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EXTENSIONS OF EULER HARMONIC SUMS

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Three new closed-form summation formulae involving harmonic numbers are established using simple arguments and they are very general extensions of Euler's famous harmonic sum identity. Some illustrative special cases as well as immediate consequences of the main results are also considered.

1. INTRODUCTION

About 235 years ago, circa 1775, Euler [**7**] (see Remark 2), produced one of his many remarkable identities, namely

$$2\sum_{n=1}^{\infty}\frac{H_n}{n^q}=(q+2)\,\zeta\,(q+1)-\sum_{m=1}^{q-2}\zeta\,(m+1)\,\zeta\,(q-m) \tag{1}$$

$$(q\in\mathbb{N}\setminus\{1\},\mathbb{N}:=\{1,2,3,\ldots\}),$$

where $H_n:=\sum\limits_{m=1}^{n}m^{-1}$, $n\in\mathbb{N}$, is the n^{th} harmonic number, while $\zeta(z)$ denotes the familiar Riemann Zeta function (for more details, see for instance, [**15**]).

The (generalized) n^{th} harmonic number of order r, $H_n^{(r)}$, is defined for positive integers n and r as

$$H_n^{(r)}:=\sum_{m=1}^{n}\frac{1}{m^r}=\frac{(-1)^{r-1}}{(r-1)!}\big(\psi^{(r-1)}(n+1)-\psi^{(r-1)}(1)\big), \tag{2}$$

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where $\psi^{(m)}$ are the polygamma functions of order m which are defined by $\psi^{(0)}(z) \equiv \psi(z)$ and $\psi^{(m)}(z) := \mathrm{d}^m\,\psi(z)/\mathrm{d}z^m$, $m \in \mathbb{N}$ and $z \neq 0, -1, -2, \dots$ [**15**, Section 2.1]. Note that $H_n := H_n^{(1)} = \psi(n+1) - \psi(1) = \psi(n+1) + \gamma$, γ being the Euler–Mascheroni constant. Here (and hereafter), $\psi(z)$ is the psi (or digamma) function, given as the logarithmic derivative of the well-known gamma function $\Gamma(z)$, *i.e.* $\psi(z) := \mathrm{d}\log\Gamma(z)/\mathrm{d}z$. In Section 2, we also use the harmonic numbers $H_x^{(r)}$ which further generalizes $H_n^{(r)}$.

The recent works of [**5, 6, 9, 10, 11, 12, 13, 14**] investigate various representations of binomial sums and zeta functions in simpler form by the use of the beta function and other techniques, and, in the process, various properties of the numbers H_n, $H_n^{(r)}$ and $H_x^{(r)}$ have been dealt with.

In this paper we intend to extend the result of Euler (1) and give three new and very general identities for the representation of

$$\sum_{n\geq 1}\frac{H_n}{(n+x)^q}, \quad \sum_{n\geq 1}\frac{H_n}{n^q\binom{an+k}{k}} \quad \text{and} \quad \sum_{n\geq 1}\frac{H_n}{n^q\binom{an+k}{k}^2} \qquad (q \in \mathbb{N}\setminus\{1\}),$$

in a closed form, where k and q are positive integers, a is a non-negative real number and x is a real number such that $x \neq -1, -2, -3, \dots$ (see Theorems 1, 2 and 3).

2. THE MAIN RESULTS

For a positive integer r and a real number x such that $x \neq -1, -2, -3, \dots$, we define and use the $x-$generalized harmonic number in power r, $H_x^{(r)}$, given in terms of the polygamma functions as

$$H_x^{(1)} := \psi(x+1) - \psi(1) = \psi(x+1) + \gamma \tag{3}$$

and, for $r = 2, 3, \dots$, as

$$H_x^{(r)} := \frac{(-1)^{r-1}}{(r-1)!}\big(\psi^{(r-1)}(x+1) - \psi^{(r-1)}(1)\big) = \zeta(r) + \frac{(-1)^{r-1}}{(r-1)!}\,\psi^{(r-1)}(x+1). \tag{4}$$

We shall need Lemma 1 in the proofs of Theorems 1, 2 and 3.

Lemma 1. *For any real α, $\alpha \neq -1, -2, -3, \dots$, we have*:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n(n+\alpha)} &= \frac{1}{2\alpha}\big[3\zeta(2) + \psi^2(\alpha) + 2\gamma\psi(\alpha) + \gamma^2 - \psi'(x)\big]\\ &= \frac{1}{2\,\alpha}\big[2\zeta(2) + (H_{\alpha-1}^{(1)})^2 + H_{\alpha-1}^{(2)}\big];\end{aligned} \tag{5}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{(n+\alpha)^2} &= \gamma\,\psi'(\alpha) + \psi(\alpha)\,\psi'(\alpha) - \frac{1}{2}\,\psi''(\alpha)\\ &= \zeta(3) + \zeta(2)H_{\alpha-1}^{(1)} - H_{\alpha-1}^{(1)}H_{\alpha-1}^{(2)} - H_{\alpha-1}^{(3)}.\end{aligned} \tag{6}$$

Proof. Since $H_n = \psi(n) + \dfrac{1}{n} + \gamma$, it is evident that both series given by Lemma 1 may be decomposed into the three components each. The four out of six series obtained in this way are well-known and very simple to sum

$$S_1(\alpha) := \sum_{n=1}^{\infty} \frac{1}{n(n+\alpha)} = \frac{1}{\alpha}[\psi(\alpha+1)+\gamma],$$

$$S_2(\alpha) := \sum_{n=1}^{\infty} \frac{1}{n^2(n+\alpha)} = \frac{1}{\alpha^2}\sum_{n=1}^{\infty}\left(\frac{\alpha}{n^2} - \frac{\alpha}{n(n+\alpha)}\right) = \frac{1}{\alpha}\,\zeta(2) - \frac{1}{\alpha^2}\big[\psi(\alpha+1)+\gamma\big],$$

$$S_3(\alpha) := \sum_{n=1}^{\infty} \frac{1}{(n+\alpha)^2} = \psi'(\alpha+1),$$

and

$$S_4(\alpha) := \sum_{n=1}^{\infty} \frac{1}{n(n+\alpha)^2} = \frac{1}{\alpha^2}\sum_{n=1}^{\infty}\left(\frac{\alpha}{n(n+\alpha)} - \frac{\alpha}{(n+\alpha)^2}\right)$$
$$= \frac{1}{\alpha^2}\Big[\psi(\alpha+1)+\gamma\Big] - \frac{1}{\alpha}\,\psi'(\alpha+1).$$

Indeed, it is easy to find their summation formulae in any standard reference book (see, for instance, [**8**, Section 5.1]) and to sum them it suffices to make a partial fraction decomposition and recall that $\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} z(n(n+z))^{-1}$, $\psi'(z+1) = \sum_{n=1}^{\infty}(n+z)^{-2}$ and $\zeta(2) = \sum_{n=1}^{\infty} n^{-2}$ (see, for example, [**15**, pp. 14, 22 and 96]). The remaining two summations,

$$S_5(\alpha) := \sum_{n=1}^{\infty} \frac{\psi(n)}{n(n+\alpha)} = \frac{1}{2\,\alpha}\big[\psi^2(\alpha+1) - \gamma^2 + \zeta(2) - \psi'(\alpha+1)\big]$$

and

$$S_6(\alpha) := \sum_{n=1}^{\infty} \frac{\psi(n)}{(n+\alpha)^2} = \psi(\alpha+1)\,\psi'(\alpha+1) - \frac{1}{2}\,\psi''(\alpha+1),$$

are also known (see [**2**, pp 432 and 433, Entries 6.2.1.6 and 6.2.1.17]), but not so easy to deduce. To evaluate $S_5(\alpha)$, notice that $\alpha(n(n+\alpha))^{-1} = \int_0^1 \mathrm{d}t(1-t^{\alpha})t^{n-1}$, next recall the known result [**2**, p. 431, Entry 6.2.1.1] (for its proof, see Section 3)

$$\sum_{n=1}^{\infty} \psi(n)\,t^{n-1} = \gamma\,\frac{1}{t-1} + \frac{1}{t-1}\,\log(1-t) \qquad (|t|<1), \tag{7}$$

then proceed as follows:

$$\alpha S_5(\alpha) = \int_0^1 \mathrm{d}t(1-t^{\alpha})\sum_{n=1}^{\infty}\psi(n)t^{n-1} = \gamma\int_0^1 \frac{1-t^{\alpha}}{t-1}\mathrm{d}t + \int_0^1 \frac{1-t^{\alpha}}{t-1}\log(1-t)\mathrm{d}t,$$

so that the proposed formula for $S_5(\alpha)$ follows upon making use of the definition of the psi function $\psi(z) = -\gamma + \int_0^1 (1-t^{z-1})(1-t)^{-1}\,\mathrm{d}t$ [**15**, p. 15] and the integral formula given by (16) and deduced in Section 3. Furthermore, the summation of $S_6(\alpha)$ is readily available upon noting the following relationship $S_6(\alpha) = \mathrm{d}(\alpha\, S_5(\alpha))/\mathrm{d}\alpha$.

Finally, we straightforwardly arrive at the formulae (5) and (6) by using the above listed summations along with (8) below as well as (3) and (4).

Remark 1. Observe that the summations (5) and (6) are themselves interesting and could be useful, however, we have failed to find them recorded in the literature. The both formulae are valid when $\alpha = 0$ and then $\sum_{n=1}^{\infty} H_n n^{-2} = 2\,\zeta(3)$. Indeed, for instance, consider (6), then, calling its right-hand side $\Phi(\alpha)$, on using

$$\psi^{(m)}(z+1) = \psi^{(m)}(z) + (-1)^m \frac{m!}{z^{m+1}} \qquad (m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{8}$$

we have

$$\Phi(\alpha) = \gamma\,\psi'(\alpha+1) + \psi(\alpha+1)\,\psi'(\alpha+1) - \frac{1}{2}\,\psi''(\alpha+1) + \frac{1}{\alpha^2}\big[\psi(\alpha+1) + \gamma - \alpha\,\psi'(\alpha+1)\big],$$

so that

$$\lim_{\alpha\to 0}\Phi(\alpha) = -\frac{1}{2}\,\psi''(1) + \lim_{\alpha\to 0}\frac{\psi(\alpha+1)+\gamma-\alpha\,\psi'(\alpha+1)}{\alpha^2} = -\,\psi''(1) = 2\,\zeta(3).$$

Now we are ready to state and prove our main results. Note that our proof of Theorem 1 is based on Lemma 1, while the main argument, besides Lemma 1, needed for the proofs of Theorems 2 and 3 is partial fraction decomposition.

Theorem 1. *Let x be a real number, $x \neq -1, -2, -3, \ldots$, and assume that $q \in \mathbb{N} \setminus \{1\}$. Then, in terms of the polygamma functions $\psi^{(m)}(x)$, we have:*

$$\sum_{n=1}^{\infty}\frac{H_n}{(n+x)^q} = \frac{(-1)^q}{(q-1)!}\Bigg[\big(\psi(x)+\gamma\big)\,\psi^{(q-1)}(x) - \frac{1}{2}\,\psi^{(q)}(x) + \sum_{m=1}^{q-2}\binom{q-2}{m}\psi^{(m)}(x)\,\psi^{(q-m-1)}(x)\Bigg]. \tag{9}$$

Proof. Clearly, if $q = 2$, then (9) reduces to (6) and there is nothing to prove. If $q > 2$, the formula (9) follows on differentiating $(q-2)$ times both sides of (6) with respect to the parameter α which is followed by a simple rearrangement.

Corollary 1. *Assume that $p \in \mathbb{N}_0$ and $q \in \mathbb{N} \setminus \{1\}$, then:*

$$2\sum_{n=1}^{\infty}\frac{H_n}{(n+p+1)^q} = 2\,H_p\big[\zeta(q) - H_p^{(q)}\big] + q\big[\zeta(q+1) - H_p^{(q+1)}\big] - \sum_{m=1}^{q-2}\big[\zeta(m+1) - H_p^{(m+1)}\big]\big[\zeta(q-m) - H_p^{(q-m)}\big].$$

Proof. In Equation (9) set $x = p+1$, recall (2), make use of the facts that $\psi(1) = -\gamma$ and $\psi^{(r)}(1) = (-1)^{r-1}\, r!\, \zeta(r+1)$, $r \in \mathbb{N}$, and note that, for an arbitrary sequence A_n, the following identity holds

$$\sum_{n=p}^{q} A_n A_{p+q-n} = \frac{2}{p+q} \sum_{n=p}^{q} n\, A_n A_{p+q-n}.$$

Remark 2. Since Euler's time [**7**], the summation (1), or equivalently (17), has been proved, in vast mathematical literature, in many different ways. For a recent and simple proof see [**4**], while for more details on this topic and extensive lists of references, the interested reader is referred to Berndt [**1**, p. 252 *et seq.*] and Choi and Srivastava [**4**, p. 58]. Note that Euler's result follows easily from all three our results, (9) above and (10) and (13) below. To obtain (1) from (9), make use of similar arguments as in the proof of Corollary 1 (also, see Section 3 and Equation (18)):

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{n^q} &= \zeta(q+1) + \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^q} = \frac{1}{2}\,(q+2)\,\zeta(q+1) - \frac{1}{q-1} \sum_{m=1}^{q-2} m\zeta(m+1)\zeta(q-m) \\ &= \frac{1}{2}\,(q+2)\,\zeta(q+1) - \frac{1}{2} \sum_{m=1}^{q-2} \zeta(m+1)\,\zeta(q-m),\end{aligned}$$

Moreover, on setting $k = 1$ and letting $a \to 0$ in (10) and (13), these expressions reduce readily to (1), since we have $\lim\limits_{x\to 0} x^s\, \psi^{(m)}\,(1/x) = 0$, $m \in \mathbb{N}_0$ and $s > 0$.

Theorem 2. *Suppose that a is a real non-negative number and let k and q be positive integers. Then:*

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q \binom{an+k}{k}} = \sum_{r=1}^{k} \binom{k}{r} (-1)^{r+q} \left(\frac{a}{r}\right)^{q-1} \Big[2\zeta(2) + \big(H^{(1)}_{r/a-1}\big)^2 + H^{(2)}_{r/a-1}\Big] \tag{10}$$

$$+ \sum_{r=1}^{k} \sum_{s=2}^{q} \binom{k}{r} (-1)^{r+q+1-s} \left(\frac{a}{r}\right)^{q-s} \left[(s+2)\,\zeta(s+1) - \sum_{m=1}^{s-2} \zeta\,(m+1)\,\zeta\,(s-m)\right].$$

Proof. Recall first that Pochhammer's symbol (or the shifted factorial) $(\alpha)_r$ is given by $(\alpha)_0 = 1$ and $(\alpha)_r = \alpha\,(\alpha+1)\,(\alpha+2)\cdots(\alpha+r-1)$, $r \in \mathbb{N}$. To prove (10), consider the series on its left-hand side and expand it as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{n^q \binom{an+k}{k}} &= \sum_{n=1}^{\infty} \frac{k!\, H_n}{n^q\, (an+1)_k} = \sum_{n=1}^{\infty} \frac{k!\, H_n}{n^q} \frac{1}{\prod_{r=1}^{k} (an+r)} \\ &= \sum_{n=1}^{\infty} \frac{k!\, H_n}{n^q} \sum_{r=1}^{k} \frac{A_r}{an+r} = \sum_{n=1}^{\infty} \frac{H_n}{n^q} \sum_{r=1}^{k} \frac{(-1)^{r+1}\, r}{an+r} \binom{k}{r},\end{aligned}$$

where

$$A_r = \lim_{n \to -\frac{r}{a}} \frac{an+r}{\prod_{r=1}^{k} (an+r)} = (-1)^{r+1}\, \frac{r}{k!} \binom{k}{r}.$$

Now, by simple rearrangement and due to the fact that

$$\frac{1}{n^q\,(an+r)} = \frac{(-a)^{q-1}}{r^{q-1} n\,(an+r)} + \sum_{s=2}^{q} \frac{(-a)^{q-s}}{r^{q+1-s}\;n^s}, \tag{11}$$

we may write

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{n^q \binom{an+k}{k}} &= \sum_{r=1}^{k} (-1)^{r+1}\, r \binom{k}{r} \sum_{n=1}^{\infty} \frac{H_n}{n^q\,(an+r)} \\ &= \sum_{r=1}^{k} (-1)^{r+1}\, r \binom{k}{r} \left[\sum_{n=1}^{\infty} H_n \left(\frac{(-a)^{q-1}}{r^{q-1} n\,(an+r)} + \sum_{s=2}^{q} \frac{(-a)^{q-s}}{r^{q+1-s}\;n^s} \right) \right],\end{aligned}$$

so that, what remains is to sum $\sum_{n=1}^{\infty} H_n\,(n(an+r))^{-1}$ and $\sum_{n=1}^{\infty} H_n n^{-s}$. It is clear that the latter series can be summed by using the Euler result (1), while for the former, by Lemma 1 (with $\alpha = r/a$), we deduce the following summation formula

$$\sum_{n=1}^{\infty} \frac{H_n}{n(an+r)} = \frac{1}{2\,r} \left[2\,\zeta(2) + (H^{(1)}_{\frac{r}{a}-1})^2 + H^{(2)}_{\frac{r}{a}-1} \right], \tag{12}$$

and thus complete the proof of (10).

Remark 3. To evaluate $H^{(1)}_{r/a-1}$, $H^{(2)}_{r/a-1}$ and $H^{(3)}_{r/a-1}$ in (10) above and (13) below, we utilize (see Equations (3) and (4))

$$H^{(1)}_{r/a-1} = \gamma + \psi\left(\frac{r}{a}\right) \quad \text{and} \quad H^{(m)}_{r/a-1} = \zeta(m) + \frac{(-1)^{m-1}}{(m-1)!}\,\psi^{(m-1)}\left(\frac{r}{a}\right) \quad (m \in \mathbb{N} \setminus \{1\}),$$

where the evaluation of $\psi^{(m)}(z)$ at rational values of the argument z can be explicitly done via formulae as given in [**3**] in terms of the polylogarithmic or other special functions.

Example 1. By Theorem 2 we have:

a) $$\begin{aligned} 2\sum_{n=1}^{\infty} \frac{H_n}{n^q(an+1)} &= (-a)^{q-1} \Big[2\zeta(2) + \big(H^{(1)}_{1/a-1}\big)^2 + H^{(2)}_{1/a-1} \Big] \\ &\quad + \sum_{s=2}^{q} (-a)^{q-s} \left[(s+2)\,\zeta\,(s+1) - \sum_{m=1}^{s-2} \zeta\,(m+1)\,\zeta\,(s-m) \right]; \end{aligned}$$

b) $$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n}{n^4 \binom{4n+3}{3}} &= 3\,\zeta(5) - \zeta(3)\zeta(2) - \frac{55}{6}\,\zeta(4) + \frac{680}{9}\,\zeta(3) + \frac{1312}{9}\,\zeta(2) \\ &\quad - \frac{2560}{9}\,\pi\ln(2) - \frac{2480}{3}\,\ln^2(2) + \frac{20480}{27}\,G, \end{aligned}$$

where G is Catalan's constant (see, for example, Srivastava and Choi [**15**, p. 29]).

Theorem 3. *Let*

$$\Theta_k^q(a) := \sum_{n=1}^{\infty} \frac{H_n}{n^q \binom{an+k}{k}^2} \qquad (a > 0; k, q \in \mathbb{N}).$$

Then we have:

$$(13)\quad 2\Theta_k^q(a) = 2\sum_{r=1}^{k} \binom{k}{r}^2 \left(\frac{-a}{r}\right)^{q-2} \left[\zeta(3) + \zeta(2) H_{r/a-1}^{(1)} - H_{r/a-1}^{(1)} H_{r/a-1}^{(2)} - H_{r/a-1}^{(3)}\right]$$
$$+ \sum_{r=1}^{k} \binom{k}{r}^2 \left(\frac{-a}{r}\right)^{q-1} \left[2\zeta(2) + \left(H_{r/a-1}^{(1)}\right)^2 + H_{r/a-1}^{(2)}\right] \left[q + 2r\left(H_{r-1}^{(1)} - H_{k-r}^{(1)}\right)\right]$$
$$+ \sum_{r=1}^{k} \sum_{s=2}^{q} \binom{k}{r}^2 \left(\frac{-a}{r}\right)^{q-s} \left[q + 1 - s + 2r\left(H_{r-1}^{(1)} - H_{k-r}^{(1)}\right)\right]$$
$$\times \left[(s+2)\zeta(s+1) - \sum_{m=1}^{s-2} \zeta(m+1)\zeta(s-m)\right].$$

Proof. Consider the following expansion:

$$(14)\qquad \Theta_k^q(a) = \sum_{n=1}^{\infty} \frac{(k!)^2 H_n}{n^q \left[(an+1)_k\right]^2} = \sum_{n=1}^{\infty} \frac{(k!)^2 H_n}{n^q} \sum_{r=1}^{k} \left[\frac{P_r}{an+r} + \frac{Q_r}{(an+r)^2}\right],$$

where

$$Q_r = \lim_{n \to -\frac{r}{a}} \frac{(an+r)^2}{\prod_{r=1}^{k} (an+r)^2} = \frac{r^2}{(k!)^2} \binom{k}{r}^2$$

and

$$P_r = \lim_{n \to -\frac{r}{a}} \frac{\partial}{\partial n} \left\{ \frac{(an+r)^2}{\prod_{r=1}^{k} (an+r)^2} \right\} = 2\, \frac{r^2}{(k!)^2} \binom{k}{r}^2 \left(H_{r-1}^{(1)} - H_{k-r}^{(1)}\right).$$

From (14) we have that

$$\Theta_k^q(a) = \sum_{n=1}^{\infty} \frac{H_n}{n^q} \sum_{r=1}^{k} r^2 \binom{k}{r}^2 \left[\frac{1}{(an+r)^2} + \frac{2}{an+r}\left(H_{r-1}^{(1)} - H_{k-r}^{(1)}\right)\right]$$

$$= \sum_{r=1}^{k} r^2 \binom{k}{r}^2 \sum_{n=1}^{\infty} H_n \left[\frac{1}{n^q (an+r)^2} + \frac{2}{n^q (an+r)}\left(H_{r-1}^{(1)} - H_{k-r}^{(1)}\right)\right],$$

thus, by (11) in conjunction with

$$\frac{1}{n^q\left(an+r\right)^2}=\frac{(-a)^{q-1}q}{r^q n\left(an+r\right)}+\frac{(-a)^q}{r^q\left(an+r\right)^2}+\sum_{s=2}^{q}\frac{(-a)^{q-s}\left(q+1-s\right)}{r^{q+2-s}n^s}$$

we obtain

$$\begin{aligned}\Theta_k^q(a)\equiv\sum_{n=1}^{\infty}\frac{H_n}{n^q\binom{an+k}{k}^2}&=\sum_{r=1}^{k}r^2\binom{k}{r}^2\frac{(-a)^q}{r^q}\sum_{n=1}^{\infty}\frac{H_n}{(an+r)^2}\\&+\sum_{r=1}^{k}r^2\binom{k}{r}^2\left\{\frac{(-a)^{q-1}}{r^{q-1}}\left[\frac{q}{r}+2\left(H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\right]\sum_{n=1}^{\infty}\frac{H_n}{n\left(an+r\right)}\right\}\\&+\sum_{r=1}^{k}r^2\binom{k}{r}^2\left\{\sum_{s=2}^{q}\frac{(-a)^{q-s}}{r^{q+1-s}}\left[\frac{q+1-s}{r}+2\left(H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\right]\sum_{n=1}^{\infty}\frac{H_n}{n^s}\right\},\end{aligned}$$

which, on summing the following series, $\sum_{n=1}^{\infty}H_n n^{-s}$, $\sum_{n=1}^{\infty}H_n\big(n(an+r)\big)^{-1}$ and $\sum_{n=1}^{\infty}H_n\big(an+r\big)^{-2}$, gives the proposed result (13). The summations of the first two series are respectively given by (1) and (12), while the summation for the third is as follows

$$\sum_{n=1}^{\infty}\frac{H_n}{(an+r)^2}=\frac{1}{a^2}\left[\zeta(3)+\zeta(2)H_{r/a-1}^{(1)}-H_{r/a-1}^{(1)}H_{r/a-1}^{(2)}-H_{r/a-1}^{(3)}\right],\tag{15}$$

and it is readily available from Lemma 1 (with $\alpha=r/a$).

Example 2. By Theorem 3 we have:

a) $$\begin{aligned}2\sum_{n=1}^{\infty}\frac{H_n}{n^q\left(an+1\right)^2}&=\Big[-aq\big(\big(H_{1/a-1}^{(1)}\big)^2+H_{1/a-1}^{(2)}\big)-2\big(H_{1/a-1}^{(1)}H_{1/a-1}^{(2)}+H_{1/a-1}^{(3)}\big)\Big]\\&\times(-a)^{q-2}+2\,(-a)^{q-2}\left(H_{1/a-1}^{(1)}-aq\right)\zeta\left(2\right)+2\,(-a)^{q-2}\left(2q-1\right)\zeta\left(3\right)\\&+\sum_{s=3}^{q}(-a)^{q-s}\left(q+1-s\right)\Big[\left(s+2\right)\zeta\left(s+1\right)-\sum_{m=1}^{s-2}\zeta\left(m+1\right)\zeta\left(s-m\right)\Big];\end{aligned}$$

b) $$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n^3\binom{4n+3}{3}^2}&=\frac{5}{4}\zeta(4)-\frac{3602}{3}\zeta(3)-\frac{52}{3}\pi^3+\big(780\ln(2)-16\big)\zeta(2)\\&-32\,\pi\ln(2)+312\,\ln^2(2)+\frac{64}{3}\big(7\pi+4+39\ln\left(2\right)\big)\,G.\end{aligned}$$

3. CONCLUDING REMARKS

In the beginning of this section, for the sake of completeness of the proof of Lemma 1, we prove the summation formula given by (7) and deduce the following integral formula

$$(16)\quad I(\alpha) := \int_0^1 \frac{1-t^\alpha}{1-t}\,\log(1-t)\,\mathrm{d}t = -\frac{1}{2}\Big[\big(\gamma+\psi(\alpha+1)\big)^2 + \zeta(2) - \psi'(\alpha+1)\Big].$$

In the light of the familiar sums, $\psi(n) = -\gamma + \sum_{m=1}^{n-1} m^{-1}$, $\sum_{n=0}^{\infty} t^n = (1-t)^{-1}$ and $\sum_{n=1}^{\infty} t^n n^{-1} = -\log(1-t)$, along with the elementary double–series identity $\sum_{n=1}^{\infty}\sum_{m=1}^{n} A(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A(m,m+n)$ [**15**, p. 337], the summation (7) follows without difficulty, as shown by the following derivation:

$$\begin{aligned}\sum_{n=1}^{\infty} \psi(n)\,t^{n-1} &= \sum_{n=1}^{\infty} t^{n-1}\left(-\gamma + \sum_{m=1}^{n-1}\frac{1}{m}\right) = \gamma\,\frac{1}{t-1} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{t^{m+n}}{m}\\ &= \gamma\,\frac{1}{t-1} + \frac{1}{t-1}\,\log(1-t).\end{aligned}$$

In order to deduce (16), first, by applying partial integration, $u(t) = 1-t^\alpha$ and $\mathrm{d}v(t) = \log(1-t)(1-t)^{-1}$, and then, on noting that $\mathrm{d}(a^t)/\mathrm{d}t = a^t\log t$, we obtain

$$I(\alpha) = -\frac{\alpha}{2}\int_0^1 t^{\alpha-1}\log^2(1-t)\,\mathrm{d}t = -\frac{\alpha}{2}\,J(\alpha) := \frac{\partial^2}{\partial\beta^2}\int_0^1 t^{\alpha-1}\Big(1-t\Big)^\beta\,\mathrm{d}t\bigg|_{\beta=0}.$$

Next, since $B(a,b) = \Gamma(a)\,\Gamma(b)/\Gamma(a+b) = \int\limits_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ where $B(a,b)$ and $\Gamma(a)$ respectively are the familiar beta and gamma function [**15**, pp. 1–12], we have

$$\begin{aligned}J(\alpha) &= \frac{\partial^2}{\partial\beta^2}\,B(\alpha,\beta+1)\bigg|_{\beta=0} = \frac{\partial^2}{\partial\beta^2}\,\frac{\Gamma(\alpha)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\bigg|_{\beta=0} = B(\alpha,\beta+1)\\ &\quad\times\Big[\big(\psi(\beta+1)-\psi(\alpha+\beta+1)\big)^2 + \psi'(\beta+1) - \psi'(\alpha+\beta+1)\Big]\bigg|_{\beta=0}\\ &= \frac{1}{\alpha}\Big[\big(\gamma+\psi(\alpha+1)\big)^2 + \zeta(2) - \psi'(\alpha+1)\Big],\end{aligned}$$

where we utilize $\psi(z) = \Gamma'(z)/\Gamma(z)$, $\psi''(z) = \psi'(z)$, $\psi(1) = -\gamma$ and $\psi'(z) = \zeta(2)$, and thus readily arrive at the integral formula (16). Note that, unlike $I(\alpha)$, the integral $J(\alpha)$ can be easily found in the literature [**8**, p. 499, Entry 2.6.9.13].

Further, we remark that Euler's identity (1) is equivalent to

$$(17)\qquad 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^q} = q\,\zeta\,(q+1) - \sum_{m=1}^{q-2}\zeta\,(m+1)\,\zeta\,(q-m)\qquad (q\in\mathbb{N}\setminus\{1\}).$$

Indeed, in view of $H_n = H_{n-1} + n^{-1}$, we have

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^q} = \sum_{n=2}^{\infty} \frac{H_n - \frac{1}{n}}{n^q} = \left(\sum_{n=1}^{\infty} \frac{H_n}{n^q} - 1 \right) - \left(\sum_{n=1}^{\infty} \frac{H_n}{n^{q+1}} - 1 \right),$$

thus

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q} = \zeta(q+1) + \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^q}. \tag{18}$$

Next observe that the integral representations of (10) and (13), respectively, can be written in the form

$$\sum_{n=1}^{\infty} \frac{H_n}{n^q \binom{an+k}{k}} = -k \int_{(0,0)}^{(1,1)} \frac{(1-y)^{k-1} \log(1-x)\ \mathrm{Li}_{q-1}(xy^a)}{x} \,\mathrm{d}x\,\mathrm{d}y$$

and

$$\Theta_k^q(a) = -k^2 \int_{(0,0,0)}^{(1,1,1)} \frac{((1-z)(1-y))^{k-1} \log(1-x)\ \mathrm{Li}_{q-1}(x(yz)^a)}{x} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where $\mathrm{Li}_m(w) := \sum_{r=1}^{\infty} w^r r^{-m}$ is the polylogarithmic function, and can be readily attained by the expansion of the binomial coefficient in terms of the beta integral function.

In this paper, we have established three new and very general extensions of Euler's harmonic sum identity (1). Clearly our results could be rewritten in the representation of the Hurwitz (or generalized) zeta function $\zeta(z, a)$, since there exists the relationship $\psi^{(r)}(z) = (-1)^{r-1}\, r!\, \zeta(r+1, z)$, $r \in \mathbb{N}$, $z \neq -1, -2, -3, \ldots$, between $\psi^{(r)}(z)$ and $\zeta(z, a)$. Here, by rewriting Corollary 1, we provide a particularly interesting example

$$\begin{aligned} 2\sum_{n=1}^{\infty} \frac{H_n}{(n+p+1)^q} &= 2\,\zeta(q, p+1)\, H_p + q\, \zeta(q+1, p+1) \\ &\quad - \sum_{m=1}^{q-2} \zeta(m+1, p+1)\, \zeta(q-m, p+1). \end{aligned}$$

It should be noted that it is easy to show that this formula is valid for any real p, $p \neq -1, -2, -3, \ldots$.

To conclude, note that it would be useful to be able to extend the approach described above to include other similar and related sums. In particular, it would be very interesting to consider sums of the form

$$\sum_{n\geq 1} \frac{H_n^{(r)}}{n^q \binom{an+k}{k}} \qquad (r \in \mathbb{N} \setminus \{1\}).$$

However, we have been unable, so far, to make any progress with this sum. Unfortunately, it appears that, even in the case $r = 2$, the method used in this work gives rise to several complex and intractable summations.

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