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A NOTE ON RECTANGULAR MORREY-CAMPANATO SPACES

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Abstract. We define rectangular central versions of the Campanato and Morrey spaces in $\mathbb{R}^2$ and consider the commutator operator of the rectangular 2-dimensional Hardy operator and a locally integrable function b acting on these Morrey spaces. We prove continuity of this operator when b belongs to a rectangular central Campanato space.

1. Introduction

The theory of Campanato spaces has its origin in the early 60's with the work of S. Campanato [**2**] and G. Stampacchia [**11**]. For $1 \leq p < \infty$ and $-1/p \leq \lambda \leq 1/n$, the classical definition for Campanato space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ is given in terms of the following norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} = \sup_B \left(\frac{1}{|B|^{1+\lambda p}} \int_B |f - f_B|^p dx \right)^{1/p},$$

where B is any ball in $\mathbb{R}^n$, $|B|$ is the Lebesgue measure of B and f_B is the average of f in B, i.e.

$$f_B = \frac{1}{|B|} \int_B f(x)dx.$$

Campanato spaces are a generalization of the space of functions with bounded mean oscillation $BMO(\mathbb{R}^n)$ defined by F. John and J. Nirenberg in 1961, which is described by the following norm

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |f - f_B| dx,$$

where B denotes a ball in $\mathbb{R}^n$.

Related to Campanato spaces we also have Morrey spaces, which were introduced in 1938 by C. Morrey [**9**] to study the local behavior of solutions to the second order elliptic partial differential equations. The norm in the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined as follows

$$\|f\|_{L^{p,\lambda}} = \sup_B \left(\frac{1}{|B|^{1+\lambda p}} \int_B |f|^p dx \right)^{1/p},$$

with $1 \leq p < \infty$ and $-1/p \leq \lambda \leq 1/n$. Is not difficult to see that $L^{p,\lambda}(\mathbb{R}^n) \subset \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$. Both, Morrey and Campanato spaces, have been generalized in various ways in order to obtain existence and uniqueness of solutions of partial differential equations and better versions of the Sobolev type embedding.

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Central Campanato and Morrey spaces are generalizations of the spaces introduced by García-Cuerva in [**6**] and S. Lu and D. Yang in [**8**]; in this paper we will focus on central rectangular versions of Campanato and Morrey spaces in $\mathbb{R}^2$. Our aim is to explore the behavior in the central rectangular Morrey space of the commutator of the rectangular 2-dimensional Hardy operator H_b^R, introduced in [**5**], when b belongs to a central rectangular Campanato space. A characterization of the central Campanato space via this commutator can be found in [**10**] in the radial context. Following the techniques developed there, we obtain our continuity results about commutators.

The notation along this manuscript will be standard and we will adopt the convention of using C to denote a constant that may be changing line by line.

2. Main Results

Assume $1 \leq p < \infty$ and $-1/p \leq \lambda \leq 0$, the central rectangular Morrey space will be defined as follows

$$\dot{\mathcal{B}}^{p,\lambda}(\mathbb{R}^2) = \{f \in L^p_{loc}(\mathbb{R}^2) : \|f\|_{\dot{\mathcal{B}}^{p,\lambda}} < \infty\},$$

where

$$\|f\|_{\dot{\mathcal{B}}^{p,\lambda}} = \sup_{R_1,R_2>0} \left(\frac{1}{(4R_1R_2)^{1+\lambda p}} \int_{[-R_1,R_1]\times[-R_2,R_2]} |f(x_1,x_2)|^p dx_1 dx_2 \right)^{1/p}.$$

It is not difficult to show that $(\dot{\mathcal{B}}^{p,\lambda}(\mathbb{R}^2), \|\cdot\|_{\dot{\mathcal{B}}^{p,\lambda}})$ is a Banach space. If we denote by $\dot{B}^{p,\lambda}(\mathbb{R}^2)$ the space obtained considering balls with center in the origin instead of rectangles in the previous definition, which bears the name of central Morrey space, we can prove that $\dot{\mathcal{B}}^{p,\lambda}(\mathbb{R}^2) \subset \dot{B}^{p,\lambda}(\mathbb{R}^2)$ with $\|\cdot\|_{\dot{B}^{p,\lambda}} < C\,\|\cdot\|_{\dot{\mathcal{B}}^{p,\lambda}}$.

Under the same conditions for p and λ, the central rectangular Campanato space, denoted by $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}(\mathbb{R}^2)$, is defined as the space consisting of all the functions f in $L^p_{loc}(\mathbb{R}^2)$ such that

$$\sup_{R_1,R_2>0} \left(\frac{1}{(4R_1R_2)^{1+\lambda p}} \int_{[-R_1,R_1]\times[-R_2,R_2]} |f - f_{R_1,R_2}|^p dx_1 dx_2 \right)^{1/p} < \infty, \quad (2.1)$$

where

$$f_{R_1,R_2} = \frac{1}{4R_1R_2} \int_{[-R_1,R_1]\times[-R_2,R_2]} f(x_1,x_2) dx_1 dx_2.$$

When it is finite, we denote the quantity in (2.1) by $\|f\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}}$.

Standard arguments allow us to show that $(\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}(\mathbb{R}^2), \|\cdot\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}})$ is a Banach space after identifying functions that differ by a constant almost everywhere in $\mathbb{R}^2$. Also, simple calculations show that $\dot{\mathcal{B}}^{p,\lambda}(\mathbb{R}^2) \subset \mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}(\mathbb{R}^2)$ and, if we denote by $C\dot{M}O^{p,\lambda}(\mathbb{R}^2)$ the Campanato space localized at the origin, we have $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}(\mathbb{R}^2) \subset C\dot{M}O^{p,\lambda}(\mathbb{R}^2)$. Actually $\|\cdot\|_{C\dot{M}O^{p,\lambda}} < C\,\|\cdot\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}}$, since the Lebesgue measure of balls and cubes are comparable.

These spaces can be seen as rectangular versions of the spaces $C\dot{M}O^{p,\lambda}(\mathbb{R}^2)$ studied by Alvarez, Guzmán-Partida and Lakey in [**1**] for $\lambda < 1/2$.

Now, let us introduce the rectangular 2-dimensional Hardy operator. For a locally integrable function f, we define $H_2^R f$ as

$$H_2^R f(x_1,x_2) = \frac{1}{|x_1||x_2|}\int_{\{(y_1,y_2):|y_j|<|x_j|\}} f(y_1,y_2)dy_1dy_2,$$

where $x_1 \neq 0$ and $x_2 \neq 0$. In [**5**] we proved continuity of the rectangular n-dimensional Hardy operator in $\dot{\mathcal{B}}^{p,0}(\mathbb{R}^2)$, $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,0}(\mathbb{R}^2)$ and more general spaces. In this work we will consider the commutator of this Hardy operator, which is defined as follows

$$H_b^R f = bH_2^R f - H_2^R(bf),$$

where b is a locally integrable function.

We wish to investigate the action of this commutator on central rectangular Morrey spaces when b belongs to a central rectangular Campanato space. Our first result will be stated under the assumption that for b, there exist a constant $C>0$ such that for any rectangle $\tilde{R}\subset\mathbb{R}^2$,

$$\sup_{(x_1,x_2)\in\tilde{R}} |b(x_1,x_2)-b_{\tilde{R}}| \leq \frac{C}{|\tilde{R}|}\int_{\tilde{R}}|b(y_1,y_2)-b_{\tilde{R}}|dy_1dy_2, \tag{2.2}$$

where $b_{\tilde{R}}$ is the average of b in $\tilde{R}$. Although this assumption may seem artificial, it is based on the condition that defines the reverse Hölder class (more information on the reverse Hölder classes may be found in [**3**], [**4**] and [**7**]).

Now we state our first result.

Theorem 2.1. *Let $1<p<\infty$, $-1/p<\lambda<0$, $-1/p_i<\lambda_i<0$, $i=1,2$, $1/p=1/p_1+1/p_2$, $\lambda=\lambda_1+\lambda_2$ and let b be a function in $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p_1,\lambda_1}$ that satisfies (2.2). Then $H_b^R:\dot{\mathcal{B}}^{p_2,\lambda_2}\to\dot{\mathcal{B}}^{p,\lambda}$ is bounded with*

$$\left\|H_b^R\right\|_{\dot{\mathcal{B}}^{p_2,\lambda_2}\to\dot{\mathcal{B}}^{p,\lambda}} \leq C\,\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p_1,\lambda_1}}.$$

Proof. Before starting, let us introduce some notation that will be helpful during the proof. For a pair of integers k_1 and k_2 we will write

(1) $I_{k_1,k_2}=[-2^{k_1},2^{k_1}]\times[-2^{k_2},2^{k_2}]$,
(2) $C_{k_1,k_2}=\{(x_1,x_2):2^{k_j-1}<|x_j|\leq 2^{k_j}, j=1,2\}$,
(3) $b_{k_1,k_2}=2^{-(k_1+k_2+2)}\int_{I_{k_1,k_2}}b(x_1,x_2)dx_1dx_2$.

Consider an arbitrary rectangle $[-R_1,R_1]\times[-R_2,R_2]\subset\mathbb{R}^2$ and take $k_1,k_2\in\mathbb{Z}$ such that $2^{k_j-1}<R_j\leq 2^{k_j}$ for $j=1,2$. Then

$$\frac{1}{(4R_1R_2)^{1+\lambda p}}\int_{[-R_1,R_1]\times[-R_2,R_2]}|f|^pdx_1dx_2 \leq C\frac{1}{2^{(k_1+k_2+2)(1+\lambda p)}}\int_{I_{k_1,k_2}}|f|^pdx_1dx_2$$

with C a constant independent of R_1 and R_2.

As a consequence, it will be enough to show that exists a constant C such that

$$\left(\frac{1}{2^{(k_1+k_2+2)(1+\lambda p)}}\int_{I_{k_1,k_2}}|H_b^R f|^pdx_1dx_2\right)^{1/p} \leq C\,\|f\|_{\dot{\mathcal{B}}^{p_2,\lambda_2}}, \tag{2.3}$$

for every pair of integers k_1 and k_2.

From the definition of H_b^R we have

$$
\begin{aligned}
&\int_{I_{K_1,K_2}} |H_b^R f(x_1,x_2)|^p dx_1 dx_2 \\
&= \int_{I_{K_1,K_2}} \left| \frac{1}{|x_1||x_2|} \int_{\{|y_j|<|x_j|\}} [b(x_1,x_2) - b(y_1,y_2)]\, f(y_1,y_2) dy_1 dy_2 \right|^p dx_1 dx_2 \\
&\le \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} \int_{C_{k_1,k_2}} \left(\frac{1}{|x_1||x_2|} \int_{I_{k_1,k_2}} |b(x_1,x_2) - b(y_1,y_2)| \right. \\
&\left. \times |f(y_1,y_2)| dy_1 dy_2 \right)^p dx_1 dx_2 \\
&\le C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} \int_{C_{k_1,k_2}} \left(\frac{1}{|x_1||x_2|} \sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \int_{C_{j_1,j_2}} |b(x_1,x_2) - b_{k_1,k_2}| \right. \\
&\left. \times |f(y_1,y_2)| dy_1 dy_2 \right)^p dx_1 dx_2 \\
&+ C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} \int_{C_{k_1,k_2}} \left(\frac{1}{|x_1||x_2|} \sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \int_{C_{j_1,j_2}} |b(y_1,y_2) - b_{k_1,k_2}| \right. \\
&\left. \times |f(y_1,y_2)| dy_1 dy_2 \right)^p dx_1 dx_2 \\
&= I + J.
\end{aligned}
$$

We can estimate the first term using Hölder's inequality for p_1/p and $(p_1/p)'$ and for p_2 and p_2' as follows

$$
\begin{aligned}
I &\le C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} 2^{-(k_1+k_2)p} \int_{I_{k_1,k_2}} |b(x_1,x_2) - b_{k_1,k_2}|^p dx_1 dx_2 \\
&\times \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \int_{I_{j_1,j_2}} |f(y_1,y_2)| dy_1 dy_2 \right)^p \\
&\le C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} 2^{-(k_1+k_2)p} \left(\int_{I_{k_1,k_2}} |b(x_1,x_2) - b_{k_1,k_2}|^{p_1} dx_1 dx_2 \right)^{p/p_1} \\
&\times |I_{k_1,k_2}|^{1/(p_1/p)'} \left[\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \left(\int_{I_{j_1,j_2}} |f(y_1,y_2)|^{p_2} dy_1 dy_2 \right)^{1/p_2} |I_{j_1,j_2}|^{1/p_2'} \right]^p \\
&\le C \, \|b\|^p_{C\dot{\mathcal{M}}\mathcal{O}^{p_1,\lambda_1}} \|f\|^p_{\dot{\mathcal{B}}^{p_2,\lambda_2}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{(\lambda_1-1)p+1} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} |I_{j_1,j_2}|^{\lambda_2+1} \right)^p
\end{aligned}
$$

$$\le C\,\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p_1,\lambda_1}}\,\|f\|^p_{\dot{\mathcal{B}}^{p_2,\lambda_2}}\sum_{k_1=-\infty}^{K_1}\sum_{k_2=-\infty}^{K_2}|I_{k_1,k_2}|^{1+\lambda p}$$
$$\le C\,\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p_1,\lambda_1}}\,\|f\|^p_{\dot{\mathcal{B}}^{p_2,\lambda_2}}\,|I_{K_1,K_2}|^{1+\lambda p}.$$

The second term can be estimated using Hölder's inequality repeatedly, first for p and p' and second for p_1/p and p_2/p, and the fact that b satisfies (2.2):

$$J \le C\sum_{k_1=-\infty}^{K_1}\sum_{k_2=-\infty}^{K_2}2^{-(k_1+k_2)p}\int_{C_{k_1,k_2}}\left(\sum_{j_1=-\infty}^{k_1}\sum_{j_2=-\infty}^{k_2}\int_{I_{j_1,j_2}}|b(y_1,y_2)-b_{k_1,k_2}|\right.$$
$$\left.\times|f(y_1,y_2)|dy_1dy_2\right)^p dx_1dx_2$$
$$\le C\sum_{k_1=-\infty}^{K_1}\sum_{k_2=-\infty}^{K_2}2^{-(k_1+k_2)p}\int_{C_{k_1,k_2}}\left(\sum_{j_1=-\infty}^{k_1}\sum_{j_2=-\infty}^{k_2}\left[\int_{I_{j_1,j_2}}|b(y_1,y_2)-b_{k_1,k_2}|^p\right.\right.$$
$$\left.\left.\times|f(y_1,y_2)|^p dy_1dy_2\right]^{1/p}|I_{j_1,j_2}|^{1/p'}\right)^p dx_1dx_2$$
$$\le C\sum_{k_1=-\infty}^{K_1}\sum_{k_2=-\infty}^{K_2}2^{-(k_1+k_2)p}\int_{C_{k_1,k_2}}\left(\sum_{j_1=-\infty}^{k_1}\sum_{j_2=-\infty}^{k_2}\left[\int_{I_{j_1,j_2}}|b(y_1,y_2)-b_{k_1,k_2}|^{p_1}\right.\right.$$
$$\left.\left.\times dy_1dy_2\right]^{1/p_1}\left[\int_{I_{j_1,j_2}}|f(y_1,y_2)|^{p_2}dy_1dy_2\right]^{1/p_2}|I_{j_1,j_2}|^{1/p'}\right)^p dx_1dx_2$$
$$\le C\,\|f\|^p_{\dot{\mathcal{B}}^{p_2,\lambda_2}}\sum_{k_1=-\infty}^{K_1}\sum_{k_2=-\infty}^{K_2}2^{-(k_1+k_2)p}\int_{C_{k_1,k_2}}\left(\sum_{j_1=-\infty}^{k_1}\sum_{j_2=-\infty}^{k_2}|I_{j_1,j_2}|^{1/p_1+1/p_2+1/p'+\lambda_2}\right.$$
$$\left.\times\left[\frac{1}{|I_{k_1,k_2}|}\int_{I_{k_1,k_2}}|b(y_1,y_2)-b_{k_1,k_2}|^{p_1}\right]^{1/p_1}\right)^p dx_1dx_2$$
$$\le C\,\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p_1,\lambda_1}}\,\|f\|^p_{\dot{\mathcal{B}}^{p_2,\lambda_2}}\sum_{k_1=-\infty}^{K_1}\sum_{k_2=-\infty}^{K_2}|I_{k_1,k_2}|^{(\lambda_1-1)p+1}\left(\sum_{j_1=-\infty}^{k_1}\sum_{j_2=-\infty}^{k_2}|I_{j_1,j_2}|^{\lambda_2+1}\right)^p$$
$$\le C\,\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p_1,\lambda_1}}\,\|f\|^p_{\dot{\mathcal{B}}^{p_2,\lambda_2}}\sum_{k_1=-\infty}^{K_1}\sum_{k_2=-\infty}^{K_2}|I_{k_1,k_2}|^{1+\lambda p}$$
$$\le C\,\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p_1,\lambda_1}}\,\|f\|^p_{\dot{\mathcal{B}}^{p_2,\lambda_2}}\,|I_{K_1,K_2}|^{1+\lambda p}.$$

Combining the above estimations for I and J, (2.3) can be proved. □

Now we wish to state a continuity result for H^R_b when b is a function for which (2.2) not necessarily holds. To do this, we need to impose stronger conditions on p and λ and the following lemma.

Lemma 2.2. *Let $1 < p < \infty$, $-1/p < \lambda < 0$, $k_j, l_j \in \mathbb{Z}$ for $j = 1,2$ and $b \in \mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}(\mathbb{R}^2)$. With the same notation in the proof of Theorem 2.1, the following holds*

$$|b(x_1,x_2)-b_{k_1,k_2}| \leq |b(x_1,x_2)-b_{l_1,l_2}|+C\,\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \max\{|I_{k_1,k_2}|^\lambda, |I_{k_1,l_2}|^\lambda, |I_{l_1,l_2}|^\lambda\}.$$

Proof. First notice that

$$\begin{aligned}|b_{k_1,k_2} - b_{k_1,k_2+1}| &\leq \frac{1}{|I_{k_1,k_2}|}\int_{I_{k_1,k_2}} |b(x_1,x_2) - b_{k_1,k_2+1}|\,dx_1dx_2\\ &\leq 2\left(\frac{1}{|I_{k_1,k_2+1}|}\int_{I_{k_1,k_2+1}} |b(x_1,x_2)-b_{k_1,k_2+1}|^p dx_1dx_2\right)^{1/p}\\ &\leq 2\,\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}}\,|I_{k_1,k_2+1}|^\lambda.\end{aligned}$$

Now observe that if $k_2 < l_2$

$$\begin{aligned}|b(x_1,x_2)-b_{k_1,k_2}| &\leq |b(x_1,x_2)-b_{k_1,l_2}| + \sum_{j=k_2}^{l_2-1}|b_{k_1,j}-b_{k_1,j+1}|\\ &\leq |b(x_1,x_2)-b_{k_1,l_2}| + 2\,\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}}\sum_{j=k_2}^{l_2-1}|I_{k_1,j+1}|^\lambda\\ &\leq |b(x_1,x_2)-b_{k_1,l_2}| + C\,\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}}\,|I_{k_1,k_2}|^\lambda.\end{aligned}$$

Similarly, if $k_2 > l_2$

$$|b(x_1,x_2)-b_{k_1,k_2}| \leq |b(x_1,x_2)-b_{k_1,l_2}| + C\,\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}}\,|I_{k_1,l_2}|^\lambda.$$

By the previous calculations we have

$$|b(x_1,x_2)-b_{k_1,k_2}| \leq |b(x_1,x_2)-b_{k_1,l_2}| + C\,\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}}\max\{|I_{k_1,k_2}|^\lambda, |I_{k_1,l_2}|^\lambda\}.$$

Proceeding in the same way for k_1 and l_1 in the first term of the right side of last inequality , we obtain

$$|b(x_1,x_2)-b_{k_1,k_2}| \leq |b(x_1,x_2)-b_{l_1,l_2}|+C\,\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \max\{|I_{k_1,k_2}|^\lambda, |I_{k_1,l_2}|^\lambda, |I_{l_1,l_2}|^\lambda\}.$$

$\square$

Theorem 2.3. *Let $2 < p < \infty$ and $-1/2p < \lambda < 0$. If $b \in \mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}(\mathbb{R}^2)$, then $H_b^R : \dot{\mathcal{B}}^{p,\lambda}(\mathbb{R}^2) \to \dot{\mathcal{B}}^{p,2\lambda}(\mathbb{R}^2)$ is bounded with*

$$\left\|H_b^R\right\|_{\dot{\mathcal{B}}^{p,\lambda}(\mathbb{R}^2)\to\dot{\mathcal{B}}^{p,2\lambda}(\mathbb{R}^2)} \leq C\,\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}}\,.$$

Proof. We will continue using the notation in the proof of Theorem 2.1. Again, it will be enough to prove that there is a constant $C > 0$ such that

$$\left(\frac{1}{|I_{K_1,K_2}|^{1+2\lambda p}}\int_{I_{K_1,K_2}} |H_b^R f(x_1,x_2)|^p dx_1dx_2\right)^{1/p} \leq C\,\|f\|_{\dot{\mathcal{B}}^{p,\lambda}}\,. \tag{2.4}$$

To obtain (2.4) we note that

$$\int_{I_{K_1,K_2}} |H_b^R f(x_1,x_2)|^p dx_1dx_2$$

$$
\begin{aligned}
&\leq \int_{I_{K_1,K_2}} \left(\frac{1}{|x_1||x_2|} \int_{\{|y_j|<|x_j|\}} |b(x_1,x_2)-b(y_1,y_2)||f(y_1,y_2)|dy_1dy_2 \right)^p dx_1dx_2 \\
&\leq C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} \int_{C_{k_1,k_2}} \left(\frac{1}{|x_1||x_2|} \int_{I_{k_1,k_2}} |b(x_1,x_2)-b_{k_1,k_2}| \right. \\
&\left. \times |f(y_1,y_2)|dy_1dy_2 \right)^p dx_1dx_2 \\
&+ C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} \int_{C_{k_1,k_2}} \left(\frac{1}{|x_1||x_2|} \int_{I_{k_1,k_2}} |b(y_1,y_2)-b_{k_1,k_2}| \right. \\
&\left. \times |f(y_1,y_2)|dy_1dy_2 \right)^p dx_1dx_2 \\
&= I + J.
\end{aligned}
$$

The term I can be handled using Hölder's inequality as follows

$$
\begin{aligned}
I &\leq C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} 2^{-(k_1+k_2)p} \int_{C_{k_1,k_2}} |b(x_1,x_2)-b_{k_1,k_2}|^p dx_1dx_2 \\
&\times \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \int_{C_{j_1,j_2}} |f(y_1,y_2)|dy_1dy_2 \right)^p \\
&\leq C \, \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{(\lambda-1)p+1} \\
&\times \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \left[\int_{I_{j_1,j_2}} |f(y_1,y_2)|^p dy_1dy_2 \right]^{1/p} |I_{j_1,j_2}|^{1/p'} \right)^p \\
&\leq C \, \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \, \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{(\lambda-1)p+1} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} |I_{j_1,j_2}|^{\lambda+1} \right)^p \\
&\leq C \, \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \, \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{1+2\lambda p} \\
&\leq C \, \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \, \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} \, |I_{K_1,K_2}|^{1+2\lambda p},
\end{aligned}
$$

where the last inequality holds since $2p\lambda > -1$.

To estimate the term J we will use Lemma 2.2 and the fact that $\max\{|I_{k_1,k_2}|^\lambda, |I_{k_1,j_2}|^\lambda, |I_{j_1,j_2}|^\lambda\} = |I_{j_1,j_2}|^\lambda$ when $k_i > j_i$, $i = 1,2$ as follows

$$
\begin{aligned}
J &\leq C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} 2^{-(k_1+k_2)p} \int_{C_{k_1,k_2}} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \int_{C_{j_1,j_2}} |b(y_1,y_2)-b_{k_1,k_2}| \right. \\
&\left. \times |f(y_1,y_2)|dy_1dy_2 \right)^p dx_1dx_2
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} 2^{-(k_1+k_2)p} \int_{C_{k_1,k_2}} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \int_{I_{j_1,j_2}} |b(y_1,y_2) - b_{k_1,k_2}| \right.\\
&\left. \times |f(y_1,y_2)| dy_1 dy_2 \right)^p dx_1 dx_2 \\
&\leq C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} 2^{-(k_1+k_2)p} \int_{C_{k_1,k_2}} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \int_{I_{j_1,j_2}} |b(y_1,y_2) - b_{j_1,j_2}| \right.\\
&\left. \times |f(y_1,y_2)| dy_1 dy_2 \right)^p dx_1 dx_2 \\
&+ C \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} 2^{-(k_1+k_2)p} \int_{C_{k_1,k_2}} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \int_{I_{j_1,j_2}} |I_{j_1,j_2}|^{\lambda} \right.\\
&\left. \times |f(y_1,y_2)| dy_1 dy_2 \right)^p dx_1 dx_2 \\
&= J_1 + J_2.
\end{aligned}
$$

Again by Hölder's inequality we obtain

$$
\begin{aligned}
J_1 &\leq C \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{-p} \int_{C_{k_1,k_2}} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \left[\int_{I_{j_1,j_2}} |b(y_1,y_2) - b_{j_1,j_2}|^{p'} \right. \right.\\
&\left. \left. \times dy_1 dy_2 \right]^{1/p'} \left[\int_{I_{j_1,j_2}} |f(y_1,y_2)|^p dy_1 dy_2 \right]^{1/p} \right)^p dx_1 dx_2 \\
&\leq C \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{-p} \int_{C_{k_1,k_2}} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} |I_{j_1,j_2}|^{\lambda+1/p'} \right.\\
&\left. \times \left[\int_{I_{j_1,j_2}} |b(y_1,y_2) - b_{j_1,j_2}|^p dy_1 dy_2 \right]^{1/p} \right)^p dx_1 dx_2 \\
&\leq C \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{-p+1} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} |I_{j_1,j_2}|^{2\lambda+1} \right)^p \\
&\leq C \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{1+2\lambda p} \\
&\leq C \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} |I_{K_1,K_2}|^{1+2\lambda p}.
\end{aligned}
$$

Finally, J_2 can be estimated in the following way

$$J_2 \leq C \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} 2^{-(k_1+k_2)p} \int_{I_{k_1,k_2}} \left(\sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} |I_{j_1,j_2}|^{\lambda+1/p'} \right.$$

$$\left. \times \left[\int_{I_{j_1,j_2}} |f(y_1,y_2)|^p dy_1 dy_2 \right]^{1/p} \right)^p dx_1 dx_2$$

$$\leq C \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{-p} \int_{I_{k_1,k_2}} \left(|I_{k_1,k_2}|^{2\lambda+1}\right)^p dx_1 dx_2$$

$$\leq C \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} \sum_{k_1=-\infty}^{K_1} \sum_{k_2=-\infty}^{K_2} |I_{k_1,k_2}|^{1+2\lambda p}$$

$$\leq C \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p,\lambda}} \|f\|^p_{\dot{\mathcal{B}}^{p,\lambda}} |I_{K_1,K_2}|^{1+2\lambda p},$$

which completes the proof of Theorem 2.3. □

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