

ON BOUNDED OPERATORS ON A BANACH SPACE AND DERIVATIONS ON PROJECTIVE TENSOR ALGEBRAS

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Abstract. We consider a Banach space \mathcal{X} endowed with a shrinking basis. Then we describe the structure of derivations on the projective Banach algebra $\mathcal{X} \hat{\otimes} \mathcal{X}^*$ which are induced by bounded linear operators on \mathcal{X} . If the underlying space has no such basis our results are no longer applicable. However, if \mathcal{X} is the space of absolutely convergent complex series we establish a relationship between bounded derivations on $l^1(\mathcal{X}^*)$ and bounded derivations on $\mathcal{X} \hat{\otimes} \mathcal{X}^*$.

1. Introduction

Given a complex Banach space \mathcal{X} let us consider the projective Banach space $\mathcal{U} \triangleq \mathcal{X} \hat{\otimes} \mathcal{X}^*$. If for $x_1, x_2 \in \mathcal{X}$, $x_1^*, x_2^* \in \mathcal{X}^*$ it is routinely checked that the equation $(x_1 \otimes x_1^*)(x_2 \otimes x_2^*) \triangleq \langle x_2, x_1^* \rangle x_1 \otimes x_2^*$ defines products turning \mathcal{U} into a Banach algebra. Let $D : \mathcal{B}(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{U})$ be the map $D(T) \triangleq T \otimes Id_{\mathcal{X}^*} - Id_{\mathcal{X}} \otimes T^*$. Then D is well defined and $\|D(T)\| \leq 2\|T\|$ for all $T \in \mathcal{B}(\mathcal{X})$, i.e. $\|D\| \leq 2$. Further, it is easy to see that $D(T)$ is a derivation on \mathcal{U} for all $T \in \mathcal{B}(\mathcal{X})$ (cf. [5], Example 2). Consequently, $D : \mathcal{B}(\mathcal{X}) \hookrightarrow \mathcal{Z}^1(\mathcal{U})$, where $\mathcal{Z}^1(\mathcal{U})$ denotes the Banach subspace of $\mathcal{B}(\mathcal{X})$ of bounded derivations on \mathcal{X} . As usual, let $\mathcal{N}^1(\mathcal{U})$ denote the class of inner derivations ad_u on \mathcal{U} , where $u \in \mathcal{U}$ and $ad_u(v) = uv - vu$ for all $v \in \mathcal{U}$. For instance, if $x_0 \in \mathcal{X}$ and $x_0^* \in \mathcal{X}^*$ let $x_0 \odot x_0^* \in \mathcal{F}(\mathcal{X})$ be the bounded finite rank operator so that $\langle x, x_0 \odot x_0^* \rangle \triangleq \langle x, x_0^* \rangle x_0$ for all $x \in \mathcal{X}$. Then

$$D(x_0 \odot x_0^*) = (x_0 \odot x_0^*) \otimes Id_{\mathcal{X}^*} - Id_{\mathcal{X}} \otimes (x_0^* \odot \hat{x}_0), \quad (1)$$

where $\hat{x}_0 \in \mathcal{X}^{**}$ is the evaluation map $\langle x^*, \hat{x}_0 \rangle \triangleq \langle x_0, x^* \rangle$ for each $x^* \in \mathcal{X}^*$. From (1) it is easy to verify that $D(x_0 \odot x_0^*) = ad_{x_0 \odot x_0^*}$. So $D[\mathcal{F}(\mathcal{X})] \subseteq \mathcal{N}^1(\mathcal{U})$.

If $T \in \mathcal{B}(\mathcal{X})$ we are concerned about the structure of $D(T)$ as well as its realization as an inner derivation on \mathcal{U} . In Th. 1 we will establish the precise relation between an inner derivation ad_u on \mathcal{U} of the type $D(T)$, the implemented vector u and the matrix representation of T with respect to a fixed shrinking basis of \mathcal{X} . For instance, this result applies to the spaces c_0 and l^p , $p > 1$, endowed with their natural basis, and also to the space $L^p([0, 1])$, $p > 1$, provided with the Haar basis. If \mathcal{X} has no shrinking basis the above result is in general no longer true. However, in Th. 3 we will give a complete solution of the problem when $\mathcal{X} = l^1(\mathbb{N})$.

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2. When X has a Shrinking Basis

Let us suppose that \mathcal{X} is endowed with a shrinking basis $\{x_n\}_{n=1}^\infty$ and an associated sequence of coefficient functionals $\{x_n^*\}_{n=1}^\infty$. Thus $\{x_n^*\}_{n=1}^\infty$ is a basis of \mathcal{X}^* and $\langle x_n, x_m^* \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{N}$. Hence, the system of all basic tensor products $x_n \otimes x_m^*$ is a basis $\{z_n\}_{n=1}^\infty$ of \mathcal{U} when arranged into a single sequence as follows: $z_1 \triangleq x_1 \otimes y_1$, and if $k, n \in \mathbb{N}$ let

$$z_k = \begin{cases} x_i \otimes y_{n+1} & \text{if } k = n^2 + i, i = 1, 2, \dots, n+1, \\ x_{n+1} \otimes y_{n+1-i} & \text{if } k = n^2 + n + 1 + i, i = 1, 2, \dots, n. \end{cases}$$

This basis has an associated sequence of coefficient functionals $\{z_k^*\}_{k=1}^\infty$ within $\mathcal{X}^* \hat{\otimes} \mathcal{X} \hookrightarrow \left(\mathcal{X} \hat{\otimes} \mathcal{X}^* \right)^*$, which can be constructed from the tensor products $x_n^* \otimes x_m$ by a similar arrangement [2]. We will say that $\{z_n\}_{n=1}^\infty$ is the tensor product basis of \mathcal{U} associated to the shrinking basis $\{x_n\}_{n=1}^\infty$.

Theorem 1. *Let \mathcal{X} be a Banach space endowed with a shrinking basis $\{x_n\}_{n=1}^\infty$ and an associated sequence of coefficient functionals $\{x_n^*\}_{n=1}^\infty$. Let $T \in \mathcal{B}(\mathcal{X})$ so that $D(T) \in \mathcal{N}^1(\mathcal{U})$ and let $u \in \mathcal{U}$ such that $D(T) = ad_u$. If $u = \sum_{m=1}^\infty u_m z_m$ and $n, m \in \mathbb{N}$ then*

$$\langle T(x_n), x_m^* \rangle = \begin{cases} u_{m^2-n+1} & \text{if } n < m, \\ 0 & \text{if } n = m, \\ u_{(n-1)^2+m} & \text{if } n > m. \end{cases}$$

Proof. If $m \in \mathbb{N}$ there is a unique $n \in \mathbb{N}$ so that $(n-1)^2 < m \leq n^2$. It is straightforward to see that $z_m = x_{\sigma_1(m)} \otimes x_{\sigma_2(m)}^*$, where $\sigma(m) = (\sigma_1(m), \sigma_2(m))$ and $\sigma : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ is the bijective function defined as

$$\sigma(m) = \begin{cases} (m - (n-1)^2, n) & \text{if } (n-1)^2 + 1 \leq m \leq (n-1)^2 + n, \\ (n, n^2 - m + 1) & \text{if } (n-1)^2 + n \leq m \leq n^2. \end{cases}$$

By [1], Prop. 6, p. 203, given $r, s \in \mathbb{N}$ we have

$$\begin{aligned} ad_u(x_r \otimes x_s^*) &= \sum_{m \in \sigma_2^{-1}(\{r\})} u_m (x_{\sigma_1(m)} \otimes x_s^*) - \sum_{m \in \sigma_1^{-1}(s)} u_m (x_r \otimes x_{\sigma_2(m)}^*) \quad (2) \\ &= \sum_{i=1}^r u_{(r-1)^2+i} (x_i \otimes x_s^*) + \sum_{i=r+1}^\infty u_{i^2-r+1} (x_i \otimes x_s^*) \\ &\quad - \sum_{i=1}^s u_{s^2-i+1} (x_r \otimes x_i^*) - \sum_{i=s}^\infty u_{i^2+s} (x_r \otimes x_{i+1}^*). \end{aligned}$$

Indeed,

$$D(T)(x_r \otimes x_s^*) = \sum_{i=1}^\infty [\langle T(x_r), x_i^* \rangle (x_i \otimes x_s^*) - \langle T(x_i), x_s^* \rangle (x_r \otimes x_i^*)]. \quad (3)$$

In particular, if $r = s$ we point out that the r th-summand of the first sum and the s th-summand of the third sum of (2) are equal. Hence, the four sums of (2) involve mutually disjoint subsets of the basis $\{z_n\}_{n=1}^\infty$. The result now follows comparing the coefficients of (2) and (3). \square

Example 2. Let $\mathcal{X} \triangleq c_0(\mathbb{N})$ be the usual Banach subspace of $l^\infty(\mathbb{N})$ of complex sequences on \mathbb{N} that converge to zero. If $n \in \mathbb{N}$ and $x_n \triangleq \{\delta_{n,m}\}_{m \in \mathbb{N}}$ then $\{x_n\}_{n \in \mathbb{N}}$ is a shrinking basis of \mathcal{X} . Indeed, \mathcal{X}^* is isometrically isomorphic to $l^1(\mathbb{N})$ and $\{x_n^*\}_{n \in \mathbb{N}}$ is identified with the standard basis of $l^1(\mathbb{N})$. Any $T \in \mathcal{B}(\mathcal{X})$ is determined with respect to the basis $\{x_n\}_{n \in \mathbb{N}}$ by a unique complex matrix $\{a_{m,n}\}_{n,m \in \mathbb{N}}$ whose columns belong to \mathcal{X} , their rows are uniformly bounded in $l^1(\mathbb{N})$ and $T(z) = \{\sum_{n=1}^{\infty} a_{m,n} z_n\}_{m=1}^{\infty}$ if $z = \{z_n\}_{n \in \mathbb{N}}$ in \mathcal{X} . Let $S \in \mathcal{B}(\mathcal{X})$ be the shift operator, i.e. $S(\{\alpha_1, \alpha_2, \dots\}) \triangleq \{0, \alpha_1, \alpha_2, \dots\}$ if $\alpha \in \mathcal{X}$. Then $D_S \notin \mathcal{N}^1(c_0(\mathbb{N}) \widehat{\otimes} l^1(\mathbb{N}))$. For, if $D_S = ad_u$ for some $u \in c_0(\mathbb{N}) \widehat{\otimes} l^1(\mathbb{N})$ let us write $u = \sum_{m=1}^{\infty} u_m z_m$, where $\{z_n\}_{n=1}^{\infty}$ is the tensor product basis associated to the standard basis of \mathcal{X} . By Th. 1 we see that $u_{n^2+n+2} = 1$ for all $n \in \mathbb{N}$, which contradicts the fact that $u_m \rightarrow 0$. Now, let $T \in \mathcal{B}(\mathcal{X})$, $T(\alpha) \triangleq \{\sum_{n=1}^{\infty} \alpha_{m+n-1}/2^n\}_{m \in \mathbb{N}}$ for $\alpha \in \mathcal{X}$. Then $D(T) = ad_v$, where

$$v \triangleq \sum_{n=1}^{\infty} \left(\sum_{m=1}^n 2^{-n+m-1} x_k \right) \otimes x_n^*.$$

3. Advances with $X = l^1(\mathbb{N})$

Theorem 3. Let $T \in \mathcal{B}(l^1(\mathbb{N}))$ be performed by the infinite complex matrix $a \triangleq \{a_{n,m}\}_{n,m \in \mathbb{N}}$ with respect to the usual basis $\{e_n\}_{n \in \mathbb{N}}$, where $e_n \triangleq \{\delta_{n,m}\}_{m \in \mathbb{N}}$ if $n \in \mathbb{N}$, i.e. its rows are uniformly bounded in $l^1(\mathbb{N})$ and if $z = \{z_n\}_{n \in \mathbb{N}}$ belongs to $l^1(\mathbb{N})$ then $T(z) = \{\sum_{n=1}^{\infty} a_{m,n} z_n\}_{m=1}^{\infty}$ (cf. [3]; [4], Th. 2.13(ii)). Then,

(i): The linear subspace \mathcal{S} of $\mathcal{U} \triangleq l^1(\mathbb{N}) \widehat{\otimes} l^\infty(\mathbb{N})$ generated by the set

$$\{e_n \otimes x^* : n \in \mathbb{N}, x^* \in l^\infty(\mathbb{N})\}$$

is dense in \mathcal{U} .

(ii): Let $s = \sum_{n=1}^{\infty} e_n \otimes x_n^*$, with $\{x_n^*\}_{n=1}^{\infty} \in l^1(l^\infty(\mathbb{N}))$. If $s = 0$ then $x_n^* = 0$ for all $n \in \mathbb{N}$.

(iii): Let $\delta(T) \in \mathcal{B}(l^1(l^\infty(\mathbb{N})))$,

$$\delta(T)(x^*) \triangleq \left\{ \left\{ \sum_{n=1}^{\infty} (a_{m,n} x_{n,k}^* - a_{n,k} x_{m,n}^*) \right\}_{k=1}^{\infty} \right\}_{m=1}^{\infty}, \quad (4)$$

where $x^* = \{x_n^*\}_{n \in \mathbb{N}}$, $x^* \in l^1(l^\infty(\mathbb{N}))$. Then $\delta(T)$ is well defined and δ becomes a bounded linear operator on $l^1(\mathbb{N})$ with values in $\mathcal{B}(l^1(l^\infty(\mathbb{N})))$.

(iv): If $x^*, y^* \in l^1(l^\infty(\mathbb{N}))$ let $x^* \cdot y^* \triangleq \{\sum_{n=1}^{\infty} x_{m,n}^* y_n^*\}_{m=1}^{\infty}$. Thus $l^1(l^\infty(\mathbb{N}))$ is an associative Banach algebra.

(v): Given $x^* \in l^1(l^\infty(\mathbb{N}))$ let $s_{x^*} \triangleq \sum_{n=1}^{\infty} e_n \otimes x_n^*$ in \mathcal{U} . Then

$$D_T(s_{x^*}) = \sum_{m=1}^{\infty} e_m \otimes \delta_m(T)(x^*). \quad (5)$$

(vi): $\delta(T) \in \mathcal{Z}^1(l^1(l^\infty(\mathbb{N})))$.

(vii): If $z^* \in l^1(l^\infty(\mathbb{N}))$, $ad_{z^*} = 0$ if and only if $z^* = 0$.

(viii): $\delta(T) \in \mathcal{N}^1(l^1(l^\infty(\mathbb{N})))$ if and only if $a^* \triangleq \{\{a_{m,n}\}_{n=1}^{\infty}\}_{m=1}^{\infty} \in l^1(l^\infty(\mathbb{N}))$.

Proof. (i) As the set $l^1(\mathbb{N}) \otimes l^\infty(\mathbb{N})$ is dense in \mathcal{U} it will suffice to prove that any basic tensor product can be approximate by elements of \mathcal{S} . For, given

$x \otimes x^* \in l^1(\mathbb{N}) \otimes l^\infty(\mathbb{N})$ then can be written as $x = \sum_{n=1}^\infty x_n e_n$ for a unique complex sequence $\{x_n\}_{n=1}^\infty$. By the continuity of the mapping $x \rightarrow x \otimes x^*$ from $l^1(\mathbb{N})$ into \mathcal{U} and using the bilinearity of the tensor product we have

$$x \otimes x^* = \left(\sum_{n=1}^\infty x_n e_n \right) \otimes x^* = \sum_{n=1}^\infty e_n \otimes (x_n x^*).$$

(ii) Given $\beta \in l^\infty(\mathbb{N})$ and a bijection $j : \mathbb{N} \rightarrow \mathbb{N}$ let us write

$$B_{\beta,j}(x, x^*) \triangleq \sum_{n=1}^\infty \beta_n x_n x_{j(n)}^*$$

for $x \in l^1(\mathbb{N})$ and $x^* \in l^\infty(\mathbb{N})$. Hence $B_{\beta,j}$ is a bounded bilinear form on $l^1(\mathbb{N}) \times l^\infty(\mathbb{N})$ and there is a unique $\tilde{B}_{\beta,j} \in \mathcal{U}^*$ so that

$$0 = \tilde{B}_{\beta,j}(s) = \sum_{n=1}^\infty B_{\beta,j}(e_n, x_n^*) = \sum_{n=1}^\infty \beta_n x_{j(n)}^*.$$

Now the conclusion is immediate.

(iii) By the isometric isomorphism $l^1(\mathbb{N})^* \approx l^\infty(\mathbb{N})$ it is straightforward to see that the columns of the matrix $\{a_{n,m}\}_{n,m \in \mathbb{N}}$ are uniformly bounded in $l^1(\mathbb{N})$ and that $T^*(x^*) = \{\sum_{n=1}^\infty a_{n,m} x_n^*\}_{m=1}^\infty$ for all $x^* \in l^\infty(\mathbb{N})$. Moreover,

$$\|T\| = \sup_{n \in \mathbb{N}} \sum_{m=1}^\infty |a_{n,m}| = \sup_{m \in \mathbb{N}} \sum_{n=1}^\infty |a_{n,m}|.$$

Let $x^* \in l^1(l^\infty(\mathbb{N}))$ and let $m, k \in \mathbb{N}$. Then

$$\begin{aligned} |\delta_m(T)(x^*)(k)| &= \left| \sum_{n=1}^\infty (a_{m,n} x_{n,k}^* - a_{n,k} x_{m,n}^*) \right| \\ &\leq \sum_{n=1}^\infty |a_{m,n}| \|x_n^*\|_\infty + \|T\| \|x_m^*\|_\infty, \end{aligned}$$

i.e. $\delta_m(T)(x^*) \in l^\infty(\mathbb{N})$ and

$$\|\delta_m(T)(x^*)\|_\infty \leq \sum_{n=1}^\infty |a_{m,n}| \|x_n^*\|_\infty + \|T\| \|x_m^*\|_\infty.$$

Further, if $M \in \mathbb{N}$ then

$$\begin{aligned} \sum_{m=1}^M \|\delta_m(T)(x^*)\|_\infty &\leq \sum_{n=1}^\infty \|x_n^*\|_\infty \sum_{m=1}^M |a_{m,n}| + \|T\| \|x^*\|_{l^1(l^\infty(\mathbb{N}))} \\ &\leq 2 \|T\| \|x^*\|_{l^1(l^\infty(\mathbb{N}))}. \end{aligned}$$

Letting $M \rightarrow \infty$ we see that $\delta(T)(x^*) \in l^1(l^\infty(\mathbb{N}))$ and

$$\|\delta(T)(x^*)\|_{l^1(l^\infty(\mathbb{N}))} \leq 2 \|T\| \|x^*\|_{l^1(l^\infty(\mathbb{N}))}.$$

Indeed, $\delta(T) \in \mathcal{B}(l^1(l^\infty(\mathbb{N})))$ and $\|\delta(T)\| \leq 2 \|T\|$. Incidentally, δ also becomes a bounded linear operator on $l^1(\mathbb{N})$ with values in $\mathcal{B}(l^1(l^\infty(\mathbb{N})))$.

(iv) It is straightforward.

(v) We have

$$\begin{aligned} D_T(s_{x^*}) &= \sum_{n=1}^{\infty} [T(e_n) \otimes x_n^* - e_n \otimes T^*(x_n^*)] \\ &= \sum_{n=1}^{\infty} \left[\left(\sum_{m=1}^{\infty} a_{m,n} e_m \right) \otimes x_n^* - e_n \otimes \left\{ \sum_{m=1}^{\infty} a_{m,k} x_{n,m}^* \right\}_{k=1}^{\infty} \right]. \end{aligned} \quad (6)$$

If $N, M \in \mathbb{N}$ we see that

$$\sum_{n=1}^N \sum_{m=1}^M |a_{m,n}| \|x_n^*\|_{\infty} \leq \|T\| \sum_{n=1}^{\infty} \|x_n^*\|_{\infty} < +\infty,$$

i.e. $\{a_{m,n}(e_m \otimes x_n^*)\}_{n,m \in \mathbb{N}}$ becomes unconditionally convergent in \mathcal{U} . Hence, by (6) we can write

$$\begin{aligned} D_T(s_{x^*}) &= \sum_{m=1}^{\infty} e_m \otimes \left[\sum_{n=1}^{\infty} a_{m,n} x_n^* - \left\{ \sum_{n=1}^{\infty} a_{n,k} x_{m,n}^* \right\}_{k=1}^{\infty} \right] \\ &= \sum_{m=1}^{\infty} e_m \otimes \left\{ \sum_{n=1}^{\infty} (a_{m,n} x_{n,k}^* - a_{n,k} x_{m,n}^*) \right\}_{k=1}^{\infty} \\ &= \sum_{m=1}^{\infty} e_m \otimes \delta_m(T)(x^*). \end{aligned}$$

(vi) Let $x^*, y^* \in l^1(l^{\infty}(\mathbb{N}))$. Then

$$\begin{aligned} \sum_{m=1}^{\infty} e_m \otimes \delta_m(T)(x^* \cdot y^*) &= D_T(s_{x^*} \cdot s_{y^*}) \\ &= D_T(s_{x^*}) \cdot s_{y^*} + s_{x^*} \cdot D_T(y^*) \\ &= \sum_{m=1}^{\infty} e_m \otimes (\delta_m(T)(x^*) \cdot y^* + x^* \cdot \delta_m(T)(y^*)) \end{aligned}$$

and by (ii) $\delta(T) \in \mathcal{Z}^1(l^1(l^{\infty}(\mathbb{N})))$.

(vii) Let $z^* \in l^1(l^{\infty}(\mathbb{N}))$ be so that $ad_{z^*} = 0$. If $n \in \mathbb{N}$ we have

$$ad_{z^*}(\{0, \dots, 0, e_n^*, 0, \dots\}) = \{z_{m,n}^* e_n^*\}_{m=1}^{\infty} - \{0, \dots, 0, z_n^*, 0, \dots\} \quad (7)$$

and $z_{m,n}^* = 0$ whenever $m \neq n$ in \mathbb{N} . Since $x^* \cdot z^* = z^* \cdot x^*$ for all $x^* \in l^1(l^{\infty}(\mathbb{N}))$ then $z_{m,m}^* x_{m,n}^* = x_{m,n}^* z_{n,n}^*$ if $m, n \in \mathbb{N}$. Consequently $z_{n,n}^* = z_{m,m}^*$ and as $\|z_m^*\|_{\infty} \rightarrow 0$ then $z^* = 0$.

(viii) Let us suppose that $\delta(T) \in \mathcal{N}^1(l^1(l^{\infty}(\mathbb{N})))$. By (vi) there is a unique $z^* \in l^1(l^{\infty}(\mathbb{N}))$ so that $\delta(T) = ad_{z^*}$. By (4) and (7) we deduce that $a_{m,n} = z_{m,n}^*$ if $m \neq n$ in \mathbb{N} . Indeed, if $x^* \in l^1(l^{\infty}(\mathbb{N}))$ and $m, k \in \mathbb{N}$ by (4) we see that

$$\delta_m(T)(x^*)_k = (a_{m,m} - a_{k,k}) x_{m,k}^* + \sum_{n \in \mathbb{N} - \{m\}} a_{m,n} x_{n,k}^* - \sum_{n \in \mathbb{N} - \{k\}} a_{n,k} x_{m,n}^*,$$

i.e. $\delta(T)(x^*)$ remains unchanged modifying the diagonal $\{a_{n,n}\}_{n=1}^{\infty}$ up to a constant. Thus, we can replace $a_{n,n} \leftrightarrow a_{n,n} - a_{1,1} + z_{1,1}^*$ for each $n \in \mathbb{N}$. Hence, it follows that $z_{m,m}^* - z_{k,k}^* = a_{m,m} - a_{k,k}$ for $m, k \in \mathbb{N}$. So, $a_{n,n} =$

$z_{n,n}^* - z_{1,1}^* + a_{1,1} = z_{n,n}^*$ for all $n \in \mathbb{N}$ and the condition is necessary. That it is also sufficient is immediate. \square

Corollary 4. *Let $D_T \in \mathcal{N}^1(\mathcal{U})$. Then it is implemented as an inner derivation by an element $u \in \mathcal{U}$ so that $u = \sum_{n=1}^{\infty} y_n \otimes y_n^*$ and*

$$\sum_{n=1}^{\infty} \sup_{m \in \mathbb{N}} \left| \sum_{k=1}^{\infty} y_{k,n} y_{k,m}^* \right| < +\infty.$$

Proof. We can assume that $\sum_{n=1}^{\infty} \|y_n\|_1 \|y_n^*\|_{\infty} < +\infty$. Further, we can assume that $\sum_{n=1}^{\infty} \|y_n^*\|_{\infty} < +\infty$ and that $\|y_n\|_1 \rightarrow 0$. If $x^* \in l^1(l^{\infty}(\mathbb{N}))$ and $m, n \in \mathbb{N}$ let us write $z_{m,n}^* \triangleq \langle y_n, x_m^* \rangle$ and $w_{m,n}^* \triangleq \sum_{k=1}^{\infty} y_{k,m} y_{k,n}^*$. If $m \in \mathbb{N}$ we have

$$\|z_m^*\|_{\infty} \leq \|x_m^*\|_{\infty} \sup_{n \in \mathbb{N}} \|y_n\|_1, \quad (8)$$

$$\|w_m^*\|_{\infty} = \sup_{n \in \mathbb{N}} |w_{m,n}^*| \leq \sum_{k=1}^{\infty} |y_{k,m}| \|y_k^*\|_{\infty}. \quad (9)$$

By (8), $z^* \in l^1(l^{\infty}(\mathbb{N}))$ and

$$\|z^*\|_{l^1(l^{\infty}(\mathbb{N}))} \leq \|x^*\|_{l^1(l^{\infty}(\mathbb{N}))} \sup_{n \in \mathbb{N}} \|y_n\|_1.$$

By (9), $w^* \in l^1(l^{\infty}(\mathbb{N}))$ and

$$\|w^*\|_{l^1(l^{\infty}(\mathbb{N}))} \leq \sum_{k=1}^{\infty} \|y_k\|_1 \|y_k^*\|_{\infty}.$$

Further,

$$\begin{aligned} \sum_{m,n,p \in \mathbb{N}} |y_{n,p} y_{n,m}^*| \|x_m^*\|_{\infty} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \|x_m^*\|_{\infty} \sum_{p=1}^{\infty} |y_{n,p} y_{n,m}^*| \\ &\leq \|x^*\|_{l^1(l^{\infty}(\mathbb{N}))} \sum_{n=1}^{\infty} \|y_n\|_1 \|y_n^*\|_{\infty} < +\infty, \end{aligned} \quad (10)$$

and

$$\sum_{m,n \in \mathbb{N}} |\langle y_n, x_m^* \rangle| \|y_n^*\|_{\infty} \leq \|x^*\|_{l^1(l^{\infty}(\mathbb{N}))} \sum_{n=1}^{\infty} \|y_n\|_1 \|y_n^*\|_{\infty} < +\infty. \quad (11)$$

By (9) and (11) we see that

$$ad_u(s_{x^*}) = \sum_{p=1}^{\infty} e_p \otimes (w^* \cdot x^* - z^* \cdot y^*)_p. \quad (12)$$

By (12), (5), Th. 3, (ii) and as $y^* \in l^1(l^\infty(\mathbb{N}))$ we have

$$\begin{aligned}
\delta(T)(x^*) &= w^* \cdot x^* - z^* \cdot y^* \\
&= \left\{ \sum_{m=1}^{\infty} w_{p,m}^* x_m^* - z_{p,m}^* y_m^* \right\}_{p=1}^{\infty} \\
&= \left\{ \left\{ \sum_{m=1}^{\infty} x_{m,k}^* \sum_{n=1}^{\infty} y_{n,p} y_{n,m}^* - \langle y_m, x_p^* \rangle y_{m,k}^* \right\}_{k=1}^{\infty} \right\}_{p=1}^{\infty} \\
&= \left\{ \left\{ \sum_{m=1}^{\infty} x_{m,k}^* \sum_{n=1}^{\infty} y_{n,p} y_{n,m}^* - y_{m,k}^* \sum_{n=1}^{\infty} y_{m,n} x_{p,n}^* \right\}_{k=1}^{\infty} \right\}_{p=1}^{\infty} \\
&= \left\{ \left\{ \sum_{m=1}^{\infty} x_{m,k}^* \sum_{n=1}^{\infty} y_{n,p} y_{n,m}^* - x_{p,m}^* \sum_{n=1}^{\infty} y_{n,m} y_{n,k}^* \right\}_{k=1}^{\infty} \right\}_{p=1}^{\infty} \\
&= \left\{ \left\{ \sum_{m=1}^{\infty} x_{m,k}^* w_{p,m}^* - x_{p,m}^* w_{m,k}^* \right\}_{k=1}^{\infty} \right\}_{p=1}^{\infty}.
\end{aligned}$$

Since $w^* \in l^1(l^\infty(\mathbb{N}))$ the claim follows by Th. 3. \square

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