

STURM-PICONE COMPARISON THEOREM FOR MATRIX SYSTEMS ON TIME SCALES

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A well-known Picone identity is extended and generalized to second-order dynamic matrix equations on arbitrary time scales. A comparison theorem is obtained in the spirit of the classical Sturm-Picone comparison theorem that extends known scalar results to matrix equations that include the linear homogeneous and inhomogeneous cases, and nonlinear unperturbed and perturbed cases.

1. INTRODUCTION

More than 170 years have past since STURM [18] published his famous comparison theorem for second order linear differential equations. There is reason to believe that in fact he first proved this result for difference equations (see REID [17]). The first nonlinear version of the Sturm-Picone identity and comparison theorem appeared in GRAEF and SPIKES [12] for second order ordinary differential equations. JAROS and KUSANO extended the identity to second order half-linear equations in [15]. Sturm type comparison theorems for higher order difference equations have appeared in [9] and [10], and a very general Sturm-Picone comparison theorem for higher order delay difference equations appeared in [11]. Here we are interested in such problems for equations on time scales.

A nonlinear Sturm-Picone comparison theorem for the scalar perturbed dynamic equation

$$(1.1) \quad (px^\Delta)^\Delta(t) + q(t, x^\sigma(t)) = r(t, x^\sigma(t)), \quad t \in \mathbb{T},$$

on time scales \mathbb{T} was obtained in BELINSKIY, GRAEF, and PETROVIĆ [4], where $x^\sigma(t) = (x \circ \sigma)(t)$. Recently, ZHANG and SUN [19] studied the linear Sturm-Picone

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comparison theorem for the scalar homogeneous dynamic equation

$$(px^\Delta)^\Delta(t) + q(t)x^\sigma(t) = 0, \quad t \in \mathbb{T};$$

note that in many ways the later paper [19] is a corollary to the earlier paper [4].

We seek to extend some of those comparison theorems to the linear homogeneous matrix dynamic equation on arbitrary time scales \mathbb{T} given by

$$(1.2) \quad (PX^\Delta)^\Delta(t) + Q(t)X^\sigma(t) = 0, \quad t \in \mathbb{T},$$

and the nonlinear perturbed dynamic equation

$$(1.3) \quad (PX^\Delta)^\Delta(t) + Q(t, X^\sigma(t))X^{\sigma\dagger}(t)X^\sigma(t) = R(t, X^\sigma(t))X^{\sigma\dagger}(t)X^\sigma(t), \quad t \in \mathbb{T},$$

where U^\dagger indicates the Moore-Penrose pseudoinverse of U , which always exists and is unique (see BEN-ISRAEL and GREVILLE [5, Theorem 1.5]). Note that (1.3) would then include as important special cases (1.1) and (1.2), and thus the continuous self-adjoint matrix differential equation

$$(PX')'(t) + Q(t)X(t) = 0, \quad t \in \mathbb{R},$$

(see REID [16]), and the corresponding discrete self-adjoint matrix difference equation

$$\Delta(P_n \Delta X_n) + Q_n X_{n+1} = 0, \quad n \in \mathbb{Z},$$

(see AHLBRANDT and PETERSON [2] and BOHNER [6]), where Δ is the forward difference operator $\Delta X_n = X_{n+1} - X_n$. We see the connection between (1.2) and (1.3) if we take $Q(t, X^\sigma(t)) = Q(t)X^\sigma(t)$, $\tilde{Q}(t, Y^\sigma(t)) = \tilde{Q}(t)Y^\sigma(t)$, and $R(\cdot, X^\sigma) \equiv 0$, so that

$$\begin{aligned} Q(t, X^\sigma(t))X^{\sigma\dagger}(t)X^\sigma(t) &= Q(t) (X^\sigma X^{\sigma\dagger} X^\sigma)(t) = Q(t)X^\sigma(t) \\ \tilde{Q}(t, Y^\sigma(t))Y^{\sigma\dagger}(t)Y^\sigma(t) &= \tilde{Q}(t) (Y^\sigma Y^{\sigma\dagger} Y^\sigma)(t) = \tilde{Q}(t)Y^\sigma(t) \end{aligned}$$

by the properties of the Moore-Penrose pseudoinverse.

Dynamic equations on time scales have been introduced within the last twenty years by HILGER and AULBACH [3, 14] to unify, extend, and generalize the theory of ordinary differential equations, difference equations, quantum equations, and all other differential systems defined over nonempty closed subsets of the real line. The linear dynamic matrix version (1.3) was introduced in AGARWAL and BOHNER [1] and studied further in BOHNER and PETERSON [7, Chapter 5] and ERBE and PETERSON [8]. BOHNER and PETERSON [7, Theorem 5.60] have a comparison theorem for (1.3) and a companion equation, the proof of which relies on disconjugacy, a related Riccati equation, a Jacobi condition involving a quadratic functional, and several other results. Our direct proof in Section 2 will be quite different, relying only on a Sturm-Picone identity dissimilar to that given in [7, Theorem 5.51]. Moreover, our result for the nonlinear perturbed matrix comparison in Section 3 is completely new. We will assume that the reader is familiar with the basic time scale calculus and notation (for example, see [7]).

2. LINEAR COMPARISON RESULTS

In this section, we extend the results of [4] and [19] by considering the pair of homogeneous linear matrix dynamic equations

$$(2.1) \quad (PX^\Delta)^\Delta(t) + Q(t)X^\sigma(t) = 0, \quad t \in \mathbb{T},$$

$$(2.2) \quad (\tilde{P}Y^\Delta)^\Delta(t) + \tilde{Q}(t)Y^\sigma(t) = 0, \quad t \in \mathbb{T},$$

where P, \tilde{P} , and Q, \tilde{Q} are right-dense continuous $n \times n$ Hermitian matrix functions (a matrix M is *Hermitian* iff $M^* = M$, where “ $*$ ” indicates conjugate transpose) with P, \tilde{P} invertible in \mathbb{T} , for $X \in \mathbb{D}$ and $Y \in \tilde{\mathbb{D}}$. Here, \mathbb{D} denotes the set of all $n \times n$ matrix-valued functions X defined on \mathbb{T} such that X^Δ is continuous on \mathbb{T}^κ and $(PX^\Delta)^\Delta$ is right-dense continuous on \mathbb{T}^{κ^2} . Then X is a *solution* of (2.1) on \mathbb{T} provided $X \in \mathbb{D}$ and X satisfies (2.1) for all $t \in \mathbb{T}^{\kappa^2}$. Similarly, $\tilde{\mathbb{D}}$ denotes the set of all $n \times n$ matrix-valued functions Y defined on \mathbb{T} such that Y^Δ is continuous on \mathbb{T}^κ and $(\tilde{P}Y^\Delta)^\Delta$ is right-dense continuous on \mathbb{T}^{κ^2} . Then Y is a *solution* of (2.2) on \mathbb{T} provided $Y \in \tilde{\mathbb{D}}$ and Y satisfies (2.2) for all $t \in \mathbb{T}^{\kappa^2}$.

Lemma 2.1. (Picone Identity) *If X is a solution of (2.1) and Y is an invertible solution of (2.2) with the product $Y^*\tilde{P}Y^\Delta$ Hermitian (i.e., Y is a prepared solution), then $G : \mathbb{T} \rightarrow M_{nn}$ defined via*

$$G(t) := (X^*PX^\Delta - X^*\tilde{P}Y^\Delta Y^{-1}X)^\Delta(t)$$

expands to

$$(2.3) \quad G = X^{\sigma*}(\tilde{Q} - Q)X^\sigma + X^{\Delta*}(P - \tilde{P})X^\Delta + (Y^{-1}X)^{\Delta*}(Y^*\tilde{P}Y^\sigma)(Y^{-1}X)^\Delta,$$

where the argument has been suppressed.

Proof. See the proof of Lemma 3.1. □

Before we present our main Sturm-Picone-type result of this section, we introduce the following key definition, an offshoot of an idea originally due to HARTMAN [13].

Definition 2.2. *A solution X of (2.1) has a generalized zero at $t \in \mathbb{T}$ iff $X(t)$ is noninvertible, or if t is a right-scattered point such that $(X^*PX^\sigma)(t)$ is invertible but $(X^*PX^\sigma)(t) \leq 0$. Similarly, a solution Y of (2.2) with the product $Y^*\tilde{P}Y^\Delta$ Hermitian has a generalized zero at $t \in \mathbb{T}$ iff $Y(t)$ is noninvertible, or if t is a right-scattered point such that $(Y^*\tilde{P}Y^\sigma)(t)$ is invertible but $(Y^*\tilde{P}Y^\sigma)(t) \leq 0$. Here $M \leq 0$ for $n \times n$ matrix M carries the standard meaning that $\gamma^*M\gamma \leq 0$ for all $\gamma \in \mathbb{C}^n$.*

Definition 2.3. *The unique solution of the initial value problem (2.1) with initial conditions $X(t_1) = 0$, $X^\Delta(t_1) = P^{-1}(t_1)$ is called the principal solution of (2.1) at t_1 .*

Theorem 2.4. (Sturm Comparison Theorem) *Assume the rd-continuous invertible Hermitian matrix functions P, \tilde{P} satisfy*

$$(2.4) \quad P(t) - \tilde{P}(t) \geq 0, \quad t \in \mathbb{T},$$

and the rd-continuous Hermitian matrix functions Q, \tilde{Q} satisfy

$$(2.5) \quad \tilde{Q}(t) - Q(t) \geq 0, \quad t \in \mathbb{T}.$$

*If the principal solution X of (2.1) at t_1 has a second generalized zero at $t_2 > t_1$ in \mathbb{T} , with X invertible and $(X^*PX^\sigma)(t) > 0$ for $t \in (t_1, t_2)_{\mathbb{T}}$, then every solution Y of (2.2) with the product $Y^*\tilde{P}Y^\Delta$ Hermitian has a generalized zero in $[t_1, t_2]_{\mathbb{T}}$.*

Proof. See the proof of Theorem 3.2. \square

3. NONLINEAR COMPARISON RESULTS

In this section, we seek to extend the results in [7, Chapter 5.3] by considering (1.1) and a possible matrix extension, namely the perturbed nonlinear matrix dynamic equation

$$(3.1) \quad (PX^\Delta)^\Delta(t) + Q(t, X^\sigma(t))X^{\sigma\dagger}(t)X^\sigma(t) = R(t, X^\sigma(t))X^{\sigma\dagger}(t)X^\sigma(t), \quad t \in \mathbb{T},$$

and the companion perturbed nonlinear matrix dynamic equation

$$(3.2) \quad (\tilde{P}Y^\Delta)^\Delta(t) + \tilde{Q}(t, Y^\sigma(t))Y^{\sigma\dagger}(t)Y^\sigma(t) = \tilde{R}(t, Y^\sigma(t))Y^{\sigma\dagger}(t)Y^\sigma(t), \quad t \in \mathbb{T},$$

where as before U^\dagger is the Moore-Penrose pseudoinverse of U , the coefficients $P, \tilde{P} : \mathbb{T} \rightarrow M_{nn}$ are right-dense continuous invertible $n \times n$ Hermitian matrix functions, but here we have the (possible) nonlinearities $Q, R, \tilde{Q}, \tilde{R} : \mathbb{T} \times M_{nn} \rightarrow M_{nn}$, for $X \in \mathbb{D}$ and $Y \in \tilde{\mathbb{D}}$, where $Q, R, \tilde{Q}, \tilde{R}$ are right-dense continuous in the first (time-scale) variable, and continuous in the second matrix variable. We let \mathbb{D} denote the set of all $n \times n$ matrix-valued functions X defined on \mathbb{T} such that X^Δ is continuous on \mathbb{T}^κ and $(PX^\Delta)^\Delta$ is right-dense continuous on \mathbb{T}^{κ^2} . Then X is a *solution* of (3.1) on \mathbb{T} provided $X \in \mathbb{D}$ and X satisfies (3.1) for all $t \in \mathbb{T}^{\kappa^2}$. Similarly, $\tilde{\mathbb{D}}$ denotes the set of all $n \times n$ matrix-valued functions Y defined on \mathbb{T} such that Y^Δ is continuous on \mathbb{T}^κ and $(\tilde{P}Y^\Delta)^\Delta$ is right-dense continuous on \mathbb{T}^{κ^2} . Then Y is a *solution* of (3.2) on \mathbb{T} provided $Y \in \tilde{\mathbb{D}}$ and Y satisfies (3.2) for all $t \in \mathbb{T}^{\kappa^2}$. Compare the following lemma for (3.1), (3.2) with Lemma 2.1.

Lemma 3.1. (Nonlinear Picone Identity) *If X is a solution of (3.1) and Y is an invertible solution of (3.2) with the product $Y^*\tilde{P}Y^\Delta$ Hermitian, then $G : \mathbb{T} \rightarrow M_{nn}$ defined via*

$$G(t) := (X^*PX^\Delta - X^*\tilde{P}Y^\Delta Y^{-1}X)^\Delta(t)$$

expands to

$$(3.3) \quad G = X^{\sigma*} \{ [\tilde{Q}(\cdot, Y^\sigma) - \tilde{R}(\cdot, Y^\sigma)] (Y^\sigma)^\dagger + [R(\cdot, X^\sigma) - Q(\cdot, X^\sigma)] (X^\sigma)^\dagger \} X^\sigma \\ + X^{\Delta*} (P - \tilde{P}) X^\Delta + (Y^{-1} X)^{\Delta*} (Y^* \tilde{P} Y^\sigma) (Y^{-1} X)^\Delta,$$

where the argument has been suppressed.

Proof. Note that under these assumptions, $(Y^\sigma)^\dagger = (Y^\sigma)^{-1}$. Suppressing the arguments, we expand the dynamic differential version of G using the delta derivative product rule and the fact that X and Y are solutions to (3.1) and (3.2), respectively, to obtain

$$G = X^{\sigma*} (P X^\Delta)^\Delta + X^{\Delta*} P X^\Delta - (X^* \tilde{P} Y^\Delta)^\Delta (Y^{-1} X)^\sigma - X^* \tilde{P} Y^\Delta (Y^{-1} X)^\Delta \\ = X^{\sigma*} R(\cdot, X^\sigma) X^{\sigma\dagger} X^\sigma - X^{\sigma*} Q(\cdot, X^\sigma) X^{\sigma\dagger} X^\sigma + X^{\Delta*} (P - \tilde{P}) X^\Delta + X^{\Delta*} \tilde{P} X^\Delta \\ - X^{\sigma*} (\tilde{P} Y^\Delta)^\Delta (Y^{-1} X)^\sigma - X^{\Delta*} \tilde{P} Y^\Delta (Y^{-1} X)^\sigma - X^* \tilde{P} Y^\Delta (Y^{-1} X)^\Delta \\ = X^{\sigma*} R(\cdot, X^\sigma) X^{\sigma\dagger} X^\sigma - X^{\sigma*} Q(\cdot, X^\sigma) X^{\sigma\dagger} X^\sigma + X^{\sigma*} \tilde{Q}(\cdot, Y^\sigma) Y^{\sigma\dagger} Y^\sigma (Y^{-1} X)^\sigma \\ + X^{\Delta*} (P - \tilde{P}) X^\Delta + X^{\Delta*} \tilde{P} X^\Delta - X^{\Delta*} \tilde{P} Y^\Delta (Y^{-1} X)^\sigma - X^* \tilde{P} Y^\Delta (Y^{-1} X)^\Delta \\ - X^{\sigma*} \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger} Y^\sigma (Y^{-1} X)^\sigma.$$

With some factoring and rearranging, we have

$$G = X^{\sigma*} [R(\cdot, X^\sigma) X^{\sigma\dagger} - \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger}] X^\sigma + X^{\sigma*} [\tilde{Q}(\cdot, Y^\sigma) (Y^\sigma)^\dagger \\ - Q(\cdot, X^\sigma) (X^\sigma)^\dagger] X^\sigma + X^{\Delta*} (P - \tilde{P}) X^\Delta - X^* \tilde{P} Y^\Delta (Y^{-1} X)^\Delta \\ + X^{\Delta*} \tilde{P} [X^\Delta - Y^\Delta (Y^\sigma)^{-1} X^\sigma] \\ = X^{\sigma*} [R(\cdot, X^\sigma) X^{\sigma\dagger} - \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger}] X^\sigma + X^{\sigma*} [\tilde{Q}(\cdot, Y^\sigma) (Y^\sigma)^\dagger \\ - Q(\cdot, X^\sigma) (X^\sigma)^\dagger] X^\sigma + X^{\Delta*} (P - \tilde{P}) X^\Delta - X^* \tilde{P} Y^\Delta (Y^{-1} X)^\Delta \\ + X^{\Delta*} \tilde{P} \{ X^\Delta + Y [-Y^{-1} Y^\Delta (Y^\sigma)^{-1} X^\sigma] \} \\ = X^{\sigma*} [R(\cdot, X^\sigma) X^{\sigma\dagger} - \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger}] X^\sigma + X^{\sigma*} [\tilde{Q}(\cdot, Y^\sigma) (Y^\sigma)^\dagger \\ - Q(\cdot, X^\sigma) (X^\sigma)^\dagger] X^\sigma + X^{\Delta*} (P - \tilde{P}) X^\Delta - X^* \tilde{P} Y^\Delta (Y^{-1} X)^\Delta \\ + X^{\Delta*} \tilde{P} [Y Y^{-1} X^\Delta + Y (Y^{-1})^\Delta X^\sigma].$$

Here we used the fact that $(Y^{-1})^\Delta = -Y^{-1} Y^\Delta (Y^\sigma)^{-1}$. Continuing in this manner, we obtain

$$G = X^{\sigma*} [R(\cdot, X^\sigma) X^{\sigma\dagger} - \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger} + \tilde{Q}(\cdot, Y^\sigma) (Y^\sigma)^\dagger - Q(\cdot, X^\sigma) (X^\sigma)^\dagger] X^\sigma \\ + X^{\Delta*} (P - \tilde{P}) X^\Delta - X^* \tilde{P} Y^\Delta (Y^{-1} X)^\Delta + X^{\Delta*} \tilde{P} Y [Y^{-1} X^\Delta + (Y^{-1})^\Delta X^\sigma] \\ = X^{\sigma*} [R(\cdot, X^\sigma) X^{\sigma\dagger} - \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger} + \tilde{Q}(\cdot, Y^\sigma) (Y^\sigma)^\dagger - Q(\cdot, X^\sigma) (X^\sigma)^\dagger] X^\sigma \\ + X^{\Delta*} (P - \tilde{P}) X^\Delta - X^* \tilde{P} Y^\Delta (Y^{-1} X)^\Delta + X^{\Delta*} \tilde{P} Y (Y^{-1} X)^\Delta \\ = X^{\sigma*} [R(\cdot, X^\sigma) X^{\sigma\dagger} - \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger} + \tilde{Q}(\cdot, Y^\sigma) (Y^\sigma)^\dagger - Q(\cdot, X^\sigma) (X^\sigma)^\dagger] X^\sigma \\ + X^{\Delta*} (P - \tilde{P}) X^\Delta + [X^{\Delta*} \tilde{P} Y - X^* (Y^*)^{-1} Y^* \tilde{P} Y^\Delta] (Y^{-1} X)^\Delta.$$

Now since Y is a solution of (3.2) with $Y^* \tilde{P} Y^\Delta$ Hermitian, and \tilde{P} is Hermitian, we have

$$\begin{aligned} G &= X^{*\sigma} [R(\cdot, X^\sigma) X^{\sigma\dagger} - \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger} + \tilde{Q}(\cdot, Y^\sigma) (Y^\sigma)^\dagger - Q(\cdot, X^\sigma) (X^\sigma)^\dagger] X^\sigma \\ &\quad + X^{\Delta*} (P - \tilde{P}) X^\Delta + [X^{\Delta*} \tilde{P} Y - X^* (Y^*)^{-1} Y^{\Delta*} \tilde{P} Y] (Y^{-1} X)^\Delta \\ &= X^{*\sigma} [R(\cdot, X^\sigma) X^{\sigma\dagger} - \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger} + \tilde{Q}(\cdot, Y^\sigma) (Y^\sigma)^\dagger - Q(\cdot, X^\sigma) (X^\sigma)^\dagger] X^\sigma \\ &\quad + X^{\Delta*} (P - \tilde{P}) X^\Delta + [X^{\Delta*} (Y^{\sigma*})^{-1} - X^* (Y^*)^{-1} Y^{\Delta*} (Y^{\sigma*})^{-1}] Y^{\sigma*} \tilde{P} Y (Y^{-1} X)^\Delta \\ &= X^{*\sigma} [R(\cdot, X^\sigma) X^{\sigma\dagger} - Q(\cdot, X^\sigma) (X^\sigma)^\dagger + \tilde{Q}(\cdot, Y^\sigma) (Y^\sigma)^\dagger - \tilde{R}(\cdot, Y^\sigma) Y^{\sigma\dagger}] X^\sigma \\ &\quad + X^{\Delta*} (P - \tilde{P}) X^\Delta + (Y^{-1} X)^{\Delta*} (Y^* \tilde{P} Y^\sigma) (Y^{-1} X)^\Delta. \end{aligned}$$

Here we have used the fact that $Y^* \tilde{P} Y^\Delta$ being Hermitian implies $Y^{*\sigma} \tilde{P} Y$ is Hermitian as well, a fact shown easily using the simple formula [14] given by $Y^\sigma = Y + \mu Y^\Delta$. \square

Theorem 3.2. (Perturbed Sturm Comparison Theorem) *Assume that*

$$(3.4) \quad P(t) - \tilde{P}(t) \geq 0$$

and

$$(3.5) \quad [R(t, U) - Q(t, U)] U^\dagger + [\tilde{Q}(t, V) - \tilde{R}(t, V)] V^\dagger \geq 0$$

for all $t \in \mathbb{T}$. If equation (3.1) has a unique principal solution X at t_1 with a second generalized zero at $t_2 > t_1$ in \mathbb{T} , with X invertible and $(X^* P X^\sigma)(t) > 0$ for $t \in (t_1, t_2)_{\mathbb{T}}$, then every solution Y of (3.2) with the product triple $Y^* \tilde{P} Y^\Delta$ Hermitian has a generalized zero in $[t_1, t_2]_{\mathbb{T}}$.

Proof. We argue by contradiction. Let X be the principal solution of (3.1) at t_1 with a second generalized zero at $t_2 > t_1$ in \mathbb{T} , and let Y be a solution of (3.2) with the product $Y^* \tilde{P} Y^\Delta$ Hermitian and with no generalized zeros in $[t_1, t_2]_{\mathbb{T}}$. It follows that Y is invertible and $(Y^* \tilde{P} Y^\sigma)(t) > 0$ for all $t \in [t_1, t_2]_{\mathbb{T}}$. Then, by Lemma 3.1, we integrate the expression for G from t_1 to t_2 to obtain

$$\begin{aligned} (3.6) \quad & (X^* P X^\Delta - X^* \tilde{P} Y^\Delta Y^{-1} X)(t) \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} (X^* P X^\Delta - X^* \tilde{P} Y^\Delta Y^{-1} X)^\Delta(t) \Delta t \\ &= \int_{t_1}^{t_2} X^{\sigma*} \{ [R(\cdot, X^\sigma) - Q(\cdot, X^\sigma)] (X^\sigma)^\dagger + [\tilde{Q}(\cdot, Y^\sigma) - \tilde{R}(\cdot, Y^\sigma)] (Y^\sigma)^\dagger \} X^\sigma(t) \Delta t \\ &\quad + \int_{t_1}^{t_2} (X^{\Delta*} (P - \tilde{P}) X^\Delta(t) + (Y^{-1} X)^{\Delta*} (Y^* \tilde{P} Y^\sigma) (Y^{-1} X)^\Delta(t)) \Delta t. \end{aligned}$$

Case I: Suppose X has a singularity t_2 . Since $X(t_1) = 0$ and $X(t_2)$ is singular, there exists a vector $\gamma \neq 0$ such that $X(t_2)\gamma = 0$, and we have

$$(3.7) \quad \gamma^* \left((X^* P X^\Delta - X^* \tilde{P} Y^\Delta Y^{-1} X)(t) \Big|_{t_1}^{t_2} \right) \gamma = 0.$$

But by (3.6) and our assumptions (3.4) and (3.5), respectively, we have

$$\begin{aligned} 0 &= \gamma^* \left(\int_{t_1}^{t_2} X^{\sigma*} \{ [R(\cdot, X^\sigma) - Q(\cdot, X^\sigma)] (X^\sigma)^\dagger \right. \\ &\quad + [\tilde{Q}(\cdot, Y^\sigma) - \tilde{R}(\cdot, Y^\sigma)] (Y^\sigma)^\dagger \} X^\sigma(t) \Delta t + \int_{t_1}^{t_2} X^{\Delta*} (P - \tilde{P}) X^\Delta(t) \Delta t \\ &\quad \left. + \int_{t_1}^{t_2} (Y^{-1}X)^\Delta (Y^* \tilde{P} Y^\sigma) (Y^{-1}X)^\Delta(t) \Delta t \right) \gamma > 0, \end{aligned}$$

which is a contradiction.

Case II: Suppose t_2 is a right-scattered point such that $(X^* P X^\sigma)(t_2)$ is invertible but $(X^* P X^\sigma)(t_2) \leq 0$. Then there exists a vector $\gamma \neq 0$ such that $\gamma^* (X^* P X^\sigma)(t_2) \gamma \leq 0$. Set $I = \int_{t_1}^{t_2} G(t) \Delta t$ for G from Lemma 3.1. Then $I > 0$ by the right-hand side of (3.6) and our assumptions (3.4) and (3.5), respectively. But

$$\begin{aligned} 0 &< \gamma^* I \gamma = \gamma^* \left(X^* P X^\Delta - X^* \tilde{P} Y^\Delta Y^{-1} X \right) (t) \gamma \Big|_{t_1}^{t_2} \\ &= \gamma^* \left(\frac{1}{\mu} X^* P (X^\sigma - X) - \frac{1}{\mu} X^* \tilde{P} (Y^\sigma - Y) Y^{-1} X \right) (t_2) \gamma \\ &= \frac{1}{\mu} \gamma^* (X^* P X^\sigma) \gamma(t_2) - \frac{1}{\mu} (X \gamma)^* (\tilde{P} Y^\sigma Y^{-1}) (X \gamma)(t_2) \\ &\quad - \frac{1}{\mu} (X \gamma)^* (P - \tilde{P}) (X \gamma)(t_2) < 0, \end{aligned}$$

another contradiction. Here we have used the fact that $Y^* \tilde{P} Y^\sigma > 0$ implies $\tilde{P} Y^\sigma Y^{-1} > 0$ as well. □

The following reformulation of Theorem 3.2 is immediate.

Theorem 3.3. *Assume P, \tilde{P} satisfy (3.4) and $Q, R, \tilde{Q}, \tilde{R}$ satisfy (3.5). If (3.2) has a solution Y with the product $Y^* \tilde{P} Y^\Delta$ Hermitian such that Y has no generalized zeros in $[t_1, t_2]_{\mathbb{T}}$ for $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$, then the principal solution X of (3.1) at t_1 satisfies $(X^* P X^\sigma)(t) > 0$ for $t \in (t_1, t_2]_{\mathbb{T}}$.*

As indicated in Section 2, equations (3.1) and (3.2) reduce to (2.1) and (2.2) in the linear homogeneous case and we obtain the standard comparison theorem. However, we should point out that the results given here in Theorems 3.2 and 3.3 allow us to compare two nonlinear unperturbed equations, a (nonlinear) unperturbed equation to a linear or nonlinear perturbed equation, etc. We leave the formulation of these results to the reader.

4. OSCILLATION

In this section, we give an oscillation result, and thus suppose that $\sup \mathbb{T} = \infty$ throughout. Let $a \in \mathbb{T}$. We say that equation (3.2) is *nonoscillatory* on $[a, \infty)_{\mathbb{T}}$ if

and only if there exist a solution Y of (3.2) with $Y^* \tilde{P} Y^\Delta$ Hermitian and a point $t_0 \in [a, \infty)_{\mathbb{T}}$ such that $(Y^* P Y^\sigma)(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$, i.e., Y has no generalized zeros on $[t_0, \infty)_{\mathbb{T}}$. Otherwise, we say that (3.2) is *oscillatory* on $[t_0, \infty)_{\mathbb{T}}$. Similarly, we say the linear homogeneous equation

$$(4.1) \quad X^{\Delta\Delta}(t) + Q(t)X^\sigma(t) = 0$$

is *nonoscillatory* on $[a, \infty)_{\mathbb{T}}$ if and only if there exist a solution X of (4.1) with $X^* X^\Delta$ Hermitian and a point $t_0 \in [a, \infty)_{\mathbb{T}}$ such that $(X^* X^\sigma)(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$, and it is *oscillatory* on $[t_0, \infty)_{\mathbb{T}}$ otherwise.

In the following theorem, we use the notation that for any $n \times n$ Hermitian matrix M , $\lambda_{\max}(M)$ denotes the largest eigenvalue of M . Moreover, we assume the time scale \mathbb{T} does not eventually become the continuous real line, in the sense that there exist two right-scattered points somewhere on any half line.

Theorem 4.1. *Assume that for any $t_0 \in [a, \infty)_{\mathbb{T}}$ there exist points $b_0 > a_0 \geq t_0$ such that $\mu(a_0) > 0$, $\mu(b_0) > 0$, and*

$$(4.2) \quad \lambda_{\max} \left(\int_{a_0}^{b_0} Q(t) \Delta t \right) \geq \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)}$$

for Q in equation (4.1). If

$$(4.3) \quad I \geq \tilde{P} \quad \text{and} \quad [\tilde{Q}(t, V) - \tilde{R}(t, V)]V^\dagger - Q(t)UU^\dagger \geq 0,$$

then (3.2) is *oscillatory*.

Proof. By [7, Theorem 5.63], condition (4.2) implies that (4.1) is oscillatory on $[a, \infty)_{\mathbb{T}}$. Then, from Theorem 3.3 and (4.3), we have that (3.2) is also oscillatory on $[a, \infty)_{\mathbb{T}}$. \square

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