NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 38 (2008), 161–169

SOME RESULTS ON WALK REGULAR AND STRONGLY REGULAR GRAPHS

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(Received August 2008)

Abstract. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where S_k denotes the neighborhood of the vertex k. We prove that a regular G is strongly regular if and only if its vertex deleted subgraph $G_i = G \setminus i$ has exactly two main eigenvalues for $i = 1, 2, \ldots, n$. In particular, we show that

$$\mu_{1,2} = \frac{\tau - \theta + r \pm \sqrt{\left(\tau - \theta - r\right)^2 - 4\,\theta}}{2},$$

where μ_1 and μ_2 are the main eigenvalues of G_i. Besides, we demonstrate that if G is a conference graph then G_i is cospectral to H_i, where H_i is switching equivalent to G_i with respect to S_i.

Let G be a simple graph of order n. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix $A = A(G)$ and is denoted by $\sigma(G)$. The Seidel spectrum of G consists of the eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of its $(0, -1, 1)$ adjacency matrix $A^* = A^*(G)$ and is denoted by $\sigma^*(G)$. Let $P_G(\lambda) = |\lambda I - A|$ and $P_G^*(\lambda) = |\lambda I - A^*|$ denote the characteristic polynomial and the Seidel characteristic polynomial, respectively.

Let $A^k = [a_{ij}^{(k)}]$ for any non-negative integer k. The number W_k of all walks of length k in G equals $\mathbf{sum}\, A^k$, where $\mathbf{sum}\, M$ is the sum of all elements in a matrix M. According to [**1**], the generating function $W_G(t)$ of the numbers W_k of length k in the graph G is defined by $W_G(t) = \sum_{k=0}^{+\infty} W_k t^k$. Besides [**1**],

$$W_G(t) = \frac{1}{t}\left[\frac{(-1)^n P_{\overline{G}}\big(-\dfrac{t+1}{t}\big)}{P_G\big(\dfrac{1}{t}\big)} - 1\right], \tag{1}$$

where $\overline{G}$ denotes the complement of G.

Let i be a fixed vertex from the vertex set $V(G) = \{1, 2, \ldots, n\}$ and let $G_i = G \setminus i$ be its corresponding vertex deleted subgraph. Let S_i denote the neighborhood of i, defined as the set of all vertices of G which are adjacent to i. Let d_i denote the degree of the vertex i and let $\Delta_i = \sum_{j \in S_i} d_j$.

1991 *Mathematics Subject Classification* 05 C 50.
Key words and phrases: Walk regular graph, strongly regular graph, main eigenvalues, conjugate adjacency matrices.
Dedicated to Professor Peter L. Hammer.

Proposition 1. *Let G be a connected or disconnected graph of order n. Then for any vertex deleted subgraph G_i we have:*

(1^0) $\Delta_j(G_i) = \Delta_j(G) - d_i(G) - a_{ij}^{(2)}(G)$ *if* $j \in S_i$;

(2^0) $\Delta_j(G_i) = \Delta_j(G) - a_{ij}^{(2)}(G)$

if $j \in T_i = V(G_i) \setminus S_i$.

Proof. Let $j \in V(G_i)$ and let S_j and $S_j^\bullet$ denote the neighborhood of the vertex j with respect to G and G_i, respectively. Let us consider the case when $j \in S_i$. In this situation $S_j = S_j^\bullet \cup \{ i \}$. Then

$$\Delta_j(G) = \sum_{k \in \Delta} d_k(G) + \sum_{k \in S_j^\bullet \setminus \Delta} d_k(G) + d_i(G)\,,$$

where $\Delta = S_i \cap S_j$. We note that $d_k(G) = d_k(G_i) + 1$ if $k \in \Delta$ and $d_k(G) = d_k(G_i)$ if $k \in S_j^\bullet \setminus \Delta$. In view of this and keeping in mind that $|S_i \cap S_j| = a_{ij}^{(2)}$, by straightforward calculation we obtain (1^0). The proof of relation (2^0) is also trivial and will be omitted. □

Further, we say that a regular graph G of order n and degree $r \geq 1$ is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, understanding that G is not the complete graph K_n. We know that a regular connected graph is strongly regular if and only if it has exactly three distinct eigenvalues.

Definition 1. *We say that $\mu \in \sigma(G)$ is the main eigenvalue if and only if $\langle \mathbf{j}, \mathbf{Pj} \rangle = n\cos^2\alpha > 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The value $\beta = |\cos\alpha|$ is called the main angle of μ.*

Proposition 2. *Let G be a disconnected regular graph of order n and degree $r \geq 1$. Then the vertex deleted subgraphs G_i have exactly two main eigenvalues for $i = 1, 2, \ldots, n$ if and and only if $G = mK_{r+1}$, where mH denotes the m-fold union of the graph H.*

Proof. First, assume that $G = mK_{r+1}$ for $m \geq 2$. Then the vertex deleted subgraph $G_i = K_r \cup (m-1)K_{r+1}$ for $i = 1, 2, \ldots, n$, which proves that it has two main eigenvalues $\mu_1 = r$ and $\mu_2 = r - 1$.

Conversely, let us assume that any vertex deleted subgraph G_i of the graph G has exactly two main eigenvalues. Let $G = G^{(1)} \cup G^{(2)} \cup \cdots \cup G^{(m)}$, where $G^{(k)}$ is the connected regular graph of order n_k and degree r for $k = 1, 2, \ldots, m$. Contrary to the statement, assume that there exist at least one graph $G^{(j)}$ which is not the complete graph K_{r+1}, which provides that $n_j \geq r + 2$. Then for any fixed vertex i from the vertex set $V(G^{(j)})$ we have $G_i = G_i^{(j)} \cup_{k \in M_j} G^{(k)}$, where $M_j = \{1, 2, \ldots, m\} \setminus \{ j \}$. Let $S_i^{(j)}$ be the neighborhood of the vertex i with respect to $G^{(j)}$ and let $T_i^{(j)} = V(G_i^{(j)}) \setminus S_i^{(j)}$. Since $|S_i^{(j)}| = r$ and $n_j \geq r + 2$ we note that $|T_i^{(j)}| \geq 1$. Finally, since $d_k(G_i^{(j)}) = r - 1$ for $k \in S_i^{(j)}$ and $d_k(G_i^{(j)}) = r$ for $k \in T_i^{(j)}$ it turns out that $G_i^{(j)}$ is not regular. Consequently, $G_i^{(j)}$ has at least two main eigenvalues $\mu_1 = \lambda_1(G_i^{(j)})$ and μ_2. So we obtain that G_i has at least three (distinct) main eigenvalues r, μ_1 and μ_2 because $r > \mu_1 > \mu_2$, a contradiction. $\square$

Let $\overline{A}^k = [\overline{a}_{ij}^{(k)}]$ for any non-negative integer k, where $\overline{A} = A(\overline{G})$. Let G be a regular graph of order n and degree r. Let

$$\Delta^{(k)} = \frac{\big(n-(r+1)\big)^k + (-1)^{k-1}(r+1)^k}{n} \qquad (k = 0, 1, 2, \dots).$$

According to [**3**],

$$\overline{a}_{ij}^{(k)} = \Delta^{(k)} + (-1)^k \sum_{m=0}^{k} \binom{k}{m} a_{ij}^{(m)} \qquad (i, j = 1, 2, \dots, n). \tag{2}$$

Theorem 1 (Hagos [**2**]). *A non-regular graph G of order n has exactly two main eigenvalues if and only if there exist two real constants p and q such that $\Delta_i + p\, d_i + q = 0$ for $i = 1, 2, \dots, n$.*

Theorem 2. *A regular graph G of order n and degree $r \geq 1$ is strongly regular if and only if its vertex deleted subgraphs G_i have exactly two main eigenvalues for $i = 1, 2, \dots, n$.*

Proof. According to Proposition 2, without loss of generality we can assume that both G and its complement $\overline{G}$ are connected. Let us assume that G_i has exactly two main eigenvalues for $i = 1, 2, \dots, n$. Using Theorem 1, we have

$$\Delta_j(G_i) + p_i\, d_j(G_i) + q_i = 0 \qquad (j \in V(G_i)), \tag{3}$$

where p_i and q_i are two fixed real values. Therefore, using (3) we get $\Delta_j(G_i) - \Delta_k(G_i) = -\,p_i\,(d_j(G_i) - d_k(G_i))$ for $j, k \in V(G_i)$. Since $d_j(G_i) = r - 1$ for $j \in S_i$ it follows that $\Delta_j(G_i) = \Delta_k(G_i)$ for $j, k \in S_i$. Since $\Delta_j(G) = r^2$ and $d_i(G) = r$ for $i, j \in V(G)$, from Proposition 1 (1^0) we obtain that $a_{ij}^{(2)} = a_{ik}^{(2)}$ for $j, k \in S_i$. Let $a_{ij}^{(2)} = \tau_i$ where τ_i is some non-negative integer for $i = 1, 2, \dots, n$. We note that $a_{ij}^{(2)} = \tau_i$ for any $j \in S_i$. Let x and y be any two adjacent vertices in G. We shall prove that $|S_x \cap S_y| = \tau$ where $\tau = \tau_1$.

Case 1.1. ($x = 1$). This case is trivial because the vertex $y \in S_1$. Thus, $a_{xy}^{(2)} = \tau$ which means that $|S_x \cap S_y| = \tau$.

Case 1.2. ($x \in S_1$). In this case $y \in S_x$ and $1 \in S_x$, which yields that $a_{xy}^{(2)} = \tau_x$ and $a_{x1}^{(2)} = \tau_x$. Since $x \in S_1$ it follows that $a_{1x}^{(2)} = \tau$, which provides that $a_{xy}^{(2)} = \tau$ because $a_{1x}^{(2)} = a_{x1}^{(2)}$.

Case 1.3. ($x \in T_1$ and $y \in T_1$). Since G is a connected graph there is a path $(1, x_1)$, (x_1, x_2), $\dots$, (x_{k-1}, x_k), (x_k, x) which connects the vertices 1 and x, where $(1, x_1)$, (x_i, x_{i+1}), $(x_k, x) \in E(G)$ for $i = 1, 2, \dots, k-1$ and $E(G)$ is the edge set of G. In view of Case 1.2 we obtain that $\tau = \tau_{x_1}$, $\tau_{x_1} = \tau_{x_2}$, $\dots$, $\tau_{x_{k-1}} = \tau_{x_k}$ and $\tau_{x_k} = \tau_x$, which means that $\tau = \tau_x$. Since $y \in S_x$ we obtain $a_{xy}^{(2)} = \tau_x$, which proves that $|S_x \cap S_y| = \tau$ for any two adjacent vertices x and y.

In order to prove that G is strongly regular it remains to show that $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j. We note first[1] that $\overline{a}_{ij}^{(2)} = \overline{\tau}$ for

[1]The proof that $\overline{a}_{ij}^{(2)} = \overline{\tau}$ for any two adjacent vertices i and j in $\overline{G}$ is based on the fact that a graph H and its complement $\overline{H}$ have the same number of main eigenvalues. Consequently, the

any two adjacent vertices i and j in $\overline{G}$ for some constant $\overline{\tau}$, because its complement $\overline{G}$ is also regular and connected.

Indeed, let i and j be two distinct non-adjacent vertices in G. Then i and j are two adjacent vertices in $\overline{G}$. In view of this and relation (2) we obtain $\overline{\tau} = \Delta^{(2)} + a_{ij}^{(2)}$, which proves that $|S_i \cap S_j| = \overline{\tau} - \Delta^{(2)}$ for any two distinct non-adjacent vertices i and j.

Conversely, assume that G is strongly regular. We shall now prove that its vertex deleted subgraphs G_i have exactly two main eigenvalues for $i = 1, 2, \ldots, n$. We note that

$$a_{ij}^{(2)} - a_{ik}^{(2)} = -(\tau - \theta)\big(d_j(G_i) - d_k(G_i)\big) \tag{4}$$

for $i = 1, 2, \ldots, n$, keeping in mind that $d_j(G_i) = r - 1$ if $j \in S_i$ and $d_j(G_i) = r$ if $j \in T_i$. Let us assume[2] that $a_{ik}^{(2)} = \theta$. Using (4), we get

$$a_{ij}^{(2)} = -(\tau - \theta)\, d_j(G_i) + (\tau - \theta)\, r + \theta\,. \tag{5}$$

Case 2.1. ($j \in S_i$). Using Proposition 1 (1^0) we get (i) $\Delta_j(G_i) = r\, d_j(G_i) - a_{ij}^{(2)}$, because $d_j(G_i) = r - 1$. Making use of (i) and (5), we arrive at

$$\Delta_j(G_i) - (\tau - \theta + r)\, d_j(G_i) + (\tau - \theta)\, r + \theta = 0 \qquad (j \in S_i)\,. \tag{6}$$

Case 2.2. ($j \in T_i$). Using Proposition 1 (2^0) we get (ii) $\Delta_j(G_i) = r\, d_j(G_i) - a_{ij}^{(2)}$, because $d_j(G_i) = r$. Making use of (ii) and (5), we arrive at

$$\Delta_j(G_i) - (\tau - \theta + r)\, d_j(G_i) + (\tau - \theta)\, r + \theta = 0 \qquad (j \in T_i)\,. \tag{7}$$

Finally, using (6) and (7) we find that $\Delta_j(G_i) + p\, d_j(G_i) + q = 0$ for $j \in V(G_i)$, where $p = -(\tau - \theta + r)$ and $q = (\tau - \theta)\, r + \theta$. In view of this and (3), we obtain that G_i has exactly two main eigenvalues for $i = 1, 2, \ldots, n$ □

Theorem 3. *Let G be a connected or disconnected strongly regular graph of order n and degree r. Then for any vertex deleted subgraph G_i we have*

$$\mu_{1,2} = \frac{\tau - \theta + r \pm \sqrt{\big(\tau - \theta - r\big)^2 - 4\,\theta}}{2}\,, \tag{8}$$

where μ_1 and μ_2 are the main eigenvalues of G_i.

Proof. In view of the proof of Theorem 2 we have $p = -(\tau - \theta + r)$ and $q = (\tau - \theta)\, r + \theta$ for any subgraph G_i of the strongly regular graph G. It was proved in [**2**] that $p = -(\mu_1 + \mu_2)$ and $q = \mu_1 \mu_2$ for any graph H with two main eigenvalues μ_1 and μ_2. Using this fact we easily obtain the statement. □

Proposition 3 (Lepović [**3**])**.** *Let G be a connected or disconnected regular graph of order n and degree r. Then*

$$P_{\overline{G}_i}(\lambda) = \frac{(-1)^{n-1}}{\lambda + r + 1}\left(\big(\lambda - \overline{r}\big)\, P_{G_i}(-\lambda - 1) - \frac{P_G(-\lambda - 1)}{\lambda + r + 1}\right), \tag{9}$$

where $\overline{r} = (n - 1) - r$.

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assumption that G_i has exactly two main eigenvalues for $i = 1, 2, \ldots, n$, provides that the vertex deleted subgraphs $\overline{G}_i$ of $\overline{G}$ also have two main eigenvalues for $i = 1, 2, \ldots, n$.

[2] If we assume that $a_{ik}^{(2)} = \tau$ then using (4) we get $a_{ij}^2 = -(\tau - \theta)\, d_j(G_i) + (\tau - \theta)(r - 1) + \tau$, which is reduced to (5). Therefore, without loss of generality we can assume that $a_{ik}^{(2)} = \theta$.

Definition 2. *A graph G of order n is walk regular if the number of closed walks of length k starting and ending at vertex i is the same for any $i = 1, 2, \ldots, n$.*

We know that a graph G of order n is walk regular if and only if its vertex deleted subgraphs G_i are cospectral for $i = 1, 2, \ldots, n$. Using (9) we obtain the following result.

Corollary 1. *Let G be a walk regular graph of order n. Then its complement $\overline{G}$ is also walk regular.*

Corollary 2. *Let G be a walk regular graph of order n. Then $W_{G_i}(t) = W_{G_j}(t)$ for $i, j = 1, 2, \ldots, n$.*

Proof. Since $P_{G_i}(\lambda) = P_{G_j}(\lambda)$ and $P_{\overline{G}_i}(\lambda) = P_{\overline{G}_j}(\lambda)$ for $i, j = 1, 2, \ldots, n$, we obtain the statement using (1). □

Theorem 4. *Let G be a connected or disconnected graph of order n. Then G is walk regular if and only if $W_{G_i}(t) = W_{G_j}(t)$ for $i, j = 1, 2, \ldots, n$.*

Proof. According to Corollary 2 it suffices to demonstrate that G is walk regular if its vertex deleted subgraphs G_i have the same generating function $W_{G_i}(t)$ for $i = 1, 2, \ldots, n$. Assume that $W_{G_i}(t) = W_{G_j}(t)$ for $i, j = 1, 2, \ldots, n$. Since $W_1(G) = W_1(G_i) + 2\, d_i(G)$ it turns out that G is regular. Consequently, making use of (1) and (9) we arrive at

$$\frac{P_G(\lambda)}{\big(\lambda - r\big)^2 P_{G_i}(\lambda)} = \frac{n}{\lambda - r} - \Big(\frac{1}{\lambda}\, W_{G_i}\big(\frac{1}{\lambda}\big)\Big),$$

from which we obtain the statement. □

Further, in order to obtain some new information on walk regular and strongly regular graphs, we need some results which are obtained by using the concept of conjugate adjacency matrices, as follows.

Let $c = a + b\sqrt{m}$ and $\overline{c} = a - b\sqrt{m}$, where a and b are two nonzero integers and m is a positive integer such that m is not a perfect square. We say that $A^c = [c_{ij}]$ is the conjugate adjacency matrix of the graph G if $c_{ij} = c$ for any two adjacent vertices i and j, $c_{ij} = \overline{c}$ for any two nonadjacent vertices i and j, and $c_{ij} = 0$ if $i = j$. The conjugate spectrum of G is the set of the eigenvalues $\lambda_1^c \geq \lambda_2^c \geq \cdots \geq \lambda_n^c$ of its conjugate adjacency matrix $A^c = A^c(G)$ and is denoted by $\sigma^c(G)$. Let $P_G^c(\lambda) = |\lambda I - A^c|$ denote the conjugate characteristic polynomial of G.

Next, replacing λ with $x + y\sqrt{m}$ the conjugate characteristic polynomial $P_G^c(\lambda)$ can be transformed into the form

$$P_G^c(x + y\sqrt{m}) = Q_n(x, y) + \sqrt{m}\, R_n(x, y)\,, \tag{10}$$

where $Q_n(x, y)$ and $R_n(x, y)$ are two polynomials of order n in variables x and y, whose coefficients are integers. Besides, according to [**4**]

$$P_{\overline{G}}^c(x - y\sqrt{m}) = Q_n(x, y) - \sqrt{m}\, R_n(x, y)\,. \tag{11}$$

We note from (10) and (11) that $x_0 + y_0\sqrt{m} \in \sigma^c(G)$ and $x_0 - y_0\sqrt{m} \in \sigma^c(\overline{G})$ if and only if x_0 and y_0 is a solution of the following system of equations

$$Q_n(x, y) = 0 \quad \text{and} \quad R_n(x, y) = 0\,. \tag{12}$$

Theorem 5 (Lepović [**4**]). *Let G and H be two graphs of order n. Then $P^c_G(\lambda) = P^c_H(\lambda)$ if and only if $P_G(\lambda) = P_H(\lambda)$ and $P_{\overline{G}}(\lambda) = P_{\overline{H}}(\lambda)$.*

Proposition 4. *Let G be a graph of order n. Then G is cospectral to its complement $\overline{G}$ if and only if $Q_n(x,-y) = Q_n(x,y)$ and $R_n(x,-y) = -R_n(x,y)$.*

Proof. Using (11) we have $P^c_{\overline{G}}(x + y\sqrt{m}) = Q_n(x,-y) - \sqrt{m}\,R_n(x,-y)$. Making use of (10) and (11) we obtain that $P^c_G(\lambda) = P^c_{\overline{G}}(\lambda)$ if and only $Q_n(x,-y) = Q_n(x,y)$ and $R_n(x,-y) = -R_n(x,y)$. Using Theorem 5 we obtain the proof. □

Corollary 3. *Let G be a graph of order n. Then G is cospectral to its complement $\overline{G}$ if and only if $Q_n(-a,-\lambda) = Q_n(-a,\lambda)$ and $R_n(-a,-\lambda) = -R_n(-a,\lambda)$.*

Definition 3. *We say that $\mu^c \in \sigma^c(G)$ is the conjugate main eigenvalue if and only if $\langle \mathbf{j}, \mathbf{P}^c\mathbf{j}\rangle = n\cos^2\gamma > 0$, where $\mathbf{P}^c$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_{A^c}(\mu^c)$. The value $\beta^c = |\cos\gamma|$ is called the conjugate main angle of μ^c.*

Let $\mathcal{M}(G)$ and $\mathcal{M}^c(G)$ be the set of all main eigenvalues and conjugate main eigenvalues of G, respectively. It was proved in [**4**] that $|\mathcal{M}^c(G)| = |\mathcal{M}(G)|$. Let

$$\sigma^c_Q(G) = \{\, x \mid Q_n(-a,x) = 0\} \quad \text{and} \quad \sigma^c_R(G) = \{\, x \mid R_n(-a,x) = 0\}\,.$$

Theorem 6. *A connected or disconnected graph G of order n has exactly k main eigenvalues if and only if $|\sigma^c_Q(G) \cap \sigma^c_R(G)| = n - k$.*

Proof. It is sufficient to show $|\sigma^c_Q(G)\cap\sigma^c_R(G)| = |\sigma^c(G)\smallsetminus\mathcal{M}^c(G)|$. Let $x \in \sigma^c_Q(G)\cap\sigma^c_R(G)$. Using (12) it follows that $\lambda^c = -a + x\sqrt{m} \in \sigma^c(G)$ and $\overline{\lambda}^c = -a - x\sqrt{m} \in \sigma^c(\overline{G})$. Since $\overline{\lambda}^c = -\lambda^c - 2a$ we obtain from [**4**] that $\lambda^c \in \sigma^c(G) \smallsetminus \mathcal{M}^c(G)$ and $\overline{\lambda}^c \in \sigma^c(\overline{G}) \smallsetminus \mathcal{M}^c(\overline{G})$.

Conversely, let $\lambda^c \in \sigma^c(G) \smallsetminus \mathcal{M}^c(G)$. Then $\overline{\lambda}^c \in \sigma^c(\overline{G}) \smallsetminus \mathcal{M}^c(\overline{G})$ where $\overline{\lambda}^c = -\lambda^c - 2a$. Since $-a + x\sqrt{m} \in \sigma^c(G)$ and $-a - x\sqrt{m} \in \sigma^c(\overline{G})$ for $x = \frac{\lambda^c + a}{\sqrt{m}}$, we have $Q_n(-a,x) = 0$ and $R_n(-a,x) = 0$, which provides that $x \in \sigma^c_Q(G) \cap \sigma^c_R(G)$. Since there exists a one-to-one correspondence between $\lambda^c \in \sigma^c(G) \smallsetminus \mathcal{M}^c(G)$ and $x \in \sigma^c_Q(G) \cap \sigma^c_R(G)$ we obtain that $|\sigma^c_Q(G) \cap \sigma^c_R(G)| = |\sigma^c(G) \smallsetminus \mathcal{M}^c(G)|$. □

Proposition 5. *Let G be a walk regular graph of order $4n+1$ and degree $r = 2n$. Let $P^c_{G_i}(x+y\sqrt{m}) = Q^{(i)}_{4n}(x,y) + \sqrt{m}\,R^{(i)}_{4n}(x,y)$ for $i = 1,2,\dots,4n+1$. If G is cospectral to its complement $\overline{G}$ then $Q^{(i)}_{4n}(x,-y) = Q^{(i)}_{4n}(x,y)$ and $R^{(i)}_{4n}(x,-y) = -R^{(i)}_{4n}(x,y)$ for $i = 1,2,\dots,4n+1$.*

Proof. In view of Proposition 4 it is sufficient to demonstrate that $P_{G_i}(\lambda) = P_{\overline{G}_i}(\lambda)$ for $i = 1,2,\dots,4n+1$. Since

$$\frac{dP_G(\lambda)}{d\lambda} = \sum_{i=1}^{4n+1} P_{G_i}(\lambda) \quad \text{and} \quad \frac{dP_{\overline{G}}(\lambda)}{d\lambda} = \sum_{i=1}^{4n+1} P_{\overline{G}_i}(\lambda)\,,$$

we obtain that $(4n+1)P_{G_i}(\lambda) = P'_G(\lambda)$ and $(4n+1)P_{\overline{G}_i}(\lambda) = P'_{\overline{G}}(\lambda)$ for $i = 1,2,\dots,4n+1$ because both G and $\overline{G}$ are walk regular, which completes the proof. □

Let S be any subset of the vertex set $V(G)$. To switch G with respect to S means to remove all edges connecting S with $T = V(G) \setminus S$, and to introduce an edge between all nonadjacent vertices in G which connect S with T. Two graphs G and H are switching equivalent if one of them is obtained from the other by switching. We know that switching equivalent graphs have the same Seidel spectrum [**1**].

Proposition 6 (Lepović [**5**]). *Let G be a connected or disconnected graph of order n. Then:*

- $P^*_G\Big(-\dfrac{\lambda}{b}\Big) = -\dfrac{\sqrt{m}\,R_n(-a,\lambda)}{(b\sqrt{m})^n}$ *if n is odd;*
- $P^*_G\Big(-\dfrac{\lambda}{b}\Big) = \dfrac{Q_n(-a,\lambda)}{(b\sqrt{m})^n}$

if n is even.

Let $G^\bullet = G \cup \bullet_x$ be the graph obtained from the graph G by adding a new isolated vertex x. We now have the following two results [**6**].

Proposition 7. *Let $P^c_{G^\bullet}(x + y\sqrt{m}\,) = Q_{n+1}(x,y) + \sqrt{m}\,R_{n+1}(x,y)$. Then:*

- $Q_{n+1}(-\,a,\lambda) = -\,a\,Q_n(-\,a,\lambda) + m(\lambda - 2b)\,R_n(-\,a,\lambda)$ *if n is even;*
- $Q_{n+1}(-\,a,\lambda) = \dfrac{mb^2}{a}\,Q_n(-\,a,\lambda) + m\lambda\,R_n(-\,a,\lambda)$

if n is odd.

Proposition 8. *Let $P^c_{G^\bullet}(x + y\sqrt{m}\,) = Q_{n+1}(x,y) + \sqrt{m}\,R_{n+1}(x,y)$. Then:*

- $R_{n+1}(-\,a,\lambda) = (\lambda - 2b)\,Q_n(-\,a,\lambda) - aR_n(-\,a,\lambda)$ *if n is odd;*
- $R_{n+1}(-\,a,\lambda) = \lambda\,Q_n(-\,a,\lambda) + \dfrac{mb^2}{a}\,R_n(-\,a,\lambda)$

if n is even.

Let $H^{(i)}$ be switching equivalent to G with respect to $S_i \subseteq V(G)$ for $i = 1,2,\ldots,n$, understanding that S_i is the neighborhood of the vertex i. Then $H^{(i)} = H_i \cup \bullet_i$ where '$\bullet_i$' is the isolated vertex denoted by 'i' in G.

Proposition 9. *Let G be a walk regular graph of order $4n+1$ and degree $r = 2n$. If G is cospectral to its complement $\overline{G}$ then $P_{H_i}(\lambda) = P_{\overline{H}_i}(\lambda)$ for $i = 1,2,\ldots,4n+1$.*

Proof. Let $P^c_{H^{(i)}}(x+y\sqrt{m}) = \mathbb{Q}^{(i)}_{4n+1}(x,y) + \sqrt{m}\,\mathbb{R}^{(i)}_{4n+1}(x,y)$ for $i = 1,2,\ldots,4n+1$. Since $H^{(i)}$ and G are switching equivalent, we obtain from Proposition 6 that $\mathbb{R}^{(i)}_{4n+1}(-\,a,\lambda) = R_{4n+1}(-\,a,\lambda)$. Let $P^c_{H_i}(x+y\sqrt{m}) = \mathbb{Q}^{(i)}_{4n}(x,y) + \sqrt{m}\,\mathbb{R}^{(i)}_{4n}(x,y)$ for $i = 1,2,\ldots,4n+1$. Since H_i is switching equivalent to G_i with respect to $S_i \subseteq V(G_i)$, we obtain from Proposition 6 that $\mathbb{Q}^{(i)}_{4n}(-\,a,\lambda) = Q^{(i)}_{4n}(-\,a,\lambda)$. Therefore, using Proposition 8,

$$R_{4n+1}(-\,a,\lambda) = \lambda\,Q^{(i)}_{4n}(-\,a,\lambda) + \frac{mb^2}{a}\,\mathbb{R}^{(i)}_{4n}(-\,a,\lambda)\,, \tag{13}$$

which provides that $\mathbb{R}^{(i)}_{4n}(-\,a,-\lambda) = -\,\mathbb{R}^{(i)}_{4n}(-\,a,\lambda)$ because $R_{4n+1}(-\,a,\lambda)$ is an odd and $Q^{(i)}_{4n}(-\,a,\lambda)$ is an even function. Using Corollary 3 we obtain $P_{H_i}(\lambda) = P_{\overline{H}_i}(\lambda)$. □

Proposition 10 (Lepović [**7**]). *Let G be a connected or disconnected regular graph of order n and degree r. Then*

$$P^c_{\overline{G}_i}(\lambda)=\frac{(-1)^{n-1}}{\lambda+\mu^c_1+2a}\left(\left(\lambda-\overline{\mu}^c_1\right)P^c_{G_i}(-\lambda-2a)-\frac{2aP^c_G(-\lambda-2a)}{\lambda+\mu^c_1+2a}\right), \tag{14}$$

where $\mu^c_1=(n-1)a+(2r-(n-1))b\sqrt{m}$ and $\overline{\mu}^c_1=(n-1)a-(2r-(n-1))b\sqrt{m}$.

Theorem 7. *Let G be a walk regular graph of order $4n+1$ and degree $r=2n$, which is cospectral to its complement $\overline{G}$. If G_i is cospectral to H_i for $i=1,2,\dots,4n+1$ then G is strongly regular.*

Proof. According to Theorem 2 it suffices to show that G_i has exactly two main eigenvalues for $i=1,2,\dots,4n+1$. First, replacing λ with $-a+\lambda\sqrt{m}$ we obtain from (14) the following system of equations

$$Q_{4n+1}(-a,\lambda)=-(4n+1)^2\,a\,Q^{(i)}_{4n}(-a,\lambda)-(4n+1)\,m\lambda\,R^{(i)}_{4n}(-a,\lambda)\,; \tag{15}$$

$$R_{4n+1}(-a,\lambda)=(4n+1)\,\lambda\,Q^{(i)}_{4n}(-a,\lambda)+\frac{m\,\lambda^2}{a}\,R^{(i)}_{4n}(-a,\lambda)\,. \tag{16}$$

Since $P_{G_i}(\lambda)=P_{H_i}(\lambda)$ and $P_{\overline{G}_i}(\lambda)=P_{\overline{H}_i}(\lambda)$ we obtain from Theorem 5 that $P^c_{G_i}(\lambda)=P^c_{H_i}(\lambda)$, which means that $\mathbb{Q}^{(i)}_{4n}(x,y)=Q^{(i)}_{4n}(x,y)$ and $\mathbb{R}^{(i)}_{4n}(x,y)=R^{(i)}_{4n}(x,y)$. In view of this and using (13) and (16), we arrive at

$$Q^{(i)}_{4n}(-a,\lambda)=-\frac{m(\lambda^2-b^2)R^{(i)}_{4n}(-a,\lambda)}{4na\lambda}\,. \tag{17}$$

We note that $0\in\sigma^c_R(G_i)$ because $R^{(i)}_{4n}(-a,\lambda)$ is an odd function. Besides, for any $\lambda^c\in\sigma^c_R(G_i)\setminus\{\,0\,\}$ we note from (17) that $\lambda^c\in\sigma^c_Q(G_i)$. Since $R^{(i)}_{4n}(-a,\lambda)$ is a polynomial of degree $4n-1$, it follows[3] that $|\sigma^c_Q(G_i)\cap\sigma^c_R(G_i)|\geq 4n-2$, from which we obtain the proof using Theorem 6. $\square$

Definition 4. *A strongly regular graph of order $4n+1$ and degree $r=2n$ with $\tau=n-1$ and $\theta=n$ is called the conference graph.*

We know that a strongly regular graph G is a conference graph if and only if it is cospectral to its complement $\overline{G}$.

Theorem 8. *Let G be a conference graph of order $4n+1$ and degree $r=2n$. Then G_i is cospectral to H_i for $i=1,2,\dots,4n+1$.*

Proof. We note that H_i is cospectral to $\overline{H}_i$ for $i=1,2,\dots,4n+1$ because G is a walk regular graph which is cospectral to its complement $\overline{G}$. In what follows we prove that $\sigma^c_Q(G_i)\cap\sigma^c_R(G_i)=\sigma^c_Q(H_i)\cap\sigma^c_R(H_i)$. Let $x\in\sigma^c_Q(G_i)\cap\sigma^c_R(G_i)$. Since $\sigma^c_Q(G_i)\cap\sigma^c_R(G_i)\subseteq\sigma^c_R(G)$ (see (16)) we get $R_{4n+1}(-a,x)=0$. Using (13) we obtain $x\in\sigma^c_Q(H_i)\cap\sigma^c_R(H_i)$. Conversely, let $x\in\sigma^c_Q(H_i)\cap\sigma^c_R(H_i)$. Using (13) and (16) we get $R_{4n+1}(-a,x)=0$ and $R^{(i)}_{4n}(-a,x)=0$, which proves the assertion. According to Theorem 2 and Theorem 6, we find that $|\sigma^c_Q(H_i)\cap\sigma^c_R(H_i)|=4n-2$. Further, we note that $\sigma^c_Q(G_i)\cap\sigma^c_R(G_i)=\sigma^c_Q(G_i)\cap\sigma^c_R(H_i)$ because $Q^{(i)}_{4n}(-a,\lambda)=$

[3]We note that $|\sigma^c_Q(G_i)\cap\sigma^c_R(G_i)|=4n-1$ is not possible because G_i is not regular. Consequently, it must be $|\sigma^c_Q(G_i)\cap\sigma^c_R(G_i)|=4n-2$.

$\mathbb{Q}_{4n}^{(i)}(-a,\lambda)$. In view of this we have the following relation $|\sigma_R^c(G_i)\cap\sigma_R^c(H_i)|\geq 4n-2$. Finally, since $0\in\sigma_R^c(G_i)\cap\sigma_R^c(H_i)$ because $R_{4n}^{(i)}(-a,\lambda)$ and $\mathbb{R}_{4n}^{(i)}(-a,\lambda)$ are two odd polynomials of degree $4n-1$, it follows that $R_{4n}^{(i)}(-a,\lambda)=\mathbb{R}_{4n}^{(i)}(-a,\lambda)$. So we arrive at $P_{G_i}^c(-a+\lambda\sqrt{m})=P_{H_i}^c(-a+\lambda\sqrt{m})$, from which we obtain the proof using Theorem 5. $\square$

Theorem 9. *Let G be a walk regular graph of order $4n+1$ and degree $r=2n$, which is cospectral to its complement $\overline{G}$. Then G is strongly regular if and only if G_i is cospectral to H_i for $i=1,2,\ldots,4n+1$.*

Proof. According to Theorem 7 it is sufficient to show that G_i is cospectral to H_i for $i=1,2,\ldots,4n+1$ if G is strongly regular. Indeed, since G is cospectral to its complement $\overline{G}$ it turns out that G is a conference graph, which provides the proof. $\square$

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