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A CURIOUS IDENTITY AND ITS APPLICATIONS TO PARTITIONS WITH BOUNDED PART DIFFERENCES

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Abstract. In this note, we present a curious q-series identity with applications to certain partitions with bounded part differences.

1. Introduction

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Recently, motivated by the work of Andrews, Beck and Robbins [**2**], Breuer and Kronholm [**3**] obtained the generating function of partitions where the difference between largest and smallest parts is at most a fixed positive integer t,

$$\sum_{n\geq 1} p_t(n)q^n = \frac{1}{1-q^t}\left(\frac{1}{(q;q)_t}-1\right), \tag{1.1}$$

where $p_t(n)$ denotes the number of such partitions of n. Here and in what follows, we use the standard q-series notation

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad \text{for } |q|<1.$$

Subsequently, the author and Yee [**4, 5**] considered an overpartition analogue of Breuer and Kronholm's result. Here an *overpartition* of n is a partition of n where the first occurrence of each distinct part may be overlined. Let $g_t(m,n)$ count the number of overpartitions of n in which there are exactly m overlined parts, the difference between largest and smallest parts is at most t, and if the difference between largest and smallest parts is exactly t, then the largest parts cannot be overlined. The author and Yee proved

$$\sum_{n\geq 1}\sum_{m\geq 0} g_t(m,n)z^m q^n = \frac{1}{1-q^t}\left(\frac{(-zq;q)_t}{(q;q)_t}-1\right). \tag{1.2}$$

Suggested by George E. Andrews, it is also natural to study other types of partitions with bounded part differences. Let $pd_t(n)$ (resp. $po_t(n)$) count the number of partitions of n in which all parts are distinct (resp. odd) and the difference between largest and smallest parts is at most t.

Theorem 1.1. *We have*

$$\sum_{n\geq 1} pd_t(n)q^n = \frac{1}{1-q^{t+1}}\left((-q;q)_{t+1}-1\right), \tag{1.3}$$

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and

$$\sum_{n\geq 1} po_{2t}(n)q^n = \frac{1}{1-q^{2t}}\left(\frac{1}{(q;q^2)_t}-1\right). \tag{1.4}$$

Noting that (1.1)–(1.4) have the same flavor, we therefore want to seek for a unified proof of these generating function identities.

Let t be a fixed positive integer. Assume that α, β, q are complex variables with $|q|<1$, $q\neq 0$, $\alpha\neq\beta q$ and $(\beta q;q)_t\neq 0$. We define the following sum

$$S(\alpha,\beta;q;t) := \sum_{r\geq 1}\frac{(1-\alpha q^r)(1-\alpha q^{r+1})\cdots(1-\alpha q^{r+t-2})}{(1-\beta q^r)(1-\beta q^{r+1})\cdots(1-\beta q^{r+t})}q^r. \tag{1.5}$$

The following curious identity provides such a unified approach.

Theorem 1.2. *We have*

$$S(\alpha,\beta;q;t) = \frac{q}{(\beta q-\alpha)(1-q^t)}\left(\frac{(\alpha;q)_t}{(\beta q;q)_t}-1\right). \tag{1.6}$$

2. Proof of Theorem 1.2

Let

$${}_{r+1}\phi_r\left(\begin{matrix}a_0,a_1,a_2\ldots,a_r\\ b_1,b_2,\ldots,b_r\end{matrix};q,z\right) := \sum_{n\geq 0}\frac{(a_0;q)_n(a_1;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_r;q)_n}z^n.$$

The following two lemmas are useful in our proof.

Lemma 2.1 (First q-Chu–Vandermonde Sum [**1**, Eq. (17.6.2)])**.** *We have*

$${}_2\phi_1\left(\begin{matrix}a,q^{-n}\\ c\end{matrix};q,\frac{cq^n}{a}\right) = \frac{(c/a;q)_n}{(c;q)_n}. \tag{2.1}$$

Lemma 2.2 (q-Analogue of the Kummer–Thomae–Whipple Transformation [**6**, p. 72, Eq. (3.2.7)])**.** *We have*

$${}_3\phi_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix};q,\frac{de}{abc}\right) = \frac{(e/a;q)_\infty(de/bc;q)_\infty}{(e;q)_\infty(de/abc;q)_\infty}{}_3\phi_2\left(\begin{matrix}a,d/b,d/c\\ d,de/bc\end{matrix};q,\frac{e}{a}\right). \tag{2.2}$$

Proof of Theorem 1.2. We have

$$
\begin{aligned}
&S(\alpha,\beta;q;t) \qquad\qquad (2.3)\\
&=\sum_{r\geq 1}\frac{(1-\alpha q^r)(1-\alpha q^{r+1})\cdots(1-\alpha q^{r+t-2})}{(1-\beta q^r)(1-\beta q^{r+1})\cdots(1-\beta q^{r+t})}q^r\\
&=\sum_{r\geq 1}\frac{(\alpha;q)_{r+t-1}(\beta;q)_r}{(\alpha;q)_r(\beta;q)_{r+t+1}}q^r\\
&=\sum_{r\geq 0}\frac{(\alpha;q)_{r+t}(\beta;q)_{r+1}}{(\alpha;q)_{r+1}(\beta;q)_{r+t+2}}q^{r+1}\\
&=\frac{q(1-\beta)(\alpha;q)_t}{(1-\alpha)(\beta;q)_{t+2}}\sum_{r\geq 0}\frac{(q;q)_r(\beta q;q)_r(\alpha q^t;q)_r}{(q;q)_r(\alpha q;q)_r(\beta q^{t+2};q)_r}q^r\\
&=\frac{q(\alpha q;q)_{t-1}}{(\beta q;q)_{t+1}}\,{}_3\phi_2\left(\begin{matrix}q,\beta q,\alpha q^t\\ \alpha q,\beta q^{t+2}\end{matrix};q,q\right)\\
&=\frac{q(\alpha q;q)_{t-1}}{(\beta q;q)_{t+1}}\frac{(\beta q^{t+1};q)_\infty(q^2;q)_\infty}{(\beta q^{t+2};q)_\infty(q;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}q,\alpha/\beta,q^{1-t}\\ \alpha q,q^2\end{matrix};q,\beta q^{t+1}\right) \quad \text{(by Eq. (2.2))}\\
&=\frac{q(\alpha q;q)_{t-1}}{(1-q)(\beta q;q)_t}\sum_{r\geq 0}\frac{(\alpha/\beta;q)_r(q^{1-t};q)_r}{(\alpha q;q)_r(q^2;q)_r}\left(\beta q^{t+1}\right)^r\\
&=\frac{q(\alpha q;q)_{t-1}}{(1-q)(\beta q;q)_t}\frac{(1-\alpha)(1-q)}{\beta q^{t+1}\left(1-\frac{\alpha}{\beta q}\right)(1-q^{-t})}\sum_{r\geq 0}\frac{\left(\frac{\alpha}{\beta q};q\right)_{r+1}(q^{-t};q)_{r+1}}{(\alpha;q)_{r+1}(q;q)_{r+1}}\left(\beta q^{t+1}\right)^{r+1}\\
&=\frac{q}{(\beta q-\alpha)(q^t-1)}\frac{(\alpha;q)_t}{(\beta q;q)_t}\left({}_2\phi_1\left(\begin{matrix}\frac{\alpha}{\beta q},q^{-t}\\ \alpha\end{matrix};q,\beta q^{t+1}\right)-1\right)\\
&=\frac{q}{(\beta q-\alpha)(q^t-1)}\frac{(\alpha;q)_t}{(\beta q;q)_t}\left(\frac{(\beta q;q)_t}{(\alpha;q)_t}-1\right) \quad \text{(by Eq. (2.1))}\\
&=\frac{q}{(\beta q-\alpha)(1-q^t)}\left(\frac{(\alpha;q)_t}{(\beta q;q)_t}-1\right).
\end{aligned}
$$

□

3. Applications

We now show how Theorem 1.2 may prove (1.1)–(1.4).

At first, we prove the two new identities (1.3) and (1.4). Note that the generating function for partitions counted by $pd_t(n)$ with smallest part equal to r is

$$q^r(1+q^{r+1})(1+q^{r+2})\cdots(1+q^{r+t}).$$

Hence

$$\sum_{n\geq 1}pd_t(n)q^n=\sum_{r\geq 1}(1+q^{r+1})(1+q^{r+2})\cdots(1+q^{r+t})q^r=S(-q,0;q;t+1).$$

It follows by Theorem 1.2 that

$$\sum_{n\geq 1}pd_t(n)q^n=S(-q,0;q;t+1)=\frac{1}{1-q^{t+1}}\left((-q;q)_{t+1}-1\right).$$

To see (1.4), one readily verifies that the generating function for partitions counted by $po_{2t}(n)$ with smallest part equal to $2r-1$ is

$$\frac{q^{2r-1}}{(1-q^{2r-1})(1-q^{2r+1})\cdots(1-q^{2r+2t-1})}.$$

Hence

$$\begin{aligned}\sum_{n\geq 1} po_{2t}(n)q^n &= \sum_{r\geq 1}\frac{1}{(1-q^{2r-1})(1-q^{2r+1})\cdots(1-q^{2r+2t-1})}q^{2r-1}\\ &= q^{-1}S(0,q^{-1};q^2;t) = \frac{1}{1-q^{2t}}\left(\frac{1}{(q;q^2)_t}-1\right).\end{aligned}$$

Here the last equality follows again from Theorem 1.2. We remark that for any positive integer t, $po_{2t}(n) = po_{2t+1}(n)$ since only odd parts are allowed in this case. Hence it suffices to consider merely the generating function of $po_{2t}(n)$.

The proofs of (1.1) and (1.2) are similar. We omit the details here.

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