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SIGNED b-MATCHINGS AND b-EDGE COVERS OF STRONG PRODUCT GRAPHS

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ABSTRACT. In this paper, we study the signed b-edge cover number and the signed b-matching number of a graph. Sharp bounds on these parameters of the strong product graphs are presented. We prove the existence of an analogue of Gallai's theorem relating maximum-size signed b-matchings and minimum-size signed b-edge covers for the complete graphs and complete bipartite graphs.

1. INTRODUCTION

Structural and algorithmic aspects of covering vertices by edges have been extensively studied in graph theory. An *edge cover* (*matching*) of a graph G is a set C of edges of G such that each vertex of G is incident to at least (at most) one edge of C. Let b be a fixed positive integer. A *simple b-edge cover* (*simple b-matching*) of a graph G is a set C of edges of G such that each vertex of G is incident to at least (at most) b edges of C. The minimum (maximum) size of a simple b-edge cover (simple b-matching) of G is called *b-edge cover number* (*b-matching number*), denoted by $\rho_b(G)$ ($\beta_b(G)$). Edge covers of bipartite graphs were studied by König [5] and Rado [7], and of general graphs by Gallai [2] and Norman and Rabin [6], and b-edge covers were studied by Gallai [2]. For an excellent survey of results on edge covers, matchings, b-edge covers and b-matchings, the reader is directed to [8].

In this paper, we study variants of the standard matching and edge cover problems. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, let $E_G(v) = \{uv \in E(G) : u \in V(G)\}$ denote the set of edges of G incident to v. The degree, $d(v)$, of v is $|E_G(v)|$. For a real-valued function $f : E(G) \to \mathbb{R}$ and for $X \subseteq E(G)$, we use $f(X)$ to denote $\sum_{e \in X} f(e)$. Let b be a fixed positive integer. A function $f : E(G) \to \{-1, 1\}$ is called a *signed b-matching* (*SbM*) of G if $f(E_G(v)) \leq b$ for every $v \in V(G)$. The maximum of the values of $f(E(G))$, taken over all signed b-matchings f of G, is called the *signed b-matching number* of G and is denoted by $\beta'_b(G)$.

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The *signed matching number* β'_1 was investigated in [10], and it has been proved that $\beta'_1(G) \geq -1$ for all connected graphs G.

A function $f : E(G) \to \{-1, 1\}$ is called a *signed b-edge cover* (*SbEC*) of G if $f(E_G(v)) \geq b$ for every $v \in V(G)$. The minimum of the values of $f(E(G))$, taken over all signed b-edge covers f of G, is called the *signed b-edge cover number* of G and is denoted by $\rho'_b(G)$. In the special case when $b = 1$, ρ'_b is the *signed star domination number* investigated in [9, 12, 13, 14].

In this paper, we investigate the parameters ρ'_b and β'_b of the strong product graphs. In Section 2, we present some bounds on these parameters. Exact values of these parameters of familar classes of graphs such as the complete graphs and complete bipartite graphs are found. As a consequence, we prove the existence of an analogue of Gallai's theorem for such graphs. In Section 3, we present sharp bounds on these parameters for the strong product graphs.

All graphs considered in this paper are finite, simple, and undirected. For a general reference on graph theory, the reader is referred to [1, 11]. For a graph G, a vertex $v \in V(G)$ is called *odd (even)* if $d(v)$ is odd (even). For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S. A *perfect matching* (1-*factor*) in G is a matching which matches all vertices of the graph. A graph G is called *1-factorable* if there is a collection of 1-factors such that every edge of G is in exactly one of these 1-factors.

The *cartesian product* $G \,\square\, H$ has $V(G \,\square\, H) = V(G) \times V(H)$, and two vertices (a, b) and (c, d) are adjacent if and only if $ac \in E(G)$ and $b = d$, or $a = c$ and $bd \in E(H)$. The *direct product* $G \times H$ has $V(G \times H) = V(G) \times V(H)$, and two vertices (a, b) and (c, d) are adjacent if and only if $ac \in E(G)$ and $bd \in E(H)$. The *strong product* $G \boxtimes H$ has $V(G \boxtimes H) = V(G) \times V(H)$, and two vertices (a, b) and (c, d) are adjacent if and only if $ac \in E(G)$ and $b = d$, or $a = c$ and $bd \in E(H)$, or $ac \in E(G)$ and $bd \in E(H)$. For a pair of vertices $u \in V(G)$ and $v \in V(H)$, denote

$$S_v = \{(u, v) : (u, v) \in V(G \boxtimes H)\} \text{ and } T_u = \{(u, v) : (u, v) \in V(G \boxtimes H)\}.$$

Denote by $(G \boxtimes H)[S_v]$ and $(G \boxtimes H)[T_u]$ the subgraphs of $G \boxtimes H$ induced by S_v and T_u, respectively. It is not hard to see that $(G \boxtimes H)[S_v] \cong G$ and $(G \boxtimes H)[T_u] \cong H$. Moreover,

$$\begin{aligned} X &= E(G \boxtimes H) - \bigcup_{v \in V(H)} E\big((G \boxtimes H)[S_v]\big) - \bigcup_{u \in V(G)} E\big((G \boxtimes H)[T_u]\big) \\ &= E(G \times H). \end{aligned}$$

Therefore, the subgraph of $G \boxtimes H$ induced by X is isomorphic to $G \times H$.

2. Bounds and exact values

Theorem 2.1. *Let b be a positive integer. For any graph G of order n,*

(1) $\beta'_b(G) \leq \lfloor bn/2 \rfloor$.
(2) $\rho'_b(G) \geq \lceil bn/2 \rceil$.

Proof. We only prove (1), as the proof of (2) is similar. For a SbM f of G, for every $v \in V(G)$, we have that

$$f\left(E_G(v)\right) \le b.$$

Hence,

$$\sum_{v \in V(G)} f\left(E_G(v)\right) \le bn.$$

In particular,

$$2f\left(E(G)\right) \le bn.$$

Thus, $\beta'_b(G) \le bn/2$, and the result follows since $\beta'_b(G)$ is an integer. □

Theorem 2.2. *Let b be a positive integer. For any integer $n \ge b$,*

(1) $\beta'_b(K_{n,n}) = \begin{cases} bn, & n-b \equiv 0 \pmod 2; \\ (b-1)n, & n-b \equiv 1 \pmod 2. \end{cases}$

(2) $\rho'_b(K_{n,n}) = \begin{cases} bn, & n-b \equiv 0 \pmod 2; \\ (b+1)n, & n-b \equiv 1 \pmod 2. \end{cases}$

Proof. We only prove (1), as the proof of (2) is similar. To prove (1), we discuss two cases.

Case 1: $n - b \equiv 0 \pmod 2$:

By [1, Theorem 9.18, p. 272], the graph $K_{n,n}$ is 1-factorable. Assigning 1 to each edge of $\frac{1}{2}(n+b)$ 1-factors and -1 to each edge of the remaining $\frac{1}{2}(n-b)$ 1-factors, we produce a SbM f such that $f\left(E(K_{n,n})\right) = bn$. So, $\beta'_b(K_{n,n}) \ge bn$. It follows that $\beta'_b(K_{n,n}) = bn$ by Theorem 2.1(1).

Case 2: $n - b \equiv 1 \pmod 2$:

Note that $\sum_{e \in E_{K_{n,n}}(v)} f(e) \le b$ implies that $\sum_{e \in E_{K_{n,n}}(v)} f(e) \le b-1$ for any SbM f of $K_{n,n}$ and for each $v \in V(K_{n,n})$. Hence,

$$f\left(E(K_{n,n})\right) \le (b-1)n,$$

which implies that $\beta'_b(K_{n,n}) \le (b-1)n$.

To show that $\beta'_b(K_{n,n}) \ge (b-1)n$, it suffices to obtain a SbM f such that $f\left(E(K_{n,n})\right) = (b-1)n$. Since $K_{n,n}$ is 1-factorable, a SbM f can be obtained by assigning 1 to each edge of $\frac{1}{2}(n+b-1)$ 1-factors and -1 to each edge of the remaining $\frac{1}{2}(n-b+1)$ 1-factors.

□

Theorem 2.3. *Let b be a positive integer. For any integer $n \ge b+1$, we have the following.*

(1) $\beta'_b(K_n) = \begin{cases} bn/2, & n \equiv 0 \ (mod\ 2),\ b \equiv 1 \ (mod\ 2); \\ (b-1)n/2, & n-b \equiv 0 \ (mod\ 4),\ b \equiv 1 \ (mod\ 2); \\ (b-1)n/2 - 1, & n-b \equiv 2 \ (mod\ 4),\ b \equiv 1 \ (mod\ 2); \\ bn/2, & n-b \equiv 1 \ (mod\ 4),\ b \equiv 0 \ (mod\ 2); \\ bn/2 - 1, & n-b \equiv 3 \ (mod\ 4),\ b \equiv 0 \ (mod\ 2); \\ (b-1)n/2, & n \equiv 0 \ (mod\ 2),\ b \equiv 0 \ (mod\ 2). \end{cases}$

$$(2)\ \rho'_b(K_n) = \begin{cases} bn/2, & n \equiv 0 \textit{ (mod 2)}, b \equiv 1 \textit{ (mod 2)}; \\ (b+1)n/2, & n-b \equiv 2 \textit{ (mod 4)}, b \equiv 1 \textit{ (mod 2)}; \\ (b+1)n/2+1, & n-b \equiv 0 \textit{ (mod 4)}, b \equiv 1 \textit{ (mod 2)}; \\ bn/2, & n-b \equiv 1 \textit{ (mod 4)}, b \equiv 0 \textit{ (mod 2)}; \\ bn/2+1, & n-b \equiv 3 \textit{ (mod 4)}, b \equiv 0 \textit{ (mod 2)}; \\ (b+1)n/2, & n \equiv 0 \textit{ (mod 2)}, b \equiv 0 \textit{ (mod 2)}. \end{cases}$$

Proof. We only prove (1), as the proof of (2) is similar. For (1), we only prove the case when b is odd, as the proof is similar when b is even. Consider the following three cases.

Case 1: n is even:

By [1, Theorem 9.19, p. 273], K_n is 1-factorable. Notice that there are $(n-1)$ 1-matchings (perfect matchings). Assigning $f(e) = 1$ to each edge e of $\frac{1}{2}(n+b-1)$ 1-factors, and $f(e') = -1$ to each edge e' of the remaining $\frac{1}{2}(n-b-1)$ 1-factors, we produce a SbM f of K_n. It is clear that $f(E(K_n)) = bn/2$. Thus, $\beta'_b(K_n) \geq bn/2$. It follows that $\beta'_b(K_n) = bn/2$ by Theorem 2.1(1).

Case 2: $n - b \equiv 0$ (mod 4) and $b \equiv 1$ (mod 2):

In this case, $n = b + 4k$ for some integer $k \geq 1$. Note that each vertex of K_n has even degree $b + 4k - 1$, and b is odd. Thus, for any SbM f and for each $v \in V(K_n)$, $f(E_{K_n}(v)) \leq b$ implies that $f(E_{K_n}(v)) \leq b - 1$. It turns out that $f(E(K_n)) \leq (b-1)n/2$. Hence,

$$\beta'_b(K_n) \leq (b-1)n/2.$$

To show that the equality holds, we need to obtain a SbM f of K_n such that $f(E(K_n)) = (b-1)n/2$. By [1, Theorem 9.21, p. 275], K_n is hamiltonian factorable with $\frac{n}{2} = 2k + \frac{b-1}{2}$ hamiltonian cycles. Denote by $C_1, \ldots, C_{2k+(b-1)/2}$ the disjoint hamiltonian cycles of K_n. We can obtain desired SbM f of K_n as follows. First, we assign 1 to each edge of $(b-1)/2$ hamiltonian cycles $C_1, \ldots, C_{(b-1)/2}$. Observe that the graph $K_n - \bigcup_{i=1}^{(b-1)/2} E(C_i)$ is Eulerian with $2kn$ edges. Let C be an Eulerian circuit of the graph $K_n - \bigcup_{i=1}^{(b-1)/2} E(C_i)$. Secondly, we assign $+1$ and -1 alternately along with C. It is straightforward to see that $f(E(K_n)) = (b-1)n/2$.

Case 3: $n - b \equiv 2$ (mod 4) and $b \equiv 1$ (mod 2):

In this case, $n = b+4k+2$ for some integer $k \geq 0$. By a similar argument as in Case 2, we have that $\beta'_b(K_n) \leq (b-1)n/2$.

We show that the equality does not hold by contradiction. Suppose that there is a SbM f of K_n such that $f(E(K_n)) = \beta'_b(K_n) = (b-1)n/2$. Let p and q be the numbers of edges in K_n with values 1 and -1, respectively. Thus,

$$p + q = (b + 4k + 1)n/2$$

and

$$p - q = (b-1)n/2.$$

Adding them together, we obtain that

$$2p = bn + 2kn,$$

which contradicts the fact that both n and b are odd. So, $\beta'_b(K_n) \leq (b-1)n/2 - 1$.
To complete the proof of this case, we need to find a SbM f of K_n such that $f(E(K_n)) = (b-1)n/2 - 1$. By [1, Theorem 9.21, p. 275], K_n is hamiltonian factorable. Denote by $C_1, \ldots, C_{2k+1+(b-1)/2}$ such disjoint hamiltonian cycles of K_n. We can form such a SbM f of K_n as follows. We assign 1 to each edge of $(b-1)/2$ hamiltonian cycles $C_1, \ldots, C_{(b-1)/2}$. Notice that the graph $K_n - \bigcup_{i=1}^{(b-1)/2} E(C_i)$ is Eulerian with $(2k+1)n$ edges. Let C be an Eulerian circuit of the graph $K_n - \bigcup_{i=1}^{(b-1)/2} E(C_i)$. We assign -1 and $+1$ alternately along with C. It is not hard to verify that $f(E(K_n)) = (b-1)n/2 - 1$.

□

The following result (see [8, Theorem 34.1, p. 165, Vol. A]) is a direct analogue of Gallai's theorem, relating maximum-size b-matchings and minimum-size b-edge covers.

Theorem 2.4. *Fix b a positive integer. If G is a graph of order n having no isolated vertices, then*

$$\rho_b(G) + \beta_b(G) = bn.$$

Surprisingly, there is no such analogue of Gallai's theorem relating maximum-size signed b-matchings and minimum-size signed b-edge covers in general, since

$$\rho'_b\left(K_{b+1,n-b-1}\right) = (b+1)(n-b-1),$$

and $\beta'_b(G) \geq -1$. However, our next theorem exhibits that there is such analogue of Gallai's theorem for the graphs K_n and $K_{n,n}$.

Theorem 2.5. *Fix b and n two positive integers with $n \geq b+1$.*

(1) $\rho'_b(K_{n,n}) + \beta'_b(K_{n,n}) = 2bn$.
(2) *If one of b and n is even, then $\rho'_b(K_n) + \beta'_b(K_n) = bn$.*

Proof. It follows from Theorems 2.2 and 2.3. □

3. Strong product graphs

In this section, we investigate the signed b-matching number and the signed b-edge cover number of the strong product graphs. The main results are the following.

Theorem 3.1. *Let G be a graph of order $n_G \geq 2$ and size m_G with k_G odd vertices. Let H be a graph of order $n_H \geq 2$ and size m_H with k_H odd* *vertices. For any integer* $b \geq 1$,

$$\beta'_b(G \boxtimes H) \geq \begin{cases} \max \ \big(\lambda(G,H), \lambda(H,G)\big) & b=1; \\ \max\limits_{1 \leq i \leq b-1} \big(\lambda(G,H), \lambda(H,G), \mu(i,G,H)\big) & b \geq 2, \end{cases}$$

where for $(P,Q) \in \{(G,H),(H,G)\}$,

$$\lambda(P,Q) = n_Q \beta'_b(P) + n_P \left(-k_Q - \frac{1-(-1)^{m_Q}}{2}\right) - k_P k_Q, \tag{3.1}$$

and for $1 \leq i \leq b-1$,

$$\mu(i,G,H) = n_H \beta'_i(G) + n_G \beta'_{b-i}(H) - k_G k_H.$$

Theorem 3.2. *Let* G *be a graph of order* $n_G \geq 2$ *and size* m_G *with* k_G *odd vertices. Let* H *be a graph of order* $n_H \geq 2$ *and size* m_H *with* k_H *odd vertices. For any integer* $b \geq 1$,

$$\rho'_b(G \boxtimes H) \leq \begin{cases} \min \ \big(\lambda(G,H), \lambda(H,G)\big) & b=1; \\ \min\limits_{1 \leq i \leq b-1} \big(\lambda(G,H), \lambda(H,G), \mu(i,G,H)\big) & b \geq 2, \end{cases}$$

where for $(P,Q) \in \{(G,H),(H,G)\}$,

$$\lambda(P,Q) = n_Q \rho'_b(P) + n_P \left(k_Q + \frac{1-(-1)^{m_Q}}{2}\right) + k_P k_Q, \tag{3.2}$$

and for $1 \leq i \leq b-1$,

$$\mu(i,G,H) = n_H \rho'_i(G) + n_G \rho'_{b-i}(H) + k_G k_H.$$

The lower bound in Theorem 3.1 and the upper bound in Theorem 3.2 are sharp, as will follow from Corollary 3.6. To prove Theorems 3.1 and 3.2, we need the following lemmas. We call a function $f : E(G) \to \{-1,1\}$ *good* (*bad*) if f satisfies that $f\left(E_G(v)\right) \geq 0$ ($f\left(E_G(v)\right) \leq 0$) for every $v \in V(G)$.

Lemma 3.3. *Let* G *be a graph of order* $n \geq 2$*, size* m *and* k *odd vertices. We have the following.*

1. *There is a good function* f *such that*
$$f(E(G)) = \sum_{e \in E(G)} f(e) \leq k + \frac{1-(-1)^m}{2}.$$
2. *There is a bad function* f *such that*
$$\sum_{e \in E(G)} f(e) \geq -k - \frac{1-(-1)^m}{2}.$$

Proof. We only prove (1), as the proof of (2) is similar. As every graph has an even number of odd vertices, k is even and $k \leq 2\lfloor \frac{n}{2} \rfloor$. Partition the odd vertices of G into $k/2$ pairs, and let H be a graph obtained by adding $k/2$ new vertices $w_1, \ldots, w_{\frac{k}{2}}$ to G, and joining each w_i to the two odd vertices of the ith pair. It is clear that H has no odd vertices and so is Eulerian. Let C

be an Eulerian circuit of H. We assign values 1 and -1 alternately along C. This defines a function $f' : E(H) \to \{1, -1\}$ such that $\sum_{e \in E_H(v)} f'(e) = 0$ for every $v \in V(H)$ and

$$\sum_{e \in E(H)} f'(e) = \sum_{e \in E(G)} f'(e) = \frac{1-(-1)^m}{2}.$$

Now we modify f' to form a good function f of G as follows: for each odd vertex v of G, change -1 to 1 exactly once on one of the edges incident with v. We need to make such changes at most $k/2$ times, as there are $k/2$ many -1's on edges to the w_i. Hence,

$$f(E(G)) = \sum_{e \in E(G)} f(e) \leq \sum_{e \in E(G)} f'(e) + 2 \cdot \frac{k}{2} = k + \frac{1-(-1)^m}{2}.$$

□

Lemma 3.4. *Let G be a graph of order $n_G \geq 2$ and size m_G with k_G odd vertices. Let H be a graph of order $n_H \geq 2$ and size m_H with k_H odd vertices.*

(1) *There is a good function f such that*

$$f(E(G \times H)) = \sum_{e \in E(G \times H)} f(e) \leq k_G k_H.$$

(2) *There is a bad function f such that*

$$f(E(G \times H)) = \sum_{e \in E(G \times H)} f(e) \geq -k_G k_H.$$

Proof. It follows by the fact that $G \times H$ is a graph of order $n_G n_H$ and size $2m_G m_H$ with $k_G k_H$ odd vertices, and Lemma 3.3. □

Proof of Theorem 3.2. We only prove the case when $b \geq 2$, as the proof is similar when $b = 1$. To show that

$$\rho_b'(G \boxtimes H) \leq \min_{1 \leq i \leq b-1} \left(\lambda(G,H), \lambda(H,G), \mu(i,G,H)\right),$$

it suffices to show

$$\rho_b'(G \boxtimes H) \leq \lambda(G,H), \tag{3.3}$$

$$\rho_b'(G \boxtimes H) \leq \lambda(H,G), \tag{3.4}$$

and for each $1 \leq i \leq b-1$,

$$\rho_b'(G \boxtimes H) \leq \mu(i,G,H). \tag{3.5}$$

The statement (3.4) will follow by symmetry from (3.3). To show that (3.3), it suffices to construct a SbEC f of $G \boxtimes H$ so that

$$\sum_{e \in E(G \boxtimes H)} f(e) \leq n_H \rho'_b(G) + n_G \left(k_H + \frac{1-(-1)^{m_H}}{2} \right) + k_G k_H.$$

Let f^G and f^H be SbECs of G and H such that

$$f^G(E(G)) = \sum_{e \in E(G)} f^G(e) = \rho'_b(G), \text{ and}$$
$$f^H(E(H)) = \sum_{e \in E(H)} f^H(e) = \rho'_b(H),$$

respectively. By Lemma 3.3(1), there exists a good function g^H of H such that

$$g^H(E(H)) = \sum_{e \in E(H)} g^H(e) \leq k_H + \frac{1-(-1)^{m_H}}{2}.$$

Let

$$X = E(G \boxtimes H) - \bigcup_{v \in V(H)} E\big((G \boxtimes H)[S_v]\big) - \bigcup_{u \in V(G)} E\big((G \boxtimes H)[T_u]\big).$$

Denote by F_X the subgraph of $G \boxtimes H$ induced by X. Since $F_X \cong G \times H$, by Lemma 3.4(1), there exists a good function h^{F_X} of F_X such that

$$h^{F_X}(E(F_X)) = \sum_{e \in E(F_X)} h^{F_X}(e) \leq k_G k_H.$$

We define f as follows. For every $v \in V(H)$, if $u_1 u_2 \in E(G)$, then

$$f((u_1, v)(u_2, v)) = f^G(u_1 u_2).$$

For every $u \in V(G)$, if $v_1 v_2 \in E(H)$, then

$$f((u, v_1)(u, v_2)) = g^H(v_1 v_2).$$

For any $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)$, then

$$f((u_1, v_1)(u_2, v_2)) = h^{F_X}((u_1, v_1)(u_2, v_2)).$$

Hence, for each $(u, v) \in V(G \boxtimes H)$, we have that

$$\begin{aligned} \sum_{e \in E_{G \boxtimes H}((u,v))} f(e) &= \sum_{e \in E_G(u)} f^G(e) + \sum_{e \in E_H(v)} g^H(e) + \sum_{e \in E_{F_X}((u,v))} h^{F_X}(e) \\ &\geq b + 0 + 0 \\ &= b, \end{aligned}$$

and

$$\begin{aligned}\sum_{e\in E(G\boxtimes H)} f(e) &= n_H \sum_{e\in E(G)} f^G(e) + n_G \sum_{e\in E(H)} g^H(e) + \sum_{e\in E(F_X)} h^{F_X}(e)\\ &\le n_H\rho'_b(G) + n_G\left(k_H + \frac{1-(-1)^{m_H}}{2}\right) + k_G k_H\\ &= \lambda(G,H).\end{aligned}$$

Thus,

$$\rho'_b(G\boxtimes H) \le \lambda(G,H),$$

and (3.3) follows.

We now prove (3.5). For each $1 \le i \le b-1$, let f_i^G be a SiEC of G such that $\sum_{e\in E(G)} f_i^G(e) = \rho'_i(G)$ and let f_{b-i}^H be a S$(b-i)$EC of H such that $\sum_{e\in E(H)} f_{b-i}^H(e) = \rho'_{b-i}(H)$. Define f as follows. For every $v \in V(H)$, if $u_1u_2 \in E(G)$, then

$$f\left((u_1,v)(u_2,v)\right) = f_i^G(u_1u_2).$$

For every $u \in V(G)$, if $v_1v_2 \in E(H)$, then

$$f\left((u,v_1)(u,v_2)\right) = f_{b-i}^H(v_1v_2).$$

For any $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$, then

$$f\left((u_1,v_1)(u_2,v_2)\right) = h^{F_X}\left((u_1,v_1)(u_2,v_2)\right).$$

Hence, for each $(u,v) \in V(G\boxtimes H)$, we have that

$$\begin{aligned}\sum_{e\in E_{G\boxtimes H}((u,v))} f(e) &= \sum_{e\in E_G(u)} f_i^G(e) + \sum_{e\in E_H(v)} f_{b-i}^H(e) + \sum_{e\in E_{F_X}((u,v))} h^{F_X}(e)\\ &\ge i + (b-i) + 0\\ &= b,\end{aligned}$$

and

$$\begin{aligned}\sum_{e\in E(G\boxtimes H)} f(e) &= n_H \sum_{e\in E(G)} f_i^G(e) + n_G \sum_{e\in E(H)} f_{b-i}^H(e) + \sum_{e\in E(F_X)} h^{F_X}(e)\\ &\le n_H\rho'_i(G) + n_G\rho'_{b-i}(H) + k_G k_H\\ &= \mu(i,G,H).\end{aligned}$$

Thus,

$$\rho'_b(G\boxtimes H) \le \mu(i,G,H),$$

and (3.5) follows. □

Proof of Theorem 3.1. The proof is quite similar to that of Theorem 3.2, except that we use bad function rather than good function, so is omitted. □

Corollary 3.5. *Let G be an Eulerian graph of order n_G and size m_G. For any graph H of order n_H, and for any integer $b \ge 1$,*

(1) $n_G\beta'_b(H) - \dfrac{1-(-1)^{m_G}}{2} n_H \le \beta'_b(G\boxtimes H) \le \left\lfloor \dfrac{b n_G n_H}{2} \right\rfloor.$

(2) $$\left\lceil \frac{bn_G n_H}{2} \right\rceil \leq \rho_b'(G \boxtimes H) \leq n_G \rho_b'(H) + \frac{1-(-1)^{m_G}}{2} n_H.$$

Proof. We only prove item (2), as the proof of item (1) is similar. The lower bound follows by Theorem 2.1(2). By hypothesis, G has no odd vertices and so

$$\lambda(H,G) = n_G \rho_b'(H) + \frac{1-(-1)^{m_G}}{2} n_H.$$

The second inequality holds by Theorem 3.2. □

By Theorem 2.2 and Corollary 3.5, we have the following.

Corollary 3.6. *Let G be an Eulerian graph of order n_G and size m_G. For all positive integers $n > b \geq 1$ satisfying $n \equiv b \pmod 2$, if m_G is even, then*

$$\rho_b'(G \boxtimes K_{n,n}) = \beta_b'(G \boxtimes K_{n,n}) = bnn_G.$$

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