatrix} Z_{j1} & Z_{j2} \\ * & Z_{j3} \end{bmatrix}$, $(j = 1,2)$, M_i, N_i, $(i = 1,2,3,4)$ with appropriate dimensions such that the LMIs in (10) are feasible.

$$\Omega(i,k) := \Omega_0 + \Omega_1^k + \Omega_2^k + \Xi_i^k < 0, (i = 1,2; k = 3,4) \quad (10)$$

with $\begin{bmatrix} \tau_3 R_3 + (1-h_d)R_\tau & [M_1 \quad N_1] \\ * & Z_1 \end{bmatrix} \ge 0$,

$$\begin{bmatrix} \tau_4 R_4 + (1-h_d)R_\tau & [M_4 \quad N_4] \\ * & Z_2 \end{bmatrix} \ge 0$$

where

$$\begin{aligned}\Omega_0 = {} & e_1(P_1A + A^TP_1 + Q_1 + Q_\delta + S_1 - R_1 + \varepsilon_1\lambda^2F^TF)e_1^T \\ & + e_1P_1A_\tau e_2^T + e_1(S_2 + R_1)e_3^T + e_1P_1(e_7^T + e_8^T) + e_1A^TP_2e_9^T \\ & - (1-\delta h_d)e_1P_2e_{10}^T + e_2(-(1-h_d)Q_\tau + \varepsilon_2\mu^2G^TG)e_2^T \\ & + e_2A_\tau^TP_2e_9^T + e_3(Q_2 - Q_1 + S_3 - S_1 - R_1 - R_2)e_3^T \\ & + e_3(-S_2 + R_2)e_4^T + e_4(Q_3 - Q_2 + Q_\tau - S_3 - R_2)e_4^T \\ & + e_5(Q_4 - Q_3)e_5^T - e_6Q_4e_6^T - \varepsilon_1e_7e_7^T - e_7P_2e_9^T - \varepsilon_2e_8e_8^T \\ & - e_8P_2e_9^T - (1-\delta h_d)e_9Q_\delta e_9^T - (1-\delta h_d)e_9P_3e_{10}^T \\ & - (1-\delta h_d)e_{10}P_\delta e_{10}^T,\end{aligned}$$

$$\begin{aligned}\Omega_1^3 = {} & \begin{bmatrix} e_4 & e_2 \end{bmatrix}\begin{bmatrix} M_1 + M_1^T & -M_1 + N_1^T \\ * & -N_1 - N_1^T \end{bmatrix}\begin{bmatrix} e_4^T \\ e_2^T \end{bmatrix} \\ & + \tau_3\begin{bmatrix} e_2 & e_5 \end{bmatrix}\begin{bmatrix} M_2 + M_2^T & -M_2 + N_2^T \\ * & -N_2 - N_2^T \end{bmatrix}\begin{bmatrix} e_2^T \\ e_5^T \end{bmatrix} \\ & - e_5(R_4)e_5^T + e_5(R_4)e_6^T - e_6(R_4)e_6^T,\end{aligned}$$

$$\Omega_1^4 = \begin{bmatrix} e_5 & e_2 \end{bmatrix}\begin{bmatrix} M_4 + M_4^T & -M_4 + N_4^T \\ * & -N_4 - N_4^T \end{bmatrix}\begin{bmatrix} e_5^T \\ e_2^T \end{bmatrix}$$

$$+\tau_4\begin{bmatrix}e_2 & e_6\end{bmatrix}\begin{bmatrix}M_3+M_3^T & -M_3+N_3^T\\ * & -N_3-N_3^T\end{bmatrix}\begin{bmatrix}e_2^T\\ e_6^T\end{bmatrix}$$
$$+\begin{bmatrix}e_4 & e_5\end{bmatrix}\left\{\begin{bmatrix}M_1+M_1^T & -M_1+N_1^T\\ * & -N_1-N_1^T\end{bmatrix}+\tau_3\begin{bmatrix}Z_{11} & Z_{12}\\ * & Z_{13}\end{bmatrix}\right\}\begin{bmatrix}e_4^T\\ e_5^T\end{bmatrix},$$

$$\Omega_2^3 := \Gamma^T(\sum_{i=1}^4 \tau_i^2 R_i + \overline{\tau} R_\tau + P_\delta)\Gamma,$$

$$\Omega_2^4 := \Gamma^T(\sum_{i=1}^4 \tau_i^2 R_i + \overline{\tau} R_\tau + P_\delta)\Gamma,\ \Gamma := Ae_1 + A_\tau e_2,$$

$$\Xi_1^k := \tau_k Z^k,\ Z^3 = \begin{bmatrix}e_4 & e_2\end{bmatrix}\begin{bmatrix}Z_{11} & Z_{12}\\ * & Z_{13}\end{bmatrix}\begin{bmatrix}e_4^T\\ e_2^T\end{bmatrix},$$

$$Z^4 = \begin{bmatrix}e_5 & e_2\end{bmatrix}\begin{bmatrix}Z_{21} & Z_{22}\\ * & Z_{23}\end{bmatrix}\begin{bmatrix}e_5^T\\ e_2^T\end{bmatrix},\ \Xi_2^k := \tau_k^2\Phi^k R_k^{-1}(\Phi^k)^T,$$

$$\Phi^3 := col\{0, M_2, 0, 0, N_2, 0, 0, 0, 0, 0\},$$

$$\Phi^4 := col\{0, M_3, 0, 0, 0, N_3, 0, 0, 0, 0\}. \tag{11}$$

(ii) In Case II, if the LMIs in (10) with $Q_\tau = 0, R_\tau = 0$, $P_\delta = 0, Q_\delta = 0$ are feasible, the system (1) is asymptotically stable for fast time-varying delay.

Proof. Choose the following delay-dependent LK functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \tag{12}$$

where

$$V_1(t) = \begin{bmatrix}x(t)\\ -x(t-\delta\tau(t))\end{bmatrix}^T P\begin{bmatrix}x(t)\\ -x(t-\delta\tau(t))\end{bmatrix} + \int_{t-\delta\tau(t)}^t \dot{x}^T(s)P_\delta\dot{x}(s)ds,$$
$$V_2(t) = \sum_{i=1}^4\int_{t-h_i}^{t-h_{i-1}} x^T(s)Q_i x(s)ds$$
$$+\int_{t-\tau(t)}^{t-h_2} x^T(s)Q_\tau x(s)ds$$
$$+\int_{t-\delta\tau(t)}^{t} x^T(s)Q_\delta x(s)ds,$$

$$V_3(t) = \int_{t-h_1}^t \begin{bmatrix}x(s)\\ x(s-\frac{\tau_0}{2})\end{bmatrix}^T\begin{bmatrix}S_1 & S_2\\ * & S_3\end{bmatrix}\begin{bmatrix}x(s)\\ x(s-\frac{\tau_0}{2})\end{bmatrix}ds,$$

$$V_4(t) = \sum_{i=1}^4 \tau_i\int_{-h_i}^{-h_{i-1}}\int_{t+\theta}^t \dot{x}^T(s)R_i\dot{x}(s)dsd\theta$$
$$+\int_{-\tau(t)}^{-h_2}\int_{t+\theta}^t \dot{x}^T(s)R_\tau\dot{x}(s)dsd\theta$$
$$+(h_4-\tau(t))\int_{t-h_2}^t \dot{x}^T(s)R_\tau\dot{x}(s)ds$$

with $P = \begin{bmatrix}P_1 & P_2\\ * & P_3\end{bmatrix} > 0$ and $P_1 > 0$, $\begin{bmatrix}S_1 & S_2\\ * & S_3\end{bmatrix} > 0$, $P_\delta \ge 0$, $Q_i > 0, R_i > 0, (i = 1,2,3,4)$, $Q_\tau \ge 0$, $Q_\delta \ge 0$, $R_\tau \ge 0$ being real symmetric matrices.

Our aim is to show that the condition $\dot{V}(t) \le -\varepsilon\|x(t)\|, (\varepsilon > 0)$ is guaranteed if the LMI in (10) hold. Then, taking the derivative of (12) with respect to t along the trajectory of system (1) has the following as

$$\begin{aligned}\dot{V}_1(t) &= \begin{bmatrix}x(t)\\ -x(t-\delta\tau(t))\end{bmatrix}^T P\begin{bmatrix}Ax(t)+A_\tau x(t-\tau(t))+f+g\\ 0\end{bmatrix}\\ &+\begin{bmatrix}Ax(t)+A_\tau x(t-\tau(t))+f+g\\ 0\end{bmatrix}^T P\begin{bmatrix}x(t)\\ -x(t-\delta\tau(t))\end{bmatrix}\\ &-(1-\delta\dot{\tau}(t))\left\{\begin{bmatrix}x(t)\\ -x(t-\delta\tau(t))\end{bmatrix}^T P\begin{bmatrix}0\\ \dot{x}(t-\delta\tau(t))\end{bmatrix}+\begin{bmatrix}0\\ \dot{x}(t-\delta\tau(t))\end{bmatrix}^T P\begin{bmatrix}x(t)\\ -x(t-\delta\tau(t))\end{bmatrix}\right\}\\ &+\dot{x}^T(t)P_\delta\dot{x}^T(t)-(1-\delta\dot{\tau}(t))\dot{x}^T(t-\delta\tau(t))P_\delta\dot{x}^T(t-\delta\tau(t))\end{aligned} \tag{13}$$

$$\begin{aligned}\dot{V}_2(t) &= \sum_{i=1}^4\left[x^T(t-h_{i-1})Q_i x(t-h_{i-1})-x^T(t-h_i)Q_i x(t-h_i)\right]\\ &+x^T(t-h_2)Q_\tau x(t-h_2)-(1-\dot{\tau}(t))x^T(t-\tau(t))Q_\tau x(t-\tau(t))\\ &\le\sum_{i=1}^4\left[x^T(t-h_{i-1})Q_i x(t-h_{i-1})-x^T(t-h_i)Q_i x(t-h_i)\right]\\ &+x^T(t-h_2)Q_\tau x(t-h_2)-(1-h_d)x^T(t-\tau(t))Q_\tau x(t-\tau(t))\\ &+x^T(t)Q_\delta x(t)-(1-\delta h_d)x^T(t-\delta\tau(t))Q_\delta x(t-\delta\tau(t))\end{aligned} \tag{14}$$

$$\begin{aligned}\dot{V}_3(t) &= \begin{bmatrix}x(t)\\ x(t-h_1)\end{bmatrix}^T\begin{bmatrix}S_1 & S_2\\ * & S_3\end{bmatrix}\begin{bmatrix}x(t)\\ x(t-h_1)\end{bmatrix}\\ &-\begin{bmatrix}x(t-h_1)\\ x(t-h_2)\end{bmatrix}^T\begin{bmatrix}S_1 & S_2\\ * & S_3\end{bmatrix}\begin{bmatrix}x(t-h_1)\\ x(t-h_2)\end{bmatrix}\end{aligned} \tag{15}$$

$$\begin{aligned}\dot{V}_4(t) &= \sum_{i=1}^4\tau_i^2\dot{x}^T(t)R_i\dot{x}(t)+(\tau(t)-h_2)\dot{x}^T(t)R_\tau\dot{x}(t)\\ &-\sum_{i=1}^4\tau_i\int_{t-h_i}^{t-h_{i-1}}\dot{x}^T(s)R_i\dot{x}(s)ds\\ &-(1-\dot{\tau}(t))\int_{t-\tau(t)}^{t-h_2}\dot{x}^T(s)R_\tau\dot{x}(s)ds\\ &+\dot{\tau}(t)\int_{t-h_2}^t\dot{x}^T(s)R_\tau\dot{x}(s)ds\\ &-\dot{\tau}(t)\int_{t-h_2}^t\dot{x}^T(s)R_\tau\dot{x}(s)ds\\ &+(h_4-\tau(t))\left[\dot{x}^T(t)R_\tau\dot{x}(t)-\dot{x}^T(t-h_2)R_\tau\dot{x}(t-h_2)\right]\end{aligned} \tag{16}$$

For any $t \ge 0$, it is the fact that $h_a \le \tau(t) \le h_a + \alpha\tau$ or $h_a + \alpha\tau \le \tau(t) \le h_b$, $(0 < \alpha < 1)$. In the case of $h_a \le \tau(t) \le h_a + \alpha\tau$, i.e., $\tau(t) \in [h_2, h_3]$, $k = 3$, suitably using the integral inequalities in Lemma 1, the following inequalities are true:

$$
\begin{aligned}
&(\tau(t)-h_2)\dot{x}^T(t)R_\tau\dot{x}(t) \\
&\quad +(h_4-\tau(t))\left[\dot{x}^T(t)R_\tau\dot{x}(t)-\dot{x}^T(t-h_2)R_\tau\dot{x}(t-h_2)\right] \\
&=\bar{\tau}\dot{x}^T(t)R_\tau\dot{x}(t) \qquad (17)\\
&\quad -(h_4-\tau(t))\dot{x}^T(t-h_2)R_\tau\dot{x}(t-h_2) \\
&\le \bar{\tau}\dot{x}^T(t)R_\tau\dot{x}(t)
\end{aligned}
$$

$$
\begin{aligned}
&-\tau_i\int_{t-h_i}^{t-h_{i-1}}\dot{x}^T(s)R_i\dot{x}(s)ds \\
&\le \begin{bmatrix} x(t-h_{i-1}) \\ x(t-h_i)\end{bmatrix}^T \begin{bmatrix} -R_i & R_i \\ * & -R_i\end{bmatrix}\begin{bmatrix} x(t-h_{i-1}) \\ x(t-h_i)\end{bmatrix} \qquad (18)\\
&(i=1,2,4)
\end{aligned}
$$

and,

$$
\begin{aligned}
&-\tau_3\int_{t-h_3}^{t-h_2}\dot{x}^T(s)R_3\dot{x}(s)ds-(1-\dot{\tau}(t))\int_{t-\tau(t)}^{t-h_2}\dot{x}^T(s)R_\tau\dot{x}(s)ds \\
&\quad +\dot{\tau}(t)\int_{t-h_2}^{t}\dot{x}^T(s)R_\tau\dot{x}(s)ds \\
&\le -\tau_3\int_{t-h_3}^{t-h_2}\dot{x}^T(s)R_3\dot{x}(s)ds-(1-h_d)\int_{t-\tau(t)}^{t-h_2}\dot{x}^T(s)R_\tau\dot{x}(s)ds \\
&=-\int_{t-\tau(t)}^{t-h_2}\dot{x}^T(s)\left(\tau_3R_3+(1-h_d)R_\tau\right)\dot{x}(s)ds \\
&\quad -\tau_3\int_{t-h_3}^{t-\tau(t)}\dot{x}^T(s)R_3\dot{x}(s)ds \\
&\le \begin{bmatrix} x(t-h_2) \\ x(t-\tau(t))\end{bmatrix}^T \left\{\begin{bmatrix} M_1+M_1^T & -M_1+N_1^T \\ * & -N_1-N_1^T\end{bmatrix} +\rho\cdot\alpha\bar{\tau}\cdot Z_1\right\}\begin{bmatrix} x(t-h_2) \\ x(t-\tau(t))\end{bmatrix} \\
&\quad +\begin{bmatrix} x(t-\tau(t)) \\ x(t-h_3)\end{bmatrix}^T \left\{\tau_3\begin{bmatrix} M_2+M_2^T & -M_2+N_2^T \\ * & -N_2-N_2^T\end{bmatrix} +(1-\rho)\cdot\alpha\bar{\tau}\cdot\begin{bmatrix} M_2 \\ N_2\end{bmatrix}R_3^{-1}\begin{bmatrix} M_2 \\ N_2\end{bmatrix}^T\right\}\begin{bmatrix} x(t-\tau(t)) \\ x(t-h_3)\end{bmatrix}
\end{aligned} \qquad (19)
$$

with $\begin{bmatrix} \tau_3R_3+(1-h_d)R_\tau & [M_1 \quad N_1] \\ * & Z_1\end{bmatrix}\ge 0$,and $\rho=\dfrac{\tau(t)-h_2}{\alpha\bar{\tau}}$, $0\le\rho\le 1$.

On the other hand, for any scalars $\varepsilon_1\ge 0,\varepsilon_2\ge 0$,it follows from (2) and (3) that

$$
0\le\varepsilon_1\left[\lambda^2x^T(t)F^TFx(t)-f^T(x(t),t)f(x(t),t)\right] \qquad (20)
$$

$$
0\le\varepsilon_2\left[\mu^2x^T(t-\tau(t))G^TGx(t-\tau(t))-g^T(x(t-\tau(t)),t)g(x(t-\tau(t)),t)\right] \qquad (21)
$$

It follows from (13)-(21) that:

$$
\begin{aligned}
\dot{V}(t)&\le\rho\xi^T(t)\begin{bmatrix}\Omega_0+\Omega_1^3+\alpha\bar{\tau}Z^3 \\ +\Gamma^T(\sum_{i=1}^{4}\tau_i^2R_i+\alpha\bar{\tau}R_\tau+P_\delta)\Gamma\end{bmatrix}\xi(t) \\
&\quad +(1-\rho)\xi^T(t)\begin{bmatrix}\Omega_0+\Omega_1^3+(\alpha\bar{\tau})^2\Phi^3R_3^{-1}(\Phi^3)^T \\ +\Gamma^T(\sum_{i=1}^{4}\tau_i^2R_i+\alpha\bar{\tau}R_\tau+P_\delta)\Gamma\end{bmatrix}\xi(t) \qquad (22)
\end{aligned}
$$

with $\begin{bmatrix} \tau_3R_3+(1-h_d)R_\tau & [M_1 \quad N_1] \\ * & Z_1\end{bmatrix}\ge 0$,where Ω_0,Ω_1^3, Γ, Z^3,Φ^3 are defined in (11).

Since $0\le\rho\le 1$,applying the convex combination method, a conclusion can be obtained that if the following LMIs

$$
\begin{cases}
\Omega_0+\Omega_1^3+\Gamma^T(\sum_{i=1}^{4}\tau_i^2R_i+\alpha\bar{\tau}R_\tau+P_\delta)\Gamma+\alpha\bar{\tau}Z^3<0 \\
\Omega_0+\Omega_1^3+\Gamma^T(\sum_{i=1}^{4}\tau_i^2R_i+\alpha\bar{\tau}R_\tau+P_\delta)\Gamma \\
\qquad +(\alpha\bar{\tau})^2\Phi^3R_3^{-1}(\Phi^3)^T<0
\end{cases} \qquad (23)
$$

hold simultaneously, then $\dot{V}(t)<-\varepsilon\|x(t)\|$ is true for a sufficiently small $\varepsilon>0$.

Meanwhile, if $h_a+\alpha\tau\le\tau(t)\le h_b$,i.e., $\tau(t)\in\left[h_3,h_4\right]$, $k=4$,similar to the above deduction process, the similar stability conditions also can be obtained.

This completes the proof.

Remark 1

It maybe noted that, in the above, no approximation of the delay term is involved excepting exploiting a convex combination of the uncertain terms involved. And, it is worth pointing out that in the case that $h_d<1$,it is clear that less conservative stability criteria can be derived by introducing the term $\int_{t-\tau(t)}^{t-h_2}x^T(s)Q_\tau x(s)ds$ in (12). However, in the case that $h_d\ge 1$,it can be seen from (Zhu and Yang,2008) that this term has no help for deriving less conservative stability criteria. Instead, in (12),some new terms $\int_{t-\delta\tau(t)}^{t}x^T(s)Q_\delta x(s)ds$, $\int_{t-\delta\tau(t)}^{t}\dot{x}^T(s)P_\delta\dot{x}(s)ds$,and $\begin{bmatrix} x(t) \\ -x(t-\delta\tau(t))\end{bmatrix}^T P\begin{bmatrix} x(t) \\ -x(t-\delta\tau(t))\end{bmatrix}$ with $0<\delta<1$, h_d satisfying $\delta h_d<1$ are added to develop some less stability criteria. Such a feature leads to less conservative results compared to the existing ones as is shown in next section using numerical examples.

Remark 2

It's worthy of mentioning that Lemma 1 plays a key effect on the present results, which is different from the common Jensen's inequality, although use of Jensen's inequality is always desired since it does not involve additional free variables besides being equivalent to (6) and (7) in Lemma 1,which can be established following the equivalency results in (Briat,2011). However, if $h(t)$ is uncertain and required to be approximated with its lower or upper bound then use of (8) and (9) would be beneficial since the free variables M,N are introduced. Moreover, partitioning of the above intervals into $n>2$ subintervals may lead to further improvements, in (Gao et al., 2006; Briat, 2011) another delay partitioning was introduced, which corresponded to the partitioning into two subintervals of $[0, h_a]$ and of $[h_a, h_b]$. Our method provides some less conservative results than the existing ones that have been recently proposed (He et al., 2007a; Shao, 2009; Gao et al., 2006; Briat, 2011; Park et al., 2011), some numerical examples below show this point.

When the lower bound of the delay is 0, that is, $h_a = 0$, the interval team $\left[0, h_a\right]$ is missing, by setting $Q_1 = 0; Q_2 = 0$; $S_i = 0; R_1 = 0; R_2 = 0$ in (12), according to Theorem 1, and then obtain the following corollary 1. For simplicity, the marks are denoted

$$\hat{\xi}(t) := col\{x(t), x(t-\tau(t)), x(t-h_3), \\ x(t-h_4), f, g, x(t-\delta\tau(t)), \dot{x}(t-\delta\tau(t))\},$$

and $\hat{e}_i, (i = 1, 2, \cdots, 8)$ are block entry matrices, for example, $\hat{e}_1^T = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Corollary 1

(i) In Case 1, for given scalars $0 = h_a \le h_b$, $0 < \alpha < 1$, and $0 < \delta < 1$, h_d satisfying $\delta h_d < 1$, the system (1) is asymptotically stable if there exist real symmetric matrices $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \ge 0$ with $P_1 > 0$, $P_\delta > 0$, $Q_\tau \ge 0$, $Q_\delta \ge 0$, $R_\tau \ge 0$, $Q_i > 0$, $R_i > 0$, $(i = 3, 4)$, and two non-negative scalars $\varepsilon_1 \ge 0, \varepsilon_2 \ge 0$ and any matrices $Z_j = \begin{bmatrix} Z_{j1} & Z_{j2} \\ * & Z_{j3} \end{bmatrix}$, $(j = 1, 2)$, $M_i, N_i, (i = 1, 2, 3, 4)$ with appropriate dimensions such that the LMIs in (24) are feasible.

$$\hat{\Omega}(i,k) := \hat{\Omega}_0 + \hat{\Omega}_1^k + \hat{\Omega}_2^k + \hat{\Xi}_i^k < 0, (i = 1, 2; k = 3, 4) \tag{24}$$

with

$$\begin{bmatrix} \tau_3 R_3 + (1-h_d) R_\tau & [M_1 \quad N_1] \\ * & Z_1 \end{bmatrix} \ge 0,$$

$$\begin{bmatrix} \tau_4 R_4 + (1-h_d) R_\tau & [M_4 \quad N_4] \\ * & Z_2 \end{bmatrix} \ge 0$$

where

$$\begin{aligned} \hat{\Omega}_0 = {} & \hat{e}_1(P_1 A + A^T P_1 + Q_\tau + Q_\delta + Q_3 + \varepsilon_1 \lambda^2 F^T F)\hat{e}_1^T + \hat{e}_1 P_1 A_\tau \hat{e}_2^T \\ & + \hat{e}_1 P(\hat{e}_5^T + \hat{e}_6^T) + \hat{e}_1 A^T P_2 \hat{e}_7^T - (1-\delta h_d)\hat{e}_1 P_2 \hat{e}_8^T \\ & + \hat{e}_2(-(1-h_d)Q_\tau + \varepsilon_2 \mu^2 G^T G)\hat{e}_2^T + \hat{e}_2 A_\tau^T P_2 \hat{e}_7^T \\ & + \hat{e}_3(Q_4 - Q_3)\hat{e}_3^T - \hat{e}_4 Q_4 \hat{e}_4^T - \varepsilon_1 \hat{e}_5 \hat{e}_5^T - \hat{e}_5 P_2 \hat{e}_7^T - \varepsilon_2 \hat{e}_6 \hat{e}_6^T \\ & - \hat{e}_6 P_2 \hat{e}_7^T - (1-\delta h_d)\hat{e}_7 Q_\delta \hat{e}_7^T - (1-\delta h_d)\hat{e}_7 P_3 \hat{e}_8^T \\ & - (1-\delta h_d)\hat{e}_8 P_\delta \hat{e}_8^T, \end{aligned}$$

$$\begin{aligned} \hat{\Omega}_1^3 = {} & \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix}^T \begin{bmatrix} M_1 + M_1^T & -M_1^T + N_1 \\ * & -N_1 - N_1^T \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} \\ & + \tau_3 \begin{bmatrix} \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}^T \begin{bmatrix} M_2 + M_2^T & -M_2 + N_2^T \\ * & -N_2 - N_2^T \end{bmatrix} \begin{bmatrix} \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \\ & - \hat{e}_3^T (R_4)\hat{e}_3 + \hat{e}_3^T (R_4)\hat{e}_4 - \hat{e}_4^T (R_4)\hat{e}_4, \end{aligned}$$

$$\begin{aligned} \hat{\Omega}_1^4 = {} & \begin{bmatrix} \hat{e}_3 \\ \hat{e}_2 \end{bmatrix}^T \begin{bmatrix} M_4 + M_4^T & -M_4 + N_4^T \\ * & -N_4 - N_4^T \end{bmatrix} \begin{bmatrix} \hat{e}_3 \\ \hat{e}_2 \end{bmatrix} \\ & + \tau_4 \begin{bmatrix} \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}^T \begin{bmatrix} M_3 + M_3^T & -M_3 + N_3^T \\ * & -N_3 - N_3^T \end{bmatrix} \begin{bmatrix} \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} \\ & + \begin{bmatrix} \hat{e}_1 \\ \hat{e}_3 \end{bmatrix}^T \left\{ \begin{bmatrix} M_1 + M_1^T & -M_1 + N_1^T \\ * & -N_1 - N_1^T \end{bmatrix} + \tau_3 \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{13} \end{bmatrix} \right\} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_3 \end{bmatrix}, \end{aligned}$$

$$\hat{\Omega}_2^3 := \hat{\Gamma}^T (\sum_{i=3}^{4} \tau_i^2 R_i + \overline{\tau} R_\tau + P_\delta)\hat{\Gamma},$$

$$\hat{\Omega}_2^4 := \hat{\Gamma}^T (\sum_{i=3}^{4} \tau_i^2 R_i + \overline{\tau} R_\tau + P_\delta)\hat{\Gamma},$$

$$\hat{\Gamma} := A\hat{e}_1 + A_\tau \hat{e}_2, \hat{\Xi}_1^k := \tau_k \hat{Z}^k, (k = 3, 4),$$

$$\hat{Z}^3 = \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix}^T \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{13} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix},$$

$$\hat{Z}^4 = \begin{bmatrix} \hat{e}_3 \\ \hat{e}_2 \end{bmatrix}^T \begin{bmatrix} Z_{21} & Z_{22} \\ * & Z_{23} \end{bmatrix} \begin{bmatrix} \hat{e}_3 \\ \hat{e}_2 \end{bmatrix}, \hat{\Xi}_2^k := \tau_k^2 \hat{\Phi}^k R_k^{-1} (\hat{\Phi}^k)^T, (k = 3, 4),$$

$$\hat{\Phi}^3 := col\{0, M_2, N_2, 0, 0, 0, 0, 0\},$$

$$\hat{\Phi}^4 := col\{0, M_3, 0, N_3, 0, 0, 0, 0\}. \tag{25}$$

(ii) In Case II, if the LMIs in (24) with $Q_\tau = 0, R_\tau = 0$, $P_\delta = 0, Q_\delta = 0$ are feasible, the system (1) is asymptotically stable for fast time-varying delay.

Remark 3

In this paper, the relationship among $x(t), x(t-\tau(t))$, $x(t-\frac{h_a}{2}), x(t-h_a)$, $x(t-h_a-\alpha\overline{\tau})$ and $x(t-h_b)$ can be full utilized to construct the LK functional, which is expected to yield less conservative results. Notice that the h_a may not be restricted to be zero. In addition, when constructing the LK Functional in (12), the information on the lower bound of the delay is token full advantage of by introducing the terms $\sum_{i=1}^{2} \int_{t-h_i}^{t-h_{i-1}} x^T(s) Q_i x(s) ds$ and $\int_{t-\tau(t)}^{t-h_2} x^T(s) R_\tau x(s) ds$ in the LK Functional. From Examples 1-2 below, it is obvious that our approaches are less conservative than the existing ones.

If there is no perturbation, that is, $f = 0$ and $g = 0$, then the stability problem of system (1) is reduced to analyzing the stability of the following linear system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + A_\tau(t)x(t-\tau(t)) \\ x(t) = \phi(t), \forall t \in [-h_b, 0], \end{cases} \tag{26}$$

This problem has been widely studied in the recent literatures (see,e.g. (Shao, 2009; Sun et al., 2010; Park et al., 2011; Shao and Han, 2012; Qian et al., 2012; Zhu et al., 2010; Tang et al., 2012)) and stability criterion for the deterministic system is stated below. Here, the marks are denoted

$$\tilde{\xi}(t) := col\{x(t), x(t-\tau(t)), x(t-h_1), x(t-h_2), \\ x(t-h_3), x(t-h_4), x(t-\delta\tau(t)), \dot{x}(t-\delta\tau(t))\}$$

and $\tilde{e}_i, (i = 1, 2, \cdots, 8)$ are block entry matrices, for example, $\tilde{e}_1^T = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Theorem 2

(i) In Case 1, for given scalars $0 < h_a \le h_b$, $0 < \alpha < 1$,and $0 < \delta < 1$, h_d satisfying $\delta h_d < 1$, the system (26) is asymptotically stable if there exist real symmetric matrices $P = \begin{bmatrix} P_1 & P_2 \\ * & P_3 \end{bmatrix} \ge 0$ with $P_1 > 0$, $P_\delta > 0$, $Q_i > 0$, $R_i > 0$, $Q_\tau \ge 0$, $Q_\delta \ge 0$, $R_\tau \ge 0$, $\begin{bmatrix} S_1 & S_2 \\ * & S_3 \end{bmatrix} > 0$,and any matrices $\tilde{Z}_j = \begin{bmatrix} \tilde{Z}_{j1} & \tilde{Z}_{j2} \\ * & \tilde{Z}_{j3} \end{bmatrix}$, $(j = 1, 2)$, $\tilde{M}_i, \tilde{N}_i$, $(i = 1, 2, 3, 4)$ with appropriate dimensions such that the LMIs in (27) are feasible.

$$\tilde{\Omega}(i,k) := \tilde{\Omega}_0 + \tilde{\Omega}_1^k + \tilde{\Omega}_2^k + \tilde{\Xi}_i^k < 0, (i = 1, 2; k = 3, 4) \tag{27}$$

with

$$\begin{bmatrix} \tau_3 R_3 + (1-h_d)R_\tau & \begin{bmatrix} \tilde{M}_1 & \tilde{N}_1 \end{bmatrix} \\ * & \tilde{Z}_1 \end{bmatrix} \ge 0,$$

$$\begin{bmatrix} \tau_4 R_4 + (1-h_d)R_\tau & \begin{bmatrix} \tilde{M}_4 & \tilde{N}_4 \end{bmatrix} \\ * & \tilde{Z}_2 \end{bmatrix} \ge 0$$

where

$$\begin{aligned} \tilde{\Omega}_0 &= \tilde{e}_1(P_1 A + A^T P_1 + Q_1 + Q_\delta + S_1 - R_1)\tilde{e}_1^T + \tilde{e}_1(P_1 A_\tau)\tilde{e}_2^T \\ &+ \tilde{e}_1(S_2 + R_1)\tilde{e}_3^T + \tilde{e}_1 A^T P_2 \tilde{e}_7^T - (1-\delta h_d)\tilde{e}_1 P_2 \tilde{e}_8^T \\ &- \tilde{e}_2(1-h_d)Q_\tau \tilde{e}_2^T + \tilde{e}_2 A_\tau^T P_2 \tilde{e}_7^T \\ &+ \tilde{e}_3(Q_2 - Q_1 + S_3 - S_1 - R_1 - R_2)\tilde{e}_3^T \\ &+ \tilde{e}_3(-S_2 + R_2)\tilde{e}_4^T + \tilde{e}_4(Q_3 - Q_2 + Q_\tau - S_3 - R_2)\tilde{e}_4^T \\ &+ \tilde{e}_5(Q_4 - Q_3)\tilde{e}_5^T - \tilde{e}_6(Q_4)\tilde{e}_6^T - (1-\delta h_d)\tilde{e}_7 Q_\delta \tilde{e}_7^T \\ &- (1-\delta h_d)\tilde{e}_7 P_3 \tilde{e}_8^T - (1-\delta h_d)\tilde{e}_8 P_\delta \tilde{e}_8^T, \end{aligned}$$

$$\begin{aligned} \tilde{\Omega}_1^3 &= \begin{bmatrix} \tilde{e}_4 & \tilde{e}_2 \end{bmatrix} \begin{bmatrix} M_1 + M_1^T & -M_1 + N_1^T \\ * & -N_1 - N_1^T \end{bmatrix} \begin{bmatrix} \tilde{e}_4^T \\ \tilde{e}_2^T \end{bmatrix} \\ &+ \tau_3 \begin{bmatrix} \tilde{e}_2 & \tilde{e}_5 \end{bmatrix} \begin{bmatrix} M_2 + M_2^T & -M_2 + N_2^T \\ * & -N_2 - N_2^T \end{bmatrix} \begin{bmatrix} \tilde{e}_2^T \\ \tilde{e}_5^T \end{bmatrix} \\ &- \tilde{e}_5 R_4 \tilde{e}_5^T + \tilde{e}_5 R_4 \tilde{e}_6^T - \tilde{e}_6 R_4 \tilde{e}_6^T, \end{aligned}$$

$$\begin{aligned} \tilde{\Omega}_1^4 &= \begin{bmatrix} \tilde{e}_5 & \tilde{e}_2 \end{bmatrix} \begin{bmatrix} \tilde{M}_4 + \tilde{M}_4^T & -\tilde{M}_4 + \tilde{N}_4^T \\ * & -\tilde{N}_4 - \tilde{N}_4^T \end{bmatrix} \begin{bmatrix} \tilde{e}_5^T \\ \tilde{e}_2^T \end{bmatrix} \\ &+ \tau_4 \begin{bmatrix} \tilde{e}_2 & \tilde{e}_6 \end{bmatrix} \begin{bmatrix} \tilde{M}_3 + \tilde{M}_3^T & -\tilde{M}_3 + \tilde{N}_3^T \\ * & -\tilde{N}_3 - \tilde{N}_3^T \end{bmatrix} \begin{bmatrix} \tilde{e}_2^T \\ \tilde{e}_6^T \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{e}_4 & \tilde{e}_5 \end{bmatrix} \left\{ \begin{bmatrix} \tilde{M}_1 + \tilde{M}_1^T & -\tilde{M}_1 + \tilde{N}_1^T \\ * & -\tilde{N}_1 - \tilde{N}_1^T \end{bmatrix} + \tau_3 \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ * & \tilde{Z}_{13} \end{bmatrix} \right\} \begin{bmatrix} \tilde{e}_4^T \\ \tilde{e}_5^T \end{bmatrix}, \end{aligned}$$

$$\tilde{\Omega}_2^3 := \tilde{\Gamma}^T (\sum_{i=1}^{4} \tau_i^2 R_i + \overline{\tau} R_\tau + P_\delta)\tilde{\Gamma},$$

$$\tilde{\Omega}_2^4 := \tilde{\Gamma}^T (\sum_{i=1}^{4} \tau_i^2 R_i + \overline{\tau} R_\tau + P_\delta)\tilde{\Gamma}, \tilde{\Gamma} := A\tilde{e}_1 + A_\tau \tilde{e}_2,$$

$$\tilde{\Xi}_1^k := \tau_k \tilde{Z}^k, \tilde{Z}^3 = \begin{bmatrix} \tilde{e}_4 & \tilde{e}_2 \end{bmatrix} \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ * & \tilde{Z}_{13} \end{bmatrix} \begin{bmatrix} \tilde{e}_4^T \\ \tilde{e}_2^T \end{bmatrix},$$

$$\tilde{Z}^4 = \begin{bmatrix} \tilde{e}_5 & \tilde{e}_2 \end{bmatrix} \begin{bmatrix} \tilde{Z}_{21} & \tilde{Z}_{22} \\ * & \tilde{Z}_{23} \end{bmatrix} \begin{bmatrix} \tilde{e}_5^T \\ \tilde{e}_2^T \end{bmatrix},$$

$$\tilde{\Xi}_2^k := \tau_k^2 \tilde{\Phi}^k R_k^{-1} (\tilde{\Phi}^k)^T, \tilde{\Phi}^3 := col\{0, \tilde{M}_2, 0, 0, \tilde{N}_2, 0, 0, 0\},$$

$$\tilde{\Phi}^4 := col\{0, \tilde{M}_3, 0, 0, 0, \tilde{N}_3, 0, 0\}. \tag{28}$$

(ii) In Case II, if the LMIs in (27) with $Q_\tau = 0, R_\tau = 0$, $P_\delta = 0, Q_\delta = 0$ are feasible, the system (26) is asymptotically stable for fast time-varying delay.

Proof. it can complete the proof as the similar line of derivation process of Theorem 1 and then omit here.

Remark 4

Theorems 1 and 2 depend on the parameter $0 < \delta < 1$ satisfying $\delta d_2 < 1$. The problem on how to choose δ to derive a better upper bound h_b for given h_a, d_1, d_2 may be solved numerically by using a numerical optimization algorithm, such as min search in the Optimization Toolbox of MATLAB, and one can refer to (Zhang and Han, 2013). Moreover, we would like to point out that our main results can be extended to more general/practical systems such as nonlinear systems or fuzzy systems, NCSs and multi-delays systems, and the corresponding results will appear in the near future.

4. NUMERICAL EXAMPLES

In this section, two numerical examples are presented to show the merit and effectiveness of the proposed method.

Example 1. Consider the system (1) with interval time-varying delay and nonlinear perturbations:

$$A = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix}, A_\tau = \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix}, F = G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which was considered in (Zuo and Wang, 2006; Han, 2004). For given values of λ, μ, h_d, and $\delta = 0.8$, the MADB of h_b for $h_a = 0$ are calculated and listed in Table 1. Moreover, when $h_d = 0.5$, for the case of $\lambda = 0$, $\mu = 0.1$ or $\lambda = 0.1$, $\mu = 0.1$, the MADB on h_b was computed as $h_b = 0.6743$ and $h_b = 0.5716$ in (Han, 2004), respectively. From Table 1, it is easy to see that the proposed approach yields less conservative than the existing ones.

Table 1. Comparison to the MADB on h_b with various h_d under $h_a = 0$ for Example 1.

Methods	$\lambda = 0$, $\mu = 0.1$			$\lambda = 0.1$, $\mu = 0.1$		
	h_d =0.5	h_d =0.9	h_d =1.1	h_d =0.5	h_d =0.9	h_d =1.1
Zuo and Wang, 2006	1.1424	0.7380	0.7355	1.0090	0.7140	0.7147
He et al., 2007b	1.4422	1.2807	1.2807	1.2848	1.2099	1.2099
Zhang et al., 2010	1.4420	1.2800	1.2800	1.2840	1.2090	1.2090
Ramakrishnan and Ray, 2011	1.4430	1.4080	1.4080	1.2870	1.2790	1.2790
Zeng et al., 2012 ($N = 3$)	1.6549	1.5964	1.5964	1.4698	1.4577	1.4577
Cor. 1 ($\alpha = 0.6$)	1.7672	1.7547	1.7547	1.6039	1.6019	1.6019
Cor. 1 ($\alpha = 0.7$)	1.7837	1.7466	1.7466	1.6116	1.6019	1.6019

Compare to the existing ones, we assume that h_d is unknown, the MADB on h_b for various h_a are also calculated using Theorem 1 and Corollary 1 ($h_a = 0$),which are listed in Table 2. Note that the criteria in (Zuo and Wang, 2006; Han, 2004; Zeng et al., 2012) can only handle the case of $h_a = 0$. Furthermore, when $h_a > 0$ and h_d is known, the MADB on h_b are computed in Table 3 for $h_a = 0.5$ and $h_a = 1.0$,respectively.

Table 2. Comparison to the MADB on h_b with various h_a under unknown h_d for Example 1

λ, μ	Methods	h_a =0	h_a =0.5	h_a =1.0
$\lambda = 0$, $\mu = 0.1$	Zuo and Wang, 2006	0.7355	——	——
	Zeng et al., 2012 ($N = 3$)	1.5964	——	——
	He et al., 2007b	1.2807	1.3083	1.5224
	Zhang et al., 2010	1.2807	1.3380	1.5430
	Ramakrishnan and Ray, 2011	1.4080	1.5580	1.7600
	Hui et al., 2013	1.4176	1.5636	1.7897
	Cor. 1 and Th. 1 ($\alpha = 0.55$)	1.7535	1.6027	1.7923
	Cor. 1 and Th. 1 ($\alpha = 0.6$)	1.7547	1.5988	1.7902
$\lambda = 0.1$, $\mu = 0.1$	Zuo and Wang, 2006	0.7147	——	——
	Zeng et al., 2012 ($N = 3$)	1.4577	——	——
	He et al., 2007b	1.2099	1.2219	1.3912
	Zhang et al., 2010	1.2099	1.2450	1.4080
	Ramakrishnan and Ray, 2011	1.2790	1.3840	1.5320
	Hui et al., 2013	1.2954	1.3858	1.5647
	Cor. 1 and Th. 1 ($\alpha = 0.55$)	1.6016	1.5078	1.5829
	Cor. 1 and Th. 1 ($\alpha = 0.6$)	1.6019	1.5050	1.5813

From the tables 2 and 3, it is clear that the proposed stability criterion is less conservative than those in (Zuo and Wang, 2006; Han, 2004; He et al., 2007b; Zhang et al., 2010; Ramakrishnan and Ray, 2011; Zeng et al., 2012; Hui et al., 2013). Meanwhile, it can be seen that the different delay decomposition parameter α yields the different LMI, and then leads to obtain the different MADB on h_b.

Table 3. Comparison to the MADB on h_b with various h_d for Example 1

h_a	Methods	$\lambda=0$, $\mu=0.1$			$\lambda=0.1$, $\mu=0.1$		
		h_d=0.5	h_d=0.9	h_d=1.1	h_d=0.5	h_d=0.9	h_d=1.1
0.5	Zhang et al., 2010	1.442	1.338	1.338	1.284	1.245	1.245
	Ramakrishnan and Ray, 2011	1.558	1.558	1.558	1.384	1.384	1.384
	Hui et al., 2013	1.5636	1.5636	1.5636	1.3858	1.3858	1.3858
	Th. 1 ($\alpha=0.7$)	1.8884	1.5784	1.5784	1.6677	1.4890	1.4890
	Th. 1 ($\alpha=0.6$)	1.8582	1.5988	1.5988	1.6489	1.5050	1.5050
1.0	Zhang et al., 2010	1.543	1.543	1.543	1.408	1.408	1.408
	Ramakrishnan and Ray, 2011	1.760	1.760	1.760	1.532	1.532	1.532
	Hui et al., 2013	1.7897	1.7897	1.7897	1.5647	1.5647	1.5647
	Th. 1 ($\alpha=0.7$)	1.7899	1.7899	1.7899	1.5749	1.5749	1.5749
	Th. 1 ($\alpha=0.6$)	1.7902	1.7902	1.7902	1.5813	1.5813	1.5813

Furthermore, other numerical example is given to show the effectiveness and merit of the proposed method.

Example 2. Consider the linear system (26) with interval time-varying delays:

$$A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}, A_\tau=\begin{bmatrix}0 & 0\\ -1 & 1\end{bmatrix}$$

When the derivative of delay is unknown or $h_d \ge 1$,for varying lower bound h_a, the MADB on h_b are computed and shown in Table 4. Note that N=2 is the number of delay partition in (Zhu et al., 2010; Tang et al., 2012). From Table 4, it can be seen that the present result is less conservative that those results in (Shao, 2009; Sun et al., 2010; Park et al., 2011; Shao and Han, 2012; Qian et al., 2012; Liu et al., 2012; Zhu et al., 2010; Tang et al., 2012; Lee and Park, 2014).

Table 4. Comparison to the MADB on h_b with various h_a under unknown h_d for Example 2

Methods	h_a=0.3	h_a=0.5	h_a=0.8	h_a=1	h_a=2
Shao, 2009	1.0715	1.2191	1.4539	1.6169	2.4798
Sun et al., 2010	1.0716	1.2196	1.4552	1.6189	2.4884
Zhu et al., 2010 (N=2)	1.1677	1.3078	1.5333	1.6910	2.5217
Tang et al., 2012 (N=2)	1.1907	1.3303	1.5550	1.7124	2.5406
P. Park et al., 2011	1.2400	1.3800	1.6000	1.7500	2.5800
Shao and Han, 2012	1.2400	1.3800	1.6000	1.7500	2.5800
Liu et al., 2012	1.2400	1.3900	1.6100	1.7700	2.5900
Lee and Park, 2014	1.2900	1.4300	1.6400	1.7900	2.6000
Qian et al., 2012	1.3500	1.4700	1.6800	1.8100	2.6100
Th. 2 ($\alpha=0.55$)	1.3588	1.4841	1.6903	1.8372	2.6458
Th. 2 ($\alpha=0.5$)	1.3626	1.4869	1.6921	1.8386	2.6463

In order to compare to those in (Shao, 2009; Sun et al., 2010; Qian et al., 2012), we assume that $h_d=0.3$ and $\delta=0.7$, and then the MADB can be calculated and listed in Table 5. Meanwhile, when $h_a=1.0$, the MADB on h_b were computed as $h_b=2.35$ and $h_b=2.3511$ in (Shao and Han, 2012) and (Zhang and Han, 2013), respectively.

Table 5. Comparison to the MADB on h_b with various h_a ($h_d = 0.3$) in Example 2

Methods	h_a=0.3	h_a=0.5	h_a=0.8	h_a=1.0
Shao, 2009	2.2224	2.2278	2.2388	2.2474
Sun et al., 2010	2.2634	2.2858	2.3078	2.3167
Qian et al., 2012	2.4910	2.4920	2.4930	2.4930
Th. 2 ($\alpha = 0.65$)	3.1876	3.1588	3.1277	3.1047

From Table 5, it also shows that the proposed result yields less conservatism than those in (Shao, 2009; Sun et al., 2010; Qian et al., 2012), especially when h_a is small, our method can obtain the larger MADB on h_b .

5. CONCLUSIONS

In this paper, a new method has been proposed to estimate the upper bound of the time derivative of LK functional without ignoring some useful terms for time-delay systems with nonlinear perturbations. Through constructing a novel LK Functional via variable delay decomposition technique, and estimating a tighter upper bound of its derivative without ignoring any terms and without including any approximation of the uncertain delay factors, and thus a less conservative stability criterion is obtained in the form of LMIs. Finally, numerical examples are given to demonstrate the effectiveness and the benefits of the proposed method. Moreover, the proposed approach is simple and may easily be extended to robust stabilization and H∞ control problems for uncertain linear/nonlinear systems and networked control systems and switch systems in (Araghi et al., 2013).

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