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# Positive stable realizations of discrete-time linear systems

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**Abstract.** The problem of existence and determination of the set of positive asymptotically stable realizations of a proper transfer function of linear discrete-time systems is formulated and solved. Necessary and sufficient conditions for existence of the set of the realizations are established. A procedure for computation of the set of realizations are proposed and illustrated by numerical examples.

**Key words:** positive, stable, realization, existence, procedure, linear, discrete-time, system.

## 1. Introduction

Determination of the state space equations for given transfer matrix is a classical problem, called realization problem, which has been addressed in many papers and books [1, 2, 3–8]. An overview on the positive realization problem is given in [1, 2, 9]. The realization problem for positive continuous-time and discrete-time linear systems has been considered in [10–16] and the positive realization problem for discrete-time systems with delays in [14, 15, 17]. The fractional positive linear systems has been addressed in [5, 18, 19]. The realization problem for fractional linear systems has been analyzed in [20] and for positive continuous-discrete systems in [21]. Stability of continuous-discrete linear systems has been considered in [22]. A method based on similarity transformation of the standard realization to the discrete positive one has been proposed in [16]. Conditions for the existence of positive stable realization with system Metzler matrix for transfer function has been established in [12]. The problem of the existence and determination of the set of Metzler matrices for given stable polynomials has been formulated and solved in [6]. The problem for computation of positive stable realizations for continuous-time linear systems has been addressed in [7].

It is well-known [1, 3, 9] that to find a realization for a given transfer function first we have to find a state matrix for a given denominator of the transfer function.

In this paper necessary and sufficient conditions for existence of the set of positive stable realizations of a proper transfer function of linear discrete-time systems are established and a procedure for computation of the set of realizations is proposed.

The paper is organized as follows. In Sec. 2 some preliminaries concerning positive linear systems are recalled and the problem formulation is given. The problem solution for systems with real negative poles of the transfer function is presented in Sec. 3. The problem of the existence and computation of the set of positive asymptotically stable realizations for single-input single-output systems with complex conjugate poles is addressed in Sec. 4. The problem for general case (multi-input multi-output) is considered in Sec. 5. Concluding remarks are given in Sec. 6.

The following notation is used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, $I_n$ – the $n\times n$ identity matrix, $A^T$ – transpose of the matrix $A$, $\Re^{n\times m}(z)$ – the set of $n\times m$ rational matrices in $z$.

## 2. Preliminaries and the problem formulation

Consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \qquad i \in Z_+ = \{0, 1, ...\} \tag{1a}$$

$$y_i = Cx_i + Du_i \tag{1b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

**Definition 1.** [1, 9] The system (1) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

**Theorem 1.** [1, 9] The system (1) is positive if and only if

$$A \in \Re_+^{n\times n}, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \tag{2}$$

**Definition 2.** [1, 9] The positive system (1) is called asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \text{for any } x_0 \in \Re_+^n. \tag{3}$$

**Theorem 2.** [1, 9] The positive system (1) is asymptotically stable if and only if all coefficients of the polynomial

$$\begin{aligned} p_n(z) &= \det[I_n(z+1) - A] \\ &= z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 \end{aligned} \tag{4}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, ..., n-1$.

**Definition 3.** [9] A matrix $P \in \Re_+^{n\times n}$ is called the monomial matrix (or generalized permutation matrix) if its every row and its every column contains only one positive entry and its remaining entries are zero.

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The inverse matrix $P^{-1}$ of the monomial matrix $P$ is equal to the transpose matrix in which every nonzero entry is replaced by its inverse and $P^{-1} \in \Re_+^{n\times n}$.

**Lemma 1.** If $A \in \Re_+^{n\times n}$ then $\overline{A} = PAP^{-1} \in \Re_+^{n\times n}$ for every monomial matrices $P \in \Re_+^{n\times n}$ and

$$\det[I_n z - \overline{A}] = \det[I_n z - A]. \tag{5}$$

**Proof.** Taking into account that if $P \in \Re_+^{n\times n}$ then $P^{-1} \in \Re_+^{n\times n}$ and $\overline{A} = PAP^{-1} \in \Re_+^{n\times n}$. It is easy to check that

$$\begin{aligned}\det[I_n z - \overline{A}] &= \det[I_n z - PAP^{-1}] = \det\{P[I_n z - A]P^{-1}\}\\ &= \det P \det[I_n z - A] \det P^{-1} = \det[I_n z - A]\end{aligned}$$

since $\det P \det P^{-1} = 1$.

The transfer matrix of the systems (1) is given by

$$T(z) = C[I_n z - A]^{-1}B + D. \tag{6}$$

The transfer matrix is called proper if

$$\lim_{z\to\infty} T(z) = K \in \Re^{p\times m} \tag{7}$$

and it is called strictly proper if $K = 0$.

**Definition 4.** Matrices (2) are called a positive realization of transfer matrix $T(z)$ if they satisfy the equality (6).

The realization is called asymptotically stable if the matrix $A$ is asymptotically stable.

**Theorem 3.** [9] The matrix $A \in \Re_+^{n\times n}$ is unstable if at least one of its diagonal entries $a_{i,i}, \quad i = 1, 2, ..., n$ is greater 1.

**Lemma 2.** The matrices

$$\begin{aligned}&\overline{A}_k = PA_kP^{-1} \in \Re_+^{n\times n}, \qquad \overline{B}_k = PB_k \in \Re_+^{n\times m},\\ &\overline{C}_k = C_kP^{-1} \in \Re_+^{p\times n}, \qquad \overline{D}_k = D_k \in \Re_+^{p\times m},\\ &k = 1, \dots, q\end{aligned} \tag{8}$$

are a positive asymptotically stable realization of the proper transfer matrix $T(z) \in \Re^{p\times m}(z)$ for any monomial matrix $P \in \Re_+^{n\times n}$ if and only if the matrices

$$\begin{aligned}&A_k \in \Re_+^{n\times n}, \quad B_k \in \Re_+^{n\times m},\\ &C_k \in \Re_+^{p\times n}, \quad D_k \in \Re_+^{p\times m},\\ &k = 1, \dots, q\end{aligned} \tag{9}$$

are its positive asymptotically stable realization.

**Proof.** Taking into account that $P \in \Re_+^{n\times n}$ is a monomial matrix then $P^{-1} \in \Re_+^{n\times n}$ is also monomial matrix and using (8) we obtain $\vec{A}_k \in \Re_+^{n\times n}$, $\overline{B}_k \in \Re_+^{n\times m}$, $\overline{C}_k \in \Re_+^{p\times n}$ if and only if (9) holds. Using (6) and (8) we obtain

$$\begin{aligned}\overline{T}(z) &= \overline{C}_k[I_n z - \overline{A}_k]^{-1}\overline{B}_k + \overline{D}_k\\ &= C_kP^{-1}[I_n z - PA_kP^{-1}]^{-1}PB_k + D_k\\ &= C_kP^{-1}\{P[I_n z - A_k]P^{-1}\}^{-1}PB_k + D_k\\ &= C_kP^{-1}P[I_n z - A_k]^{-1}P^{-1}PB_k + D_k\\ &= C_k[I_n z - A_k]^{-1}B_k + D_k = T(z).\end{aligned} \tag{10}$$

Therefore, the matrices (2.8) are a positive asymptotically stable realization of $T(z)$ if and only if the matrices (2.9) are also its positive asymptotically stable realization.

The problem under considerations can be stated as follows: Given a rational proper matrix $T(z) \in \Re^{p\times m}(z)$, find the set of its positive asymptotically stable realizations (8).

In this paper necessary and sufficient conditions for existence of the set of the positive asymptotically stable realizations for a given $T(z)$ will be established and a procedure for computation of the set of realizations will be proposed.

## 3. SISO systems with only real positive poles

In this section the single-input single-output (SISO) discrete-time linear systems with the proper transfer function

$$T(z) = \frac{b_n z^n + b_{n-1}z^{n-1} + ... + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0} \tag{11}$$

having only real positive poles (not necessarily distinct) $\alpha_1$, $\alpha_2, \dots, \alpha_n$, i.e.

$$\begin{aligned}d_n(z) &= (z - \alpha_1)(z - \alpha_2)...(z - \alpha_n)\\ &= z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0,\\ a_{n-1} &= -(\alpha_1 + \alpha_2 + ... + \alpha_n),\\ a_{n-2} &= -\alpha_1(\alpha_2 + \alpha_3 + ... + \alpha_n)\\ &\quad -\alpha_2(\alpha_3 + \alpha_4 + ... + \alpha_n)\\ &\quad -... - \alpha_{n-1}\alpha_n, ...,\\ a_0 &= (-1)^n \alpha_1\alpha_2...\alpha_n\end{aligned} \tag{12}$$

will be considered.

**Theorem 4.** For the proper transfer function

$$T(z) = \frac{b_1 z + b_0}{z + a} \tag{13}$$

there exists the set of positive asymptotically stable realizations

$$\begin{aligned}&\overline{A}_k = PA_kP^{-1}, \qquad \overline{B}_k = PB_k,\\ &\overline{C}_k = C_kP^{-1}, \qquad \overline{D}_k = D_k, \quad k = 1, 2\end{aligned} \tag{14}$$

for any positive parameter $P > 0$ and $A_k$, $B_k$, $C_k$, $D_k$ having one of the forms

$$\begin{aligned}&A_1 = [-a], \qquad B_1 = [1],\\ &C_1 = [b_0 - ab_1], \qquad D_1 = [b_1]\end{aligned} \tag{15a}$$

or

$$\begin{aligned}&A_2 = [-a], \qquad B_2 = [b_0 - ab_1],\\ &C_2 = [1], \qquad D_2 = [b_1]\end{aligned} \tag{15b}$$

if and only if

$$-1 < a \le 0, \qquad b_1 \ge 0, \qquad b_0 - ab_1 \ge 0. \tag{16}$$

**Proof.** The matrix $A_1 \in \Re_+^{1\times 1}$ and is asymptotically stable if $-1 < a \le 0$. The matrices $C_1 \in \Re_+^{1\times 1}$, $D_1 \in \Re_+^{1\times 1}$ if and only if $b_1 \ge 0, \quad b_0 - ab_1 \ge 0$. By Lemma 2 the matrices (14) are a positive asymptotically stable realization of (13) for any $P > 0$ if and only if the matrices (15a) are its positive asymptotically stable realization. Proof for matrices (15b) is similar.

**Lemma 3.** The nonnegative matrix

$$A = \begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \in \Re_+^{2\times 2} \tag{17}$$

has only real eigenvalues $z_1$, $z_2$ such that $z_1 + z_2 \geq 0$ and one of the zeros is negative if and only if $a_{12}a_{21} > a_{11}a_{22}$.

**Proof.** The characteristic polynomial of the matrix (17)

$$\begin{aligned} \det[I_2 z - A] &= \begin{vmatrix} z-a & -a_{12} \\ -a_{21} & z-a_{22} \end{vmatrix} \\ &= z^2 - (a_{11}+a_{22})z + a_{11}a_{22} - a_{12}a_{21} \\ &= z^2 + a_1 z + a_0 = (z-z_1)(z-z_2), \\ a_1 &= -(a_{11}+a_{22}) = -(z_1+z_2), \\ a_0 &= a_{11}a_{22} - a_{12}a_{21} = z_1 z_2 \end{aligned} \tag{18}$$

has only real zeros since

$$\begin{aligned} a_1^2 - 4a_0 &= (a_{11}+a_{22})^2 + 4(a_{11}a_{22} - a_{12}a_{21}) \\ &= (a_{11}^2 + a_{22}^2 - 2a_{11}a_{22}) + 4a_{12}a_{21} \\ &= (a_{11}-a_{22})^2 + 4a_{12}a_{21} \geq 0. \end{aligned} \tag{19}$$

Taking into account that $a_1 = -(a_{11}+a_{22}) = -(z_1+z_2)$ we conclude that at least one of the zeros is positive such that $z_1 + z_2 \geq 0$. From the equality $a_{11}a_{22} - a_{12}a_{21} = z_1 z_2$ it follows that one of the zeros is negative if and only if $a_{11}a_{22} < a_{12}a_{21}$.

**Theorem 5.** For the transfer function

$$T(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^2 + a_1 z + a_0} \tag{20}$$

there exists the set of positive asymptotically stable realizations

$$\begin{aligned} \overline{A}_k &= PA_kP^{-1} \in \Re_+^{2\times 2}, \quad \overline{B}_k = PB_k \in \Re_+^{2\times 1}, \\ \overline{C}_k &= C_kP^{-1} \in \Re_+^{1\times 2}, \quad \overline{D}_k = D_k \in \Re_+^{1\times 1}, \\ k &= 1,2 \end{aligned} \tag{21}$$

for any monomial matrix $P \in \Re_+^{2\times 2}$ and $A_k$, $B_k$, $C_k$, $D_k$ having one of the forms

$$\begin{aligned} A_1 &= \begin{bmatrix} z_1 & 1 \\ 0 & z_2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C_1 &= [\; b_2 z_1^2 + b_1 z_1 + b_0 \quad b_2(z_1+z_2) + b_1 \;], \\ D_1 &= [b_2] \end{aligned} \tag{22a}$$

or

$$\begin{aligned} A_2 &= A_1^T = \begin{bmatrix} z_1 & 0 \\ 1 & z_2 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} b_2 z_1^2 + b_1 z_1 + b_0 \\ b_2(z_1+z_2) + b_1 \end{bmatrix}, \\ C_2 &= [\; 0 \quad 1 \;], \quad D_2 = [b_2] \end{aligned} \tag{22b}$$

if and only if the polynomial

$$d_2(z) = z^2 + a_1 z + a_0 \tag{23}$$

has two positive zeros $z_1$, $z_2$ satisfying the condition

$$|z_k| < 1 \qquad \text{for} \quad k = 1,2 \tag{24}$$

and

$$\begin{aligned} &b_2 \geq 0, \\ &b_2 z_1^2 + b_1 z_1 + b_0 \geq 0, \quad b_2(z_1+z_2) + b_1 \geq 0. \end{aligned} \tag{25}$$

**Proof.** The matrix $A_1$ is asymptotically stable and nonnegative if and only if the polynomial (23) has real positive zeros $z_1$, $z_2$ satisfying the condition (24). The matrix

$$D_1 = \lim_{z\to\infty} T(z) = [b_2] \in \Re_+^{1\times 1} \tag{26}$$

if and only if $b_2 \geq 0$. The strictly proper transfer function has the form

$$T_{sp}(z) = T(z) - D_1 = \frac{\overline{b}_1 z + \overline{b}_0}{z^2 + a_1 z + a_0} \tag{27}$$

where

$$\overline{b}_1 = b_1 - a_1 b_2, \qquad \overline{b}_0 = b_0 - a_0 b_2. \tag{28}$$

Taking into account the forms of $A_1$ and $B_1$ given by (22a) we obtain

$$\begin{aligned} T_{sp}(z) &= C_1[I_2 z - A_1]^{-1} B_1 \\ &= [\; c_1 \quad c_2 \;] \begin{bmatrix} z - z_1 & 1 \\ 0 & z - z_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{c_1 + c_2(z - z_1)}{z^2 + a_1 z + a_0} = \frac{\overline{b}_1 z + \overline{b}_0}{z^2 + a_1 z + a_0} \end{aligned} \tag{29a}$$

and

$$\begin{aligned} c_2 &= \overline{b}_1 = b_1 - a_1 b_2 = b_2(z_1+z_2) + b_1, \\ c_1 &= \overline{b}_0 + c_2 z_1 \\ &= b_0 - z_1 z_2 b_2 + [b_2(z_1+z_2) + b_1]z_1 \\ &= b_2 z_1^2 + b_1 z_1 + b_0. \end{aligned} \tag{29b}$$

From (29b) it follows that the matrix $C \in \Re_+^{1\times 2}$ if and only if the conditions (25) are satisfied. The proof for (22b) is similar.

**Remark 1.** If the polynomial (23) has two zeros $z_k < 1$ for $k = 1,2$ and $b_k \geq 0$ for $k = 0,1,2$ then the transfer function (20) has the set of positive asymptotically table realizations (21).

**Lemma 4.** If the polynomial

$$\begin{aligned} d_n(z) = z^n &+ (-1)^1 \widetilde{a}_{n-1} z^{n-1} + (-1)^2 \widetilde{a}_{n-2} z^{n-2} \\ &+ ... + (-1)^n \widetilde{a}_0 \end{aligned} \tag{30}$$

has only real positive zeros $\alpha_k > 0$, $\quad k = 1, ..., n$ then

$$\widetilde{a}_{n-k} > 0 \quad \text{for} \quad k = 1, 2, \ldots, n. \tag{31}$$

**Proof.** Proof will be accomplished by induction. The hypothesis is valid for $n = 1$ and $n = 2$. For $n = 1$ we have $z - \alpha_1 = z + (-1)^1\alpha_1 = z + (-1)^1\widetilde{a}_0$, $\widetilde{a}_0 = \alpha_1 > 0$. Similarly for $n = 2$ we have

$$\begin{aligned} (z-\alpha_1)(z-\alpha_2) &= z^2 - (\alpha_1+\alpha_2)z + \alpha_1\alpha_2 \\ &= z^2 + (-1)^1\widetilde{a}_1 z + (-1)^2\widetilde{a}_0 \end{aligned}$$

and

$$\widetilde{a}_1 = (\alpha_1 + \alpha_2) > 0, \qquad \widetilde{a}_0 = \alpha_1\alpha_2 > 0.$$

Assuming that the hypothesis is true for $k > 1$ ($k \in N = \{1, 2, ...\}$)

$$(z-\alpha_1)(z-\alpha_2)...(z-\alpha_k) = z^k + (-1)^1\widetilde{a}_{k-1}z^{k-1} + (-1)^2\widetilde{a}_{k-2}z^{k-2} + ... + (-1)^k\widetilde{a}_0, \quad \widetilde{a}_{k-j} > 0, \quad j = 1, 2, \ldots, n \tag{32}$$

we shall show that the hypothesis is also valid for $k+1$. Using (32) we obtain

$$\begin{aligned}(z-\alpha_1)&(z-\alpha_2)...(z-\alpha_k)(z-\alpha_{k+1})\\ &= (z^k + (-1)^1\widetilde{a}_{k-1}z^{k-1} + (-1)^2\widetilde{a}_{k-2}z^{k-2}\\ &+ ... + (-1)^k\widetilde{a}_0)(z-\alpha_{k+1})\\ &= z^{k+1} + (-1)^1(\widetilde{a}_{k-1} + \alpha_{k+1})z^k\\ &+ (-1)^2(\widetilde{a}_{k-2} + \widetilde{a}_{k-1}\alpha_{k+1})z^{k-1}\\ &+ ... + (-1)^k(\widetilde{a}_0 + \widetilde{a}_1\alpha_{k+1})z + (-1)^{k+1}\widetilde{a}_0\alpha_{k+1}\end{aligned} \tag{33}$$

and

$$(\widetilde{a}_{k-1} + \alpha_{k+1}) > 0, \quad (\widetilde{a}_{k-2} + \widetilde{a}_{k-1}\alpha_{k+1}) > 0, \ldots,$$

$$(\widetilde{a}_0 + \widetilde{a}_1\alpha_{k+1}) > 0, \quad \widetilde{a}_0\alpha_{k+1} > 0.$$

This completes the proof.

**Theorem 6.** The polynomial

$$d_n(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 \tag{34}$$

has only real positive zeros satisfying the condition

$$z_k < 1 \quad \text{for} \quad k = 1, 2, \ldots, n \tag{35}$$

if and only if all coefficients of the polynomial

$$\begin{aligned}\overline{d}_n(z) &= d_n(z+1) = (z+1)^n + a_{n-1}(z+1)^{n-1}\\ &+ ... + a_1(z+1) + a_0\\ &= z^n + \overline{a}_{n-1}z^{n-1} + ... + \overline{a}_1 z + \overline{a}_0,\\ \overline{a}_{n-1} &= n + a_{n-1}, ..., \overline{a}_0 = 1 + a_0 + a_1 + ... + a_{n-1}\end{aligned} \tag{36}$$

are positive, i.e.

$$\overline{a}_k > 0 \quad \text{for} \quad k = 0, 1, \ldots, n-1. \tag{37}$$

**Proof.** By Theorem 2 the asymptotically stable polynomial (34) has positive zeros satisfying the condition (35) if and only if all coefficients of the polynomial (36) are positive.

**Theorem 7.** There exists the set of positive asymptotically stable realizations

$$\begin{aligned}\overline{A}_k &= PA_kP^{-1} \in \Re_+^{n\times n}, \quad \overline{B}_k = PB_k \in \Re_+^{n\times 1},\\ \overline{C}_k &= C_kP^{-1} \in \Re_+^{1\times n}, \quad \overline{D}_k = D_k \in \Re_+^{1\times 1},\\ k &= 1, 2\end{aligned} \tag{38}$$

for any monomial matrix $P \in \Re_+^{n\times n}$ and $A_k$, $B_k$, $C_k$, $D_k$ having one of the forms

$$A_1 = \begin{bmatrix} \alpha_1 & 1 & 0 & ... & 0\\ 0 & \alpha_2 & 1 & ... & 0\\ \vdots & \vdots & \vdots & ... & \vdots\\ 0 & 0 & 0 & ... & \alpha_n \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0\\ \vdots\\ 0\\ 1 \end{bmatrix},$$

$$C_1^T = \begin{bmatrix} b_0 - a_0 b_n - \widehat{a}_{20}c_2 - \widehat{a}_{30}c_3 - ... - \widehat{a}_{n,0}c_n\\ \vdots\\ b_{n-2} - a_{n-2}b_n - \widehat{a}_{n,n-2}c_n\\ b_{n-1} - a_{n-1}b_n \end{bmatrix},$$

$$D_1 = [b_n] \tag{39a}$$

or

$$A_2 = A_1^T, \quad B_2 = C_1^T, \quad C_2 = B_1^T, \quad D_2 = D_1 \tag{39b}$$

of the transfer function (11) with only real poles $\alpha_1$, $\alpha_2$, …, $\alpha_n$ if and only if the conditions

$$\begin{aligned}c_n &= b_{n-1} - a_{n-1}b_n \geq 0\\ c_{n-1} &= b_{n-2} - a_{n-2}b_n - \widehat{a}_{n,n-2}c_n \geq 0\\ &\vdots\\ c_1 &= b_0 - a_0 b_n - \widehat{a}_{20}c_2 - \widehat{a}_{30}c_3 - ... - \widehat{a}_{n,0}c_n \geq 0\end{aligned} \tag{40a}$$

where

$$\begin{aligned}\widehat{a}_{20} &= -\alpha_1, \quad \widehat{a}_{30} = \alpha_1\alpha_2,\\ \widehat{a}_{n,0} &= (-1)^{n-1}\alpha_1\alpha_2...\alpha_{n-1}, ...,\\ \widehat{a}_{31} &= -(\alpha_1 + \alpha_2), ..., \widehat{a}_{n,n-2}\\ &= -(\alpha_1 + \alpha_2 + ... + \alpha_{n-1})\end{aligned} \tag{40b}$$

are met.

**Proof.** The matrix $A_1 \in \Re_+^{n\times n}$ is asymptotically stable if and only if its eigenvalues $z_k = \alpha_k$, $k$ = 1,2,…,$n$ are only real positive and satisfy the condition (35).The matrix

$$D_1 = \lim_{z\to\infty} T(z) = [b_n] \in \Re_+^{1\times 1} \tag{41}$$

if and only if $b_n \geq 0$. The strictly proper transfer function has the form

$$\begin{aligned}T_{sp}(z) &= T(z) - D_1\\ &= \frac{\overline{b}_{n-1}z^{n-1} + ... + \overline{b}_1 z + \overline{b}_0}{z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0},\end{aligned} \tag{42a}$$

where

$$\overline{b}_k = b_k - a_k b_n \quad \text{for} \quad k = 0, 1, \ldots, n-1. \tag{42b}$$

Assuming $B_1^T = [\ 0 \quad ... \quad 0 \quad 1\ ] \in \Re_+^{n\times 1}$ we obtain

$$T_{sp}(z) = C_1[I_n z - A_1]^{-1}B_1 = [\ c_1 \quad ... \quad c_n\ ]$$
$$\begin{bmatrix} z-\alpha_1 & -1 & 0 & ... & 0 \\ 0 & z-\alpha_2 & -1 & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ 0 & 0 & 0 & ... & z-\alpha_n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{43a}$$
$$= \frac{[\ c_1 \quad ... \quad c_n\ ]}{d_n(z)} \begin{bmatrix} p_1(z) \\ p_2(z) \\ \vdots \\ p_n(z) \end{bmatrix}$$
$$= \frac{c_1p_1(z) + c_2p_2(z) + ... + c_np_n(z)}{d_n(z)}$$

where

$$\begin{gathered} d_n(z) = (z-\alpha_1)(z-\alpha_2)...(z-\alpha_k) \\ = z^n + (-1)^1\widetilde{a}_{n-1}z^{n-1} + (-1)^2\widetilde{a}_{n-2}z^{n-2} + ... + (-1)^n\widetilde{a}_0 \\ p_1(z) = 1, \\ p_2(z) = z - \alpha_1 = z + \widehat{a}_{20}, \\ \widehat{a}_{20} = -\alpha_1, \\ p_3(z) = (z-\alpha_1)(z-\alpha_2) = z^2 + \widehat{a}_{31}z + \widehat{a}_{30}, \\ \widehat{a}_{31} = -(\alpha_1+\alpha_2), \quad \widehat{a}_{30} = \alpha_1\alpha_2, \\ \vdots \\ p_n(z) = (z-\alpha_1)(z-\alpha_2)...(z-\alpha_{n-1}) \\ = z^{n-1} + \widehat{a}_{n,n-2}z^{n-2} + ... + \widehat{a}_{n,1}z + \widehat{a}_{n,0}, \\ \widehat{a}_{n,n-2} = -(\alpha_1+\alpha_2+...+\alpha_{n-1}), ..., \\ \widehat{a}_{n,0} = (-1)^{n-1}\alpha_1\alpha_2..., \alpha_{n-1}. \end{gathered} \tag{43b}$$

From comparison of (3.33a) and (3.32a) we have

$$\begin{gathered} c_n = \overline{b}_{n-1} = b_{n-1} - a_{n-1}b_n, \\ c_{n-1} = \overline{b}_{n-2} - \widehat{a}_{n,n-2}c_n = b_{n-2} - a_{n-2}b_n - \widehat{a}_{n,n-2}c_n, \\ \vdots \\ c_1 = \overline{b}_0 - \widehat{a}_{20}c_2 - \widehat{a}_{30}c_3 - ... - \widehat{a}_{n,0}c_n \\ = b_0 - a_0b_n - \widehat{a}_{20}c_2 - \widehat{a}_{30}c_3 - ... - \widehat{a}_{n,0}c_n. \end{gathered} \tag{44}$$

From (44) it follows that $C_1 \in \Re_+^{1\times n}$ if and only if the conditions (40a) are met. The proof for (39b) follows immediately from the equality

$$\begin{aligned} T(z) = [T(z)]^T &= [C_1[I_nz - A_1]^{-1}B_1 + D_1]^T \\ &= B_1^T[I_nz - A_1^T]^{-1}C_1^T + D_1 \\ &= C_2[I_nz - A_2]^{-1}B_2 + D_2. \end{aligned} \tag{45}$$

By Lemma 2 the matrices (38) are a positive asymptotically stable realization of (11) for any monomial matrix $P \in \Re_+^{n\times n}$ if and only if the matrices (39) are its positive asymptotically stable realization.

From above considerations we have the following procedure for computation of the set of positive asymptotically stable realizations (38) of the transfer function (11) with real negative poles.

**Procedure 1.**

Step 1. Check if the denominator

$$\begin{aligned} d_n(z) &= (z-\alpha_1)(z-\alpha_2)...(z-\alpha_k) \\ &= z^n + (-1)^1\widetilde{a}_{n-1}z^{n-1} \\ &+ (-1)^2\widetilde{a}_{n-2}z^{n-2} + ... + (-1)^n\widetilde{a}_0 \end{aligned} \tag{46}$$

of the transfer function (11) satisfies the conditions (31). If the conditions are satisfied go to the step 2, if not then does not exist the set of positive asymptotically stable realizations (38).

Step 2. Check if all coefficients of the polynomial (36) are positive. If the conditions are satisfied go to the step 4, if not then does not exist the set of positive asymptotically stable realizations (38).

Step 3. Check the conditions (40). If the conditions are satisfied go to the step 4, if not then does not exist the set of positive asymptotically stable realizations (38).

Step 4. Compute the zeros $\alpha_k$, $k = 1, 2, \dots, n$ of the polynomial 46 and find the matrices (39a) or (39b).

Step 5. Using (38) compute the desired set of realizations.

**Example 1.** Compute the set of positive asymptotically stable realizations of the transfer function

$$T(z) = \frac{0.1z^3 + z^2 + 2z + 3}{z^3 - 1.1z^2 + 0.35z - 0.025}. \tag{47}$$

Using Procedure 1 we obtain the following.

Step 1. The denominator

$$d_3(z) = z^3 - 1.1z^2 + 0.35z - 0.025 \tag{48}$$

of (47) satisfies the conditions (32) since $\widetilde{a}_2 = 1.1 > 0$, $\widetilde{a}_1 = 0.35 > 0$ and $\widetilde{a}_0 = 0.025 > 0$.

Step 2. All coefficients of the polynomial

$$\begin{aligned} \overline{d}_3(z) = d_3(z+1) &= (z+1)^3 \\ -1.1(z+1)^2 &+ 0.35(z+1) - 0.025 \\ &= z^3 + 1.9z^2 + 1.15z + 0.225 \end{aligned} \tag{49}$$

are positive.

Step 3. The conditions (40) are also met since

$$\begin{gathered} c_3 = b_2 - a_2b_3 = 1.11 > 0, \\ c_2 = b_1 - a_1b_3 - \widehat{a}_{31}c_3 = 3.075 > 0, \\ c_1 = b_0 - a_0b_3 - \widehat{a}_{20}c_2 - \widehat{a}_{30}c_3 = 4.2625 > 0. \end{gathered} \tag{50}$$

Step 4. The zeros of the polynomial (48) are $z_1 = z_2 = \alpha_1 = \alpha_2 = 0.5$, $z_3 = \alpha_3 = 0.1$. Using (39a) and (50) we obtain

$$A_1 = \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C_1 = [\ 4.2625 \quad 3.075 \quad 1.11\ ], \quad D_1 = [0.1]. \tag{51}$$

Step 3. The desired set of realizations is given by

$$\overline{A}_1 = P \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0.1 \end{bmatrix} P^{-1},$$
$$\overline{B}_1 = P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{52}$$
$$\overline{C}_1 = [\ 4.2625 \quad 3.075 \quad 1.11\ ]P^{-1},$$
$$\overline{D}_1 = D_1 = [0.1]$$

for any monomial matrix $P \in \Re_+^{3\times 3}$.

**Remark 2.** If the conditions (31), (40) are met and the conditions (37) are not satisfied then there exists the set of positive but unstable realizations (38) of the transfer function (11).

**Example 2.** The transfer function

$$T(z) = \frac{z^3 - z^2 + 2z}{z^3 - 3z^2 + 2.25z - 0.5} \tag{53}$$

satisfies the conditions (32) and (40) since $\widetilde{a}_2 = 3 > 0$, $\widetilde{a}_1 = 2.25 > 0$, $\widetilde{a}_0 = 0.5 > 0$ and

$$\begin{aligned} c_3 &= b_2 - a_2 b_3 = 2 > 0, \\ c_2 &= b_1 - a_1 b_3 - \widehat{a}_{31} c_3 = 1.75 > 0, \\ c_1 &= b_0 - a_0 b_3 - \widehat{a}_{20} c_2 - \widehat{a}_{30} c_3 = 0.875 > 0. \end{aligned} \tag{54}$$

The conditions (37) are not satisfied since

$$\begin{aligned} \overline{d}_3(z) &= (z+1)^3 - 3(z+1)^2 + 2.25(z+1) - 0.5 \\ &= z^3 - 0.75z - 0.25. \end{aligned} \tag{55}$$

The poles of (53) are $z_1 = z_2 = \alpha_1 = \alpha_2 = 0.5$, $z_3 = \alpha_3 = 2$. Using (39a) and (54) we obtain

$$A_1 = \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C_1 = [\ 0.875 \quad 1.75 \quad 2\ ], \quad D_1 = [1]. \tag{56}$$

and the set of positive but unstable realizations is given by

$$\overline{A}_1 = P \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 2 \end{bmatrix} P^{-1}, \quad \overline{B}_1 = P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$\overline{C}_1 = [\ 0.875 \quad 1.75 \quad 2\ ]P^{-1}, \quad \overline{D}_1 = D_1 = [1] \tag{57}$$

for any monomial matrix $P \in \Re_+^{3\times 3}$.

## 4. Systems with complex conjugate poles

In this section the single-input single-output linear discrete-time system with the proper transfer function (11) having at least one pair of complex conjugate poles will be considered.

First we shall consider the system with the transfer function

$$T(z) = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0} \tag{58}$$

having one real pole $z_1 = \alpha$ and a pair of complex conjugate poles $z_2 = \alpha_1 + j\beta_1$, $z_3 = \alpha_1 - j\beta_1$, i.e.

$$\begin{aligned} d_3(z) &= (z-\alpha)(z-\alpha_1+j\beta_1)(z-\alpha_1-j\beta_1) \\ &= z^3 + a_2 z^2 + a_1 z + a_0, \end{aligned} \tag{59a}$$

where

$$\begin{aligned} a_2 = -2\alpha_1 - \alpha, \qquad a_1 = \alpha_1^2 + \beta_1^2 + 2\alpha\alpha_1, \\ a_0 = -\alpha(\alpha_1^2 + \beta_1^2). \end{aligned} \tag{59b}$$

**Lemma 5.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \Re_+^{3\times 3} \tag{60}$$

and

$$\begin{aligned} d_3(z) &= \det[I_3 z - A] \\ &= \begin{vmatrix} z-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & z-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & z-a_{33} \end{vmatrix} \\ &= z^3 + a_2 z^2 + a_1 z + a_0 \end{aligned} \tag{61}$$

The eigenvalues $z_1$,$z_2$,$z_3$ of the matrix (60) are located in the open unit circle on the complex plane $z$ if and only if all coefficients of the polynomial

$$\overline{d}_3(w) = d_3(w+1) = w^3 + \overline{a}_2 w^2 + \overline{a}_1 w + \overline{a}_0 \tag{62}$$

are positive, i.e.

$$\begin{aligned} \overline{a}_2 &= 3 + a_2 > 0, \\ \overline{a}_1 &= 3 + 2a_2 + a_1 > 0, \\ \overline{a}_0 &= a_0 + a_1 + a_2 + 1 > 0. \end{aligned} \tag{63}$$

**Proof.** It is well-known [1, 9] that the matrix (60) is asymptotically stable (Schur matrix) if and only if the matrix $A - I_3$ is an asymptotically stable Metzler matrix, i.e. $A - I_3 \in M_{3s}$ and this matrix is asymptotically stable if and only if all coefficients of its characteristic polynomial are positive. Using (60) and (63) we obtain

$$\begin{aligned} \overline{d}_3(w) &= d_3(w+1) = \det[I_3(w+1) - A] \\ &= (w+1)^3 + a_2(w+1)^2 + a_1(w+1) + a_0 \\ &= w^3 + (3+a_2)w^2 + (3+2a_2+a_1)w + a_0 \\ &\quad + a_1 + a_2 + 1 = w^3 + \overline{a}_2 w^2 + \overline{a}_1 w + \overline{a}_0 \end{aligned} \tag{64}$$

and the conditions (63).

**Lemma 6.** The characteristic polynomial (61) of asymptotically stable matrix (60) with positive trace, i.e.

$$\text{trace}\, A = a_{11} + a_{22} + a_{33} > 0, \tag{65}$$

has $a_2 < 0$ satisfying the condition

$$3 + a_2 > 0. \tag{66}$$

**Proof.** It is well-known that

$$a_2 = -\text{trace } A = -(a_{11} + a_{22} + a_{33}). \tag{67}$$

From (65) it follows if (62) holds then $a_2 < 0$. By conditions (63) of Lemma 5 the matrix (60) is asymptotically stable only if the condition (66) is satisfied.

**Remark 3.** From (59b) it follows that $a_2 < 0$ if and only if $\alpha > -2\alpha_1$.

The characteristic polynomial of

$$A_1 = \begin{bmatrix} a_{11} & 1 & a_{13} \\ 0 & a_{22} & a_{23} \\ 1 & 0 & a_{33} \end{bmatrix} \in \Re_+^{3\times 3} \tag{68}$$

has the form

$$\begin{aligned} \det[I_3 z - A_1] &= \begin{vmatrix} z - a_{11} & -1 & -a_{13} \\ 0 & z - a_{22} & -a_{23} \\ -1 & 0 & z - a_{33} \end{vmatrix} \\ &= z^3 + a_2 z^2 + a_1 z + a_0, \end{aligned} \tag{69}$$

where

$$\begin{aligned} a_2 &= -(a_{11} + a_{22} + a_{33}), \\ a_1 &= a_{11}(a_{22} + a_{33}) + a_{22}a_{33} - a_{13}, \\ a_0 &= -a_{11}a_{22}a_{33} + a_{22}a_{13} - a_{23}. \end{aligned} \tag{70}$$

Knowing $a_0$, $a_2$, $a_3$ and choosing $a_{11}$, $a_{22}$, $a_{33}$ so that $a_{11} + a_{22} + a_{33} = -a_2$ from (70) we may find

$$\begin{aligned} a_{13} &= a_{11}(a_{22} + a_{33}) + a_{22}a_{33} - a_1, \\ a_{23} &= -a_{11}a_{22}a_{33} + a_{22}a_{13} - a_0. \end{aligned} \tag{71}$$

**Theorem 8.** There exists the set of positive asymptotically stable realizations

$$\begin{aligned} \overline{A}_k &= PA_kP^{-1} \in \Re_+^{n\times n}, \\ \overline{B}_k &= PB_k \in \Re_+^{n\times 1}, \\ \overline{C}_k &= C_kP^{-1} \in \Re_+^{1\times n}, \\ \overline{D}_k &= D_k = [b_3] \in \Re_+^{1\times 1}, \quad k = 1, 2 \end{aligned} \tag{72}$$

for any monomial matrix $P \in \Re_+^{3\times 3}$ and the matrices $A_k$, $B_k$, $C_k$, $D_k$ having one of the forms

$$\begin{gathered} A_1 = \begin{bmatrix} a_{11} & 1 & a_{13} \\ 0 & a_{22} & a_{23} \\ 1 & 0 & a_{33} \end{bmatrix}, \\ B_1 = \begin{bmatrix} b_1 + (a_{11} + a_{22})b_2 - [(a_{11} + a_{22})a_2 + a_1]b_3 \\ b_0 + a_{22}b_1 + a_{22}^2 b_2 - (a_0 + a_{22}^2 a_2 + a_{22}a_1)b_3 \\ b_2 - a_2 b_3 \end{bmatrix}, \\ C_1 = [\ 0 \quad 0 \quad 1\ ], \quad D_1 = [b_3] \end{gathered} \tag{73a}$$

or

$$A_2 = A_1^T, \quad B_2 = C_1^T, \quad C_2 = B_1^T, \quad D_2 = D_1 \tag{73b}$$

of the transfer function (58) if and only if

$$a_2^2 - 3a_1 > 0, \qquad 2a_2^3 - 9a_1a_2 - 27a_0 > 0, \tag{74a}$$

$$\begin{gathered} a_2 < 0, \qquad 3 + a_2 > 0, \quad 3 + 2a_2 + a_1 > 0, \\ a_0 + a_1 + a_2 + 1 > 0 \end{gathered} \tag{74b}$$

and

$$\begin{gathered} b_1 + (a_{11} + a_{22})b_2 - [(a_{11} + a_{22})a_2 + a_1]b_3 \geq 0, \\ b_0 + a_{22}b_1 + a_{22}^2 b_2 - (a_0 + a_{22}^2 a_2 + a_{22}a_1)b_3 \geq 0, \\ b_2 - a_2 b_3 \geq 0. \end{gathered} \tag{75}$$

**Proof.** By Lemma 5 and 5 the matrix $A_1$ corresponding to the denominator (61) of (58) is asymptotically stable if and only if the conditions (74b) are met. It is well-known [7] that the function $a_{11}(a_{22}+a_{33})+a_{22}a_{33}$ for $a_{11}+a_{22}+a_{33} = a_2$ reach its maximal values for

$$a_{11} = a_{22} = a_{33} = \frac{a_2}{3}.$$

From (71) we obtain

$$a_{13} = 3\left(\frac{a_2}{3}\right)^2 - a_1 \geq 0 \quad \text{or} \quad a_2^2 - 3a_1 > 0.$$

Similarly

$$\begin{aligned} a_{23} &= -\left(\frac{a_2}{3}\right)^3 + \left(\frac{a_2}{3}\right)\left(\frac{a_2^2}{3} - a_1\right) - a_0 \\ &= 2\left(\frac{a_2}{3}\right)^3 - \frac{a_1a_2}{3} - a_0 \geq 0 \\ &\text{or} \quad 2a_2^3 - 9a_1a_2 - 27a_0 > 0. \end{aligned}$$

Therefore, there exist $a_{13} > 0$ and $a_{23} > 0$ if and only if the conditions (74a) are satisfied. The matrix

$$D_1 = \lim_{z\to\infty} T(z) = [b_3] \in \Re_+^{1\times 1} \tag{76}$$

if and only if $b_3 \geq 0$. The strictly proper transfer function has the form

$$T_{sp}(z) = T(z) - D_1 = \frac{\overline{b}_2 z^2 + \overline{b}_1 z + \overline{b}_0}{z^3 + a_2z^2 + a_1z + a_0} \tag{77a}$$

where

$$\overline{b}_2 = b_2 - a_2b_3, \quad \overline{b}_1 = b_1 - a_1b_3, \quad \overline{b}_0 = b_0 - a_0b_3. \tag{77b}$$

Assuming $C_1 = [\ 0 \quad 0 \quad 1\ ]$ we obtain

$$\begin{aligned} T_{sp}(z) &= C_1[I_3z - A_1]^{-1}B_1 \\ &= [\ 0 \quad 0 \quad 1\ ] \begin{bmatrix} z - a_{11} & -1 & -a_{13} \\ 0 & z - a_{22} & -a_{23} \\ -1 & 0 & z - a_{33} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix} \\ &= \frac{[\ z - a_{22} \quad 1 \quad z^2 - (a_{11} + a_{22})z + a_{11}a_{22}\ ]}{z^3 + a_2z^2 + a_1z + a_0} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix} \\ &= \frac{b_{13}z^2 + [b_{11} - (a_{11} + a_{22})b_{13}]z + b_{12} - a_{22}b_{11} + a_{11}a_{22}b_{13}}{z^3 + a_2z^2 + a_1z + a_0}. \end{aligned} \tag{78}$$

Comparison of (77a) and (78) yields

$$\begin{aligned} b_{13} &= \overline{b}_2 = b_2 - a_2 b_3, \\ b_{11} &= \overline{b}_1 + (a_{11} + a_{22}) b_{13} \\ &= b_1 + (a_{11} + a_{22}) b_2 - [a_2(a_{11} + a_{22}) + a_1] b_3, \\ b_{12} &= \overline{b}_0 + a_{22} b_{11} - a_{11} a_{22} b_{13} = b_0 + a_{22} b_1 \\ &+ a_{22}^2 b_2 - (a_0 + a_{22}^2 a_2 + a_{22} a_1) b_3. \end{aligned} \tag{79}$$

From (73a) it follows that $B_1 \in \Re_+^{3\times 1}$ if and only if the conditions (75) are met. The proof for (73b) follows immediately from the equality (45). By Lemma 2 the matrices (72) are a positive asymptotically stable realization for any monomial matrix $P \in \Re_+^{3\times 3}$ of (48) if and only if the matrices (73) are its positive asymptotically stable realization.

**Remark 4.** If $|a_0| + |a_1| + |a_2| < 1$ then the matrix $A_1$ is asymptotically stable and the conditions (74b) are met.

From considerations we have the following procedure for computation of the set of positive asymptotically stable realizations (72) for the transfer function (58).

**Procedure 2.**

Step 1. Check the conditions (74) and (75). If the conditions are met, go to Step 2, if not then does not exist the set of realizations (72) of (58).

Step 2. Using (71) and (73a) compute $a_{13}$, $a_{23}$ and the matrices $A_1$, $B_1$, $C_1$, $D_1$.

Step 3. Using (72) compute the desired set of realizations.

**Example 3.** Compute the set of positive asymptotically table realizations of the transfer function

$$T(z) = \frac{4z^3 - z^2 + 2z - 0.1}{z^3 - 0.4z^2 - 0.03z - 0.232}. \tag{80}$$

Using Procedure 2 we obtain the following.

Step 1. The transfer function (80) satisfies the conditions (64) and (65) since

$$\begin{aligned} a_2^2 - 3a_1 &= 0.25 > 0, \\ 2a_2^3 - 9a_1 a_2 - 27a_0 &= 6.028 > 0, \end{aligned} \tag{81a}$$

$$\begin{aligned} a_2 &= -0.4 < 0, \\ 3 + a_2 &= 2.6 > 0, \\ 3 + 2a_2 + a_1 &= 2.17 > 0, \\ a_0 + a_1 + a_2 + 1 &= 0.338 > 0 \end{aligned} \tag{81b}$$

and

$$\begin{aligned} b_2 - a_2 b_3 &= 0.6 > 0, \\ b_1 + (a_{11} &+ a_{22}) b_2 \\ -[a_2(a_{11} + a_{22}) + a_1] b_3 &= 2.24 > 0, \\ b_0 + a_{22} b_1 &+ a_{22}^2 b_2 \\ -(a_0 + a_{22}^2 a_2 + a_{22} a_1) b_3 &= 1.022 > 0. \end{aligned} \tag{81c}$$

Step 2. Using (68) and (77) we obtain

$$\begin{aligned} &a_{11} = a_{22} = 0.1, \qquad a_{33} = 0.2, \\ &a_{13} = 0.08, \qquad a_{23} = 0.238, \\ &A_1 = \begin{bmatrix} 0.1 & 1 & 0.8 \\ 0 & 0.1 & 0.238 \\ 1 & 0 & 0.2 \end{bmatrix}, \\ &B_1 = \begin{bmatrix} 2.24 \\ 1.022 \\ 0.6 \end{bmatrix}, \\ &C_1^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D_1 = [4]. \end{aligned} \tag{82}$$

Step 3. Using (68) and (78) we obtain the desired set of realizations

$$\begin{aligned} \overline{A}_1 &= P A_1 P^{-1}, & \overline{B}_1 &= P B_1, \\ \overline{C}_1 &= C_1 P^{-1}, & \overline{D}_1 &= D_1 \end{aligned} \tag{83}$$

for any monomial matrix $P \in \Re_+^{3\times 3}$.

Note that the set of realizations (83) depends on three arbitrary parameters which are the entries of the matrix $P$. The set of realizations depends on five parameters if we choose $a_{11} = p_1$, $a_{22} = p_2$, $a_{33} = a_2 - p_1 - p_2$. In this case using (71) we obtain

$$\begin{aligned} a_{13} &= p_1(a_2 - p_1) + p_2(a_2 - p_1 - p_2) - a_1, \\ a_{23} &= -(a_2 - p_1 - p_2) p_1 p_2 + p_2[p_1(a_2 - p_1) \\ &+ p_2(a_2 - p_1 - p_2) - a_1] - a_0 \end{aligned} \tag{84}$$

and the matrices $A_1$, $B_1$, $C_1$, $D_1$ have the forms

$$\begin{aligned} A_1 &= \begin{bmatrix} p_1 & 1 & a_{13} \\ 0 & p_2 & a_{23} \\ 1 & 0 & a_2 - p_1 - p_2 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} b_1 + (p_1 + p_2) b_2 - [a_2(p_1 + p_2) + a_1] b_3 \\ b_0 + p_2 b_1 + p_2^2 b_2 - (a_0 + a_2 p_2^2 + a_1 p_2) b_3 \\ b_2 - a_2 b_3 \end{bmatrix}, \\ C_1 &= [\begin{matrix} 0 & 0 & 1 \end{matrix}], \quad D_1 = [b_3] \end{aligned} \tag{85}$$

where $0 < a_2 - p_1 - p_2 < a_2$.

**Remark 5.** The matrix $A_1$ in Theorem 8 can be replaced by the matrices

$$\begin{aligned} A_3 &= \begin{bmatrix} p_1 & 0 & 1 \\ a_{21} & p_2 & 0 \\ a_{31} & 1 & a_2 - p_1 - p_2 \end{bmatrix}, \\ A_4 &= \begin{bmatrix} p_1 & a_{12} & 0 \\ 0 & p_2 & 1 \\ 1 & a_{32} & a_2 - p_1 - p_2 \end{bmatrix} \end{aligned} \tag{86}$$

and the matrix $A_2$ by $A_3^T$, $A_4^T$. For $A_3$ the matrices $B_3$, $C_3$ have the forms

$$B_3 = \begin{bmatrix} b_2 - a_2 b_3 \\ b_0 + p_1 b_1 + p_1^2 b_2 + (a_1 p_1 - a_0 - p_1^2 a_2) b_3 \\ b_1 - (p_1 + p_2) b_2 + [(p_1 + p_2) a_2 - a_1] b_3 \end{bmatrix}, \quad C_3 = [\ 1 \quad 0 \quad 0\ ] \tag{87}$$

and the $A_4$ the matrices $B_4$, $C_4$ have the forms

$$B_4 = \begin{bmatrix} b_0 + (p_1 + p_2 - a_2) b_1 + [(p_1 + p_2)^2 - a_2^2] b_2 \\ +(p_1 + p_2 - a_2)[a_2^2 - a_1 - a_2(p_1 + p_2)] b_3 \\ b_2 - a_2 b_3 \\ b_1 + (p_1 + a_2) b_2 + (a_2^2 - a_2 p_1 - a_1) b_3 \end{bmatrix}, \quad C_4 = [\ 0 \quad 1 \quad 0\ ]. \tag{88}$$

## 5. General case of SISO systems

In general case it is assumed that the transfer function (11) has at least one pair of complex conjugate poles.

**Theorem 9.** There exists the set of positive asymptotically stable realizations

$$\begin{aligned} \overline{A}_k &= P A_k P^{-1} \in \Re_+^{n \times n}, & \overline{B}_k &= P B_k \in \Re_+^{n \times 1}, \\ \overline{C}_k &= C_k P^{-1} \in \Re_+^{1 \times n}, & \overline{D}_k &= D_k \in \Re_+^{1 \times 1} \end{aligned} \tag{89}$$

for any monomial matrix $P \in \Re_+^{n \times n}$ and $A_k$, $B_k$, $C_k$, $D_k$ having one of the forms

$$A_1 = \begin{bmatrix} p_1 & 1 & 0 & \dots & 0 & a_{1,n} \\ 0 & p_2 & 1 & \dots & 0 & a_{2,n} \\ 0 & 0 & p_3 & \dots & 0 & a_{3,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{n-2,n} \\ 0 & 0 & 0 & \dots & p_{n-1} & a_{n-1,n} \\ 1 & 0 & 0 & \dots & 0 & a_{n-1} - p_1 - \dots - p_{n-1} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} b_{n-1} - a_{n-1} b_n \\ b_{n-2} - a_{n-2} b_n - \widehat{a}_{n-1,n-2} b_{1,n} \\ \vdots \\ b_0 - a_0 b_n - \widehat{a}_{n,0} b_{1,n} - \widehat{a}_{10} b_{11} - \dots - \widehat{a}_{n-2,0} b_{1,n-2} \end{bmatrix},$$

$$C_1^T = [\ 0 \quad \dots \quad 0 \quad 1\ ], \quad D_1 = [b_n] \tag{90a}$$

where $0 < p_1 + p_2 + \ldots + p_{n-1} < -a_{n-1}$ or

$$A_2 = A_1^T, \quad B_2 = C_1^T, \quad C_2 = B_1^T, \quad D_2 = D_1 \tag{90b}$$

and

$$\begin{aligned} a_{1,n} &= -p_1(a_{n-1} + p_1) - p_2(a_{n-1} + p_1 + p_2) \\ &\quad - \dots - p_{n-1}(a_{n-1} + p_1 + \dots + p_{n-1}) - a_{n-2}, \\ &\vdots \\ a_{n-1,n} &= p_1 \dots p_{n-1}(a_{n-1} + p_1 + \dots + p_{n-1}) \\ &\quad - \widehat{a}_{10} a_{1,n} - \dots - \widehat{a}_{n-2,0} a_{n-2,n} \end{aligned} \tag{91}$$

of the transfer function (11) if and only if the coefficients of its denominator

$$d_n(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 \tag{92}$$

satisfies the conditions

$$n + a_{n-1} > 0, \dots, a_0 + a_1 + \dots + a_{n-1} + 1 > 0 \tag{93a}$$

$$\begin{gathered} C_2^n \left( \frac{a_{n-1}}{n} \right)^2 - a_{n-2} \geq 0, \\ C_3^n \left( \frac{a_{n-1}}{n} \right)^3 - \left[ C_2^n \left( \frac{a_{n-1}}{n} \right)^2 - a_{n-2} \right] \\ \cdot C_1^{n-2} \left( \frac{a_{n-1}}{n} \right) - a_{n-3} \geq 0, \\ \vdots \\ C_n^n \left( \frac{a_{n-1}}{n} \right)^n - \left[ C_2^n \left( \frac{a_{n-1}}{n} \right)^2 - a_{n-2} \right] \\ \cdot C_1^{n-2} \left( \frac{a_{n-1}}{n} \right)^{n-2} - \dots - C_1^1 \left( \frac{a_{n-1}}{n} \right) - a_0 \geq 0 \\ C_k^n = \frac{n!}{k!(n-k)!} \end{gathered} \tag{93b}$$

and

$$\begin{gathered} b_{n-1} - a_{n-1} b_n \geq 0, \\ b_{n-2} - a_{n-2} b_n - \widehat{a}_{n-1,n-2} b_{1,n} \geq 0, \\ \vdots \\ b_0 - a_0 b_n - \widehat{a}_{n,0} b_{1,n} - \widehat{a}_{10} b_{11} - \dots - \widehat{a}_{n-2,0} b_{1,n-2} \geq 0, \end{gathered} \tag{94a}$$

where

$$\begin{aligned} \widehat{a}_{1,n-3} &= -(a_{22} + a_{33} + \dots + a_{n-1,n-1}), \dots, \widehat{a}_{10} \\ &= (-1)^{n-2} a_{22} a_{33} \dots a_{n-1,n-1}, \\ \widehat{a}_{2,n-4} &= -(a_{33} + a_{44} + \dots + a_{n-1,n-1}), \dots, \widehat{a}_{20} \\ &= (-1)^{n-3} a_{33} a_{44} \dots a_{n-1,n-1}, \\ &\vdots \\ \widehat{a}_{n,n-2} &= -(a_{11} + a_{22} + \dots + a_{n-1,n-1}), \dots, \widehat{a}_{n,0} \\ &= (-1)^{n-1} a_{11} a_{22} \dots a_{n-1,n-1}. \end{aligned} \tag{94b}$$

**Proof.** The matrix $\overline{A}_k \in \Re_+^{n \times n}$ corresponding to the denominator (46) is asymptotically stable if and only if all coefficients of the polynomial

$$\begin{aligned} \overline{d}_n(w) &= d_n(w+1) = \det[I_n(w+1) - A_1] \\ &= (w+1)^n + a_{n-1}(w+1)^{n-1} + \dots + a_1(w+1) + a_0 \\ &= w^n + \overline{a}_{n-1} w^{n-1} + \dots + \overline{a}_1 w + \overline{a}_0 \end{aligned} \tag{95}$$

are positive, i.e. (93a) holds. The characteristic polynomial of $A_1$ has the form

$$\det[I_n z - A_1] = \begin{vmatrix} z-a_{11} & -1 & 0 & ... & 0 & -a_{1,n} \\ 0 & z-a_{22} & -1 & ... & 0 & -a_{2,n} \\ 0 & 0 & z-a_{33} & ... & 0 & -a_{3,n} \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & -1 & -a_{n-2,n} \\ 0 & 0 & 0 & ... & z-a_{n-1,n-1} & -a_{n-1,n} \\ -1 & 0 & 0 & ... & 0 & z-a_{n,n} \end{vmatrix}$$

$$= (z-a_{11})(z-a_{22})...(z-a_{n,n}) + (-1)^{n+2} \begin{vmatrix} -1 & 0 & ... & 0 & -a_{1,n} \\ z-a_{22} & -1 & ... & 0 & -a_{2,n} \\ 0 & z-a_{33} & ... & 0 & -a_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & -1 & -a_{n-2,n} \\ 0 & 0 & ... & z-a_{n-1,n-1} & -a_{n-1,n} \end{vmatrix} \tag{96a}$$

$$= (z-a_{11})(z-a_{22})...(z-a_{n,n}) - a_{1,n}(z-a_{22})(z-a_{33})...(z-a_{n-1,n-1})$$

$$-a_{2,n}(z-a_{33})(z-a_{44})...(z-a_{n-1,n-1}) - ... - a_{n-2,n}(z-a_{n-1,n-1}) - a_{n-1,n}$$

$$= z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0,$$

where

$$\begin{gathered} a_{n-1} = -(a_{11} + a_{22} + ... + a_{n,n}), \\ a_{n-2} = a_{11}(a_{22} + a_{33} + ... + a_{n,n}) \\ +a_{22}(a_{33} + a_{44} + ... + a_{n,n}) + ... \\ +a_{n-2,n-2}(a_{n-1,n-1} + a_{n,n}) \\ +a_{n-1,n-1}a_{n,n} - a_{1,n}, \\ \vdots \\ a_1 = (-1)^{n-1}a_{11}a_{22}a_{33}...a_{n-1,n-1} \\ +(-1)^{n-1}a_{11}a_{22}...a_{n-2,n-2}a_{n,n} \\ +(-1)^{n-1}a_{22}a_{33}...a_{n,n} \\ -a_{1,n}((-1)^{n-2}a_{22}a_{33}...a_{n-2,n-2} + ... \\ +(-1)^{n-3}a_{33}a_{44}...a_{n-1,n-1}) \\ -a_{2,n}((-1)^{n-4}a_{33}a_{44}...a_{n-2,n-2} \\ +... + (-1)^{n-4}a_{44}a_{55}...a_{n-1,n-1}) \\ +... + a_{n-3,n}a_{n-2,n-2} + a_{n-2,n}, \\ a_0 = (-1)^{n-1}a_{11}a_{22}...a_{n,n} \\ -(-1)^{n-2}a_{1,n}a_{22}...a_{n-1,n-1} \\ -(-1)^{n-3}a_{2,n}a_{33}...a_{n-1,n-1} \\ +a_{n-2,n}a_{n-1,n-1} - a_{n-1,n}. \end{gathered} \tag{96b}$$

From (96b) we have

$$\begin{gathered} a_{1,n} = a_{11}(a_{22} + a_{33} + ... + a_{n,n}) \\ +a_{22}(a_{33} + a_{44} + ... + a_{n,n}) + ... \\ +a_{n-2,n-2}(a_{n-1,n-1} + a_{n,n}) \\ +a_{n-1,n-1}a_{n,n} - a_{n-2} \geq 0, \\ \vdots \\ a_{n-2,n} = a_{11}a_{22}a_{33}...a_{n-1,n-1} \\ +a_{11}a_{22}...a_{n-2,n-2}a_{n,n} \\ +a_{22}a_{33}...a_{n,n} - a_{1,n}(a_{22}a_{33}...a_{n-2,n-2} \\ +... + a_{33}a_{44}...a_{n-1,n-1}) \\ -a_{2,n}(a_{33}a_{44}...a_{n-2,n-2} + ... \\ +a_{44}a_{55}...a_{n-1,n-1}) - ... \\ -a_{n-3,n}a_{n-2,n-2} - a_1 \geq 0, \\ a_{n-1,n} = a_{11}a_{22}...a_{n,n} \\ -a_{1,n}a_{22}...a_{n-1,n-1} - a_{2,n}a_{33}...a_{n-1,n-1} \\ -a_{n-2,n}a_{n-1,n-1} - a_0 \geq 0. \end{gathered} \tag{97}$$

The functions $a_{11}(a_{22}+a_{33}+...+a_{n,n})+a_{22}(a_{33}+a_{44}+...+a_{n,n})+...+a_{n-2,n-2}(a_{n-1,n-1}+a_{n,n})$, $\ldots$, $a_{11}a_{22}...a_{n,n}$ for $a_{11}+a_{22}+...+a_{n,n} = a_{n-1}$ (given) reach their maximal values if [7]

$$a_{11} = a_{22} = ... = a_{n,n} = \frac{a_{n-1}}{n}. \tag{98}$$

Substitution of (98) into (97) yields

$$\begin{gathered} a_{1,n} = C_2^n \left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2} \geq 0, \\ a_{2,n} = C_3^n \left(\frac{a_{n-1}}{n}\right)^3 - \left[C_2^n \left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2}\right] \\ \cdot C_1^{n-2} \left(\frac{a_{n-1}}{n}\right) - a_{n-3} \geq 0, \\ \vdots \\ a_{n-1,n} = C_n^n \left(\frac{a_{n-1}}{n}\right)^n - \left[C_2^n \left(\frac{a_{n-1}}{n}\right)^2 - a_{n-2}\right] \\ \cdot C_1^{n-2} \left(\frac{a_{n-1}}{n}\right)^{n-2} - ... - C_1^1 \left(\frac{a_{n-1}}{n}\right) - a_0 \geq 0. \end{gathered} \tag{99}$$

The conditions (99) are equivalent to the conditions (93b). The matrix

$$D_1 = \lim_{z \to \infty} T(z) = [b_n] \in \Re_+^{1 \times 1} \tag{100}$$

if and only if $b_n \geq 0$. The strictly proper transfer function has the form

$$T_{sp}(z) = T(z) - D_1 = \frac{\overline{b}_{n-1} z^{n-1} + ... + \overline{b}_1 z + \overline{b}_0}{z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0}, \tag{101a}$$

where

$$\overline{b}_k = b_k - a_k b_n \quad \text{for} \quad k = 1, 2, \ldots, n-1. \tag{101b}$$

Assuming $C_1 = [\ 0 \ \ ... \ \ 0 \ \ 1\ ] \in \Re_+^{1 \times n}$ we obtain

$$\begin{aligned} T_{sp}(z) &= C_1 [I_n z - A_1]^{-1} B_1 \\ &= [\ 0 \ \ ... \ \ 0 \ \ 1\ ] \begin{bmatrix} z - a_{11} & -1 & 0 & ... & 0 & -a_{1,n} \\ 0 & z - a_{22} & -1 & ... & 0 & -a_{2,n} \\ 0 & 0 & z - a_{33} & ... & 0 & -a_{3,n} \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & -1 & -a_{n-2,n} \\ 0 & 0 & 0 & ... & z - a_{n-1,n-1} & -a_{n-1,n} \\ -1 & 0 & 0 & ... & 0 & z - a_{n,n} \end{bmatrix}^{-1} \begin{bmatrix} b_{11} \\ \vdots \\ b_{1,n-1} \\ b_{1,n} \end{bmatrix} \end{aligned} \tag{102a}$$

$$= \frac{[\ p_1(z) \ \ ... \ \ p_n(z)\ ]}{d_n(z)} \begin{bmatrix} b_{11} \\ \vdots \\ b_{1,n-1} \\ b_{1,n} \end{bmatrix} = \frac{p_1(z) b_{11} + p_2(z) b_{12} + ... + p_n(z) b_{1,n}}{d_n(z)}$$

where

$$\begin{gathered} p_1(z) = (z - a_{22})(z - a_{33})...(z - a_{n-1,n-1}) = z^{n-2} + \widehat{a}_{1,n-3} z^{n-3} + ... + \widehat{a}_{11} z + \widehat{a}_{10}, \\ p_2(z) = (z - a_{33})(z - a_{44})...(z - a_{n-1,n-1}) = z^{n-3} + \widehat{a}_{2,n-4} z^{n-4} + ... + \widehat{a}_{21} z + \widehat{a}_{20}, \\ \vdots \\ p_{n-1}(z) = 1, \\ p_n(z) = (z - a_{11})(z - a_{22})...(z - a_{n-1,n-1}) = z^{n-1} + \widehat{a}_{n,n-2} z^{n-2} + ... + \widehat{a}_{n,1} z + \widehat{a}_{n,0}, \end{gathered} \tag{102b}$$

$$\begin{gathered} \widehat{a}_{1,n-3} = -(a_{22} + a_{33} + ... + a_{n-1,n-1}), ..., \widehat{a}_{10} = (-1)^{n-2} a_{22} a_{33} ... a_{n-1,n-1}, \\ \widehat{a}_{2,n-4} = -(a_{33} + a_{44} + ... + a_{n-1,n-1}), ..., \widehat{a}_{20} = (-1)^{n-3} a_{33} a_{44} ... a_{n-1,n-1}, \\ \vdots \\ \widehat{a}_{n,n-2} = -(a_{11} + a_{22} + ... + a_{n-1,n-1}), ..., \widehat{a}_{n,0} = (-1)^{n-1} a_{11} a_{22} ... a_{n-1,n-1}. \end{gathered} \tag{102c}$$

From comparison of (101a) and (102a) we have

$$\begin{aligned} b_{1,n} &= \overline{b}_{n-1} = b_{n-1} - a_{n-1}b_n, \\ b_{1,n-1} &= \overline{b}_{n-2} - \widehat{a}_{n-1,n-2}b_{1,n} \\ &= b_{n-2} - a_{n-2}b_n - \widehat{a}_{n-1,n-2}b_{1,n}, \\ &\vdots \\ b_{1,1} &= b_0 - a_0b_n - \widehat{a}_{n,0}b_{1,n} \\ &-\widehat{a}_{10}b_{11} - ... - \widehat{a}_{n-2,0}b_{1,n-2}. \end{aligned} \tag{103}$$

From (103) it follows that $B_1 \in \Re_+^{n\times 1}$ if and only if the conditions (94a) are met. The proof for (94b) follows immediately from (45). By Lemma 2 the matrices (89) are a positive asymptotically stable realization of (3.1) for any monomial matrix $P \in \Re_+^{n\times n}$ if and only if the matrices (90) are its positive asymptotically stable realization.

## 6. Concluding remarks

The problem of existence and computation of the set of positive asymptotically stable realizations of a proper transfer function of linear discrete-time systems has been formulated and solved. Necessary and sufficient conditions for existence of the set of realizations have been established (Theorems 4–9). The procedure for computation of the set of realizations for transfer functions with only real negative poles and with at least one pair of complex conjugate poles have been proposed (Procedures 1 and 2). The effectiveness of the procedures have been demonstrated on numerical examples. The presented methods can be extended to positive asymptotically stable discrete-time linear systems and also to multi-input multi-output continuous-time and discrete-time linear systems. An open problem is an existence of these considerations to fractional linear systems [5].

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REFERENCES

[1] L. Farina and S. Rinaldi, *Positive Linear Systems. Theory and Applications*, J. Wiley, New York, 2000.

[2] L. Benvenuti and L. Farina, "A tutorial on the positive realization problem", *IEEE Trans. Autom. Control* 49 (5), 651–664 (2004).

[3] T. Kaczorek, *Linear Control Systems*, vol. 1, Research Studies Press and J. Wiley, New York, 1992.

[4] T. Kaczorek, *Polynomial and Rational Matrices*, Springer-Verlag, London, 2009.

[5] T. Kaczorek, *Selected Problems in Fractional Systems Theory*, Springer-Verlag, Berlin, 2011.

[6] T. Kaczorek, "Existence and determination of the set of Metzler matrices for given stable polynomials", *Int. J. Appl. Comput. Sci.* 22 (2), 389–399 (2012).

[7] T. Kaczorek, "Positive stable realizations for fractional descriptor continuous-time linear systems", *Archives of Control Sciences* 22 (4), (2012), (to be published).

[8] U. Shaker and M. Dixon, "Generalized minimal realization of transfer-function matrices", *Int. J. Contr.* 25 (5), 785–803 (1977).

[9] T. Kaczorek, *Positive 1D and 2D Systems*, Springer-Verlag, London, 2002.

[10] T. Kaczorek, "A realization problem for positive continuous-time linear systems with reduced numbers of delays", *Int. J. Appl. Math. Comp. Sci.* 16 (3), 325–331 (2006).

[11] T. Kaczorek, "Computation of realizations of discrete-time cone systems", *Bull. Pol. Ac.: Tech.* 54 (3), 347–350 (2006).

[12] T. Kaczorek, "Computation of positive stable realizations for linear continuous-time systems", *Bull. Pol. Acad. Sci. Techn.* 59 (3), 273–281 (2011).

[13] T. Kaczorek, "Positive stable realizations of fractional continuous-time linear systems", *Int. J. Appl. Math. Comp. Sci.* 21 (4), 697–702 (2011).

[14] T. Kaczorek, "Realization problem for positive multivariable discrete-time linear systems with delays in the state vector and inputs", *Int. J. Appl. Math. Comp. Sci.* 16 (2), 101–106 (2006).

[15] T. Kaczorek, "Realization problem for positive discrete-time systems with delay", *System Science* 30 (4), 117–130 (2004).

[16] T. Kaczorek, "Positive stable realizations with system Metzler matrices", *Archives of Control Sciences* 21 (2), 167–188 (2011).

[17] T. Kaczorek, "Positive minimal realizations for singular discrete-time systems with delays in state and delays in control", *Bull. Pol. Ac.: Tech.* 53 (3), 293–298 (2005).

[18] T. Kaczorek, "Fractional positive continuous-time linear systems and their reachability", *Int. J. Appl. Math. Comput. Sci.* 18 (2), 223–228 (2008).

[19] T. Kaczorek, "Fractional positive linear systems" *Kybernetes: Int. J. Systems & Cybernetics* 38 (7/8), 1059–1078 (2009).

[20] T. Kaczorek, "Realization problem for fractional continuous-time systems", *Archives of Control Sciences* 18 (1), 43–58 (2008).

[21] T. Kaczorek, "Positive fractional 2D continuous-discrete linear systems", *Bull. Pol. Ac.: Tech.* 59 (4), 575–580 (2011).

[22] T. Kaczorek, "Stability of continuous-discrete linear systems described by the general model", *Bull. Pol. Ac.: Tech.* 59 (2), 189–193 (2011).