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SOME CHARACTERIZATIONS OF THE EXPONENTIAL DISTRIBUTION BASED ON ORDER STATISTICS

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In this paper some new characterizing theorems of the exponential distribution based on the order statistics are presented. Some existing results are generalized and the open conjecture by Arnold and Villasenor is solved.

1. INTRODUCTION

There is an abundance of characterizations of the exponential distribution and among them, a considerable part is based on the properties of order statistics. A selection of these properties can be found in [**1**], [**3**], [**7**] and [**10**]. Recently, Arnold and Villasenor [**4**] proposed a series of characterizations based on the sample of size two and stated some conjectures on their generalization. They also proposed a new method of proof which can be used when the density in question is analytic. Yanev and Chakraborty [**15**] used this method to prove two characterization theorems concerning the maximum of a sample of size three as well as the characterization based on consecutive maxima [**6**]. Obradović [**13**] considered the characterizations of the same kind which include the median of a sample of size three.

In this paper, we extend the generalizations to arbitrary order statistics. We consider the case of consecutive order statistics via convolution with independent random variables from the same distribution. Similar problems have been studied in [**14**] and [**5**], however, their formulations are slightly different, in terms that the convolution in question includes a random variable with fixed distribution.

The other case we consider is the characterization based on representation of the k-th order statistic of a sample of size n as a weighted sum of k sample

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members. This problem has a history, AHSANULLAH and RAHMAN [**2**] proved the theorem when the representation is valid for all k. Later, HUANG [**9**] showed that the condition in general cannot be relaxed to just one value of k. We prove that under the assumption of analyticity of the density function, the theorem is valid even under this relaxed condition. As a corollary, we solve the conjecture of ARNOLD and VILLASENOR regarding representation of sample maximum stated in [**4**].

2. AUXILIARY RESULTS

In this section we present four combinatorial identities that will be used in the proofs of the characterization theorems. All of them contain Stirling numbers of the second kind. A Stirling number of the second kind, denoted by $\left\{ a \atop b \right\}$, represents the number of ways to partition a set of a objects into b non-empty subsets. In the proofs of our lemmas we use the following well-known identities (see e.g. [**8**], ch. 5).

$$\left\{ a \atop b \right\} = \left\{ a-1 \atop b-1 \right\} + b\left\{ a-1 \atop b \right\}, \tag{1}$$

$$\left\{ a+1 \atop b+1 \right\} = \sum_{\ell=0}^{a} \binom{a}{\ell} \left\{ \ell \atop b \right\}, \tag{2}$$

$$\left\{ a+b+1 \atop b \right\} = \sum_{\ell=0}^{b} \ell \left\{ a+\ell \atop \ell \right\}, \tag{3}$$

$$a^b = \sum_{\ell=0}^{b} \left\{ b \atop \ell \right\} a(a-1)\cdots(a-\ell+1). \tag{4}$$

We proceed with the lemmas necessary for the proofs of the characterization theorems.

Lemma 1. *For integers k, n, r such that $1 < k \leq n$ and $r \geq 0$ it holds that*

$$\begin{aligned} &\sum_{j=k-2}^{k+r-1} \sum_{i=0}^{j-k+2} \binom{n-k}{i} (i+k-2)! \left\{ j+1 \atop i+k-1 \right\} (k-1) n^{k+r-1-j} \\ &\qquad = \sum_{i=0}^{r+1} \binom{n-k}{i} (i+k-1)! \left\{ k+r+1 \atop i+k \right\}. \end{aligned} \tag{5}$$

Proof. We prove the lemma by induction on r. For $r = 0$ the equality (5) reduces to

$$(k-1)!\left(n + \left\{ k \atop k-1 \right\} + (k-1)(n-k)\right) = (k-1)!\left(\left\{ k+1 \atop k \right\} + k(n-k)\right),$$

which is true because of (1). Thus the statement of the lemma holds for $r = 0$ for all $1 < k \leq n$.

Let us now suppose that (5) is satisfied for $r-1$ for all $1<k\leq n$. The left hand side of (5) can be split into

$$\sum_{j=k-2}^{k+r-2}\sum_{i=0}^{j-k+2}\binom{n-k}{i}(i+k-2)!\left\{{j+1 \atop i+k-1}\right\}(k-1)n^{k+r-1-j}$$
$$+\sum_{i=0}^{r+1}\binom{n-k}{i}(i+k-2)!\left\{{k+r \atop i+k-1}\right\}(k-1).$$

Using the induction hypothesis on the first summand we have that the expression above is equal to

$$\sum_{i=0}^{r}\binom{n-k}{i}(i+k-1)!\left\{{k+r \atop i+k}\right\}n+\sum_{i=0}^{r+1}\binom{n-k}{i}(i+k-2)!\left\{{k+r \atop i+k-1}\right\}(k-1). \tag{6}$$

It remains to prove that (6) is equal to

$$\sum_{i=0}^{r+1}\binom{n-k}{i}(i+k-1)!\left\{{k+r+1 \atop i+k}\right\},$$

which can be written as

$$\sum_{i=0}^{r+1}\binom{n-k}{i}(i+k-1)!\left\{{k+r \atop i+k-1}\right\}+\sum_{i=0}^{r}\binom{n-k}{i}(i+k)!\left\{{k+r \atop i+k}\right\}. \tag{7}$$

Grouping the corresponding summands from (6) and (7), we get

$$\sum_{i=0}^{r}\binom{n-k}{i}(i+k-1)!\left\{{k+r \atop i+k}\right\}(n-k-i)=\sum_{i=0}^{r+1}\binom{n-k}{i}(i+k-2)!\left\{{k+r \atop i+k-1}\right\}i.$$

The last equality is easily proved by putting $j=i+1$ in the first sum.

Lemma 2. *For integers k,n,r such that $1<k\leq n$ and $r\geq 0$ it holds that*

$$\begin{aligned}&\sum_{j=k-2}^{k+r-1}\sum_{i=0}^{j-k+2}\binom{n-k+1}{i}(i+k-2)!\left\{{j+1 \atop i+k-1}\right\}(k-1)(n-k+1)^{k+r-1-j}\\ &\qquad=\sum_{i=0}^{r+1}\binom{n-k}{i}(i+k-1)!\left\{{k+r+1 \atop i+k}\right\},\end{aligned} \tag{8}$$

The proof of this lemma is analogous to the proof of Lemma 1 so we omit it here.

Lemma 3. *For integers k, n, r such that $1 < k \le n$ and $r \ge 0$ it holds that*

$$\sum_{i=0}^{r+1}\binom{n-k}{i}\sum_{s=1}^{k}(n-k+s)\frac{(i+s-1)!}{(s-1)!}\left\{{s+r \atop i+s}\right\} = \sum_{i=0}^{r+1}\binom{n-k}{i}\frac{(i+k-1)!}{(k-1)!}\left\{{k+r+1 \atop i+k}\right\}. \tag{9}$$

Proof. We prove the lemma by induction on n. For any r and k and $n = k$ the expression (9) reduces to identity (3).

Suppose now that the equality (9) is true for any k, any r, and $n-1$. We need to prove that it is also true for n. Transforming the left hand side of (9) we get

$$\begin{aligned}
&\sum_{i=0}^{r+1}\binom{n-k}{i}\left((n-k-i)\sum_{s=1}^{k}\frac{(i+s-1)!}{(s-1)!}\left\{{s+r \atop i+s}\right\} + \sum_{s=1}^{k}\frac{(i+s)!}{(s-1)!}\left\{{s+r \atop i+s}\right\}\right)\\
&= \sum_{i=0}^{r+1}(n-k)\binom{n-k-1}{i}\sum_{s=1}^{k}\frac{(i+s-1)!}{(s-1)!}\left\{{s+r \atop i+s}\right\}\\
&\quad + \sum_{i=0}^{r+1}\binom{n-k-1}{i}\sum_{s=1}^{k}\frac{(i+s)!}{(s-1)!}\left\{{s+r \atop i+s}\right\} + \sum_{i=0}^{r+1}\binom{n-k-1}{i-1}\sum_{s=1}^{k}\frac{(i+s)!}{(s-1)!}\left\{{s+r \atop i+s}\right\}\\
&= \sum_{i=0}^{r+1}(n-k)\binom{n-k-1}{i}\sum_{s=1}^{k}\frac{(i+s-1)!}{(s-1)!}\left\{{s+r \atop i+s}\right\}\\
&\quad + \sum_{i=0}^{r+1}\binom{n-k-1}{i}\sum_{s=1}^{k}\frac{(i+s)!}{(s-1)!}\left(\left\{{s+r \atop i+s}\right\} + (i+s+1)\left\{{s+r \atop i+1+s}\right\}\right).
\end{aligned}$$

Applying (1), shifting the index s in the last inner sum, and separating the newly obtained term for $s = k+1$, the expression above becomes

$$\begin{aligned}
&\sum_{i=0}^{r+1}(n-k)\binom{n-k-1}{i}\sum_{s=1}^{k}\frac{(i+s-1)!}{(s-1)!}\left\{{s+r \atop i+s}\right\}\\
&\qquad + \sum_{i=0}^{r+1}\binom{n-k-1}{i}\sum_{s=2}^{k}\frac{(i+s-1)!}{(s-2)!}\left\{{s+r \atop i+s}\right\}\\
&\qquad + \sum_{i=0}^{r+1}\binom{n-k-1}{i}\frac{(i+k)!}{(k-1)!}\left\{{k+1+r \atop i+k+1}\right\}.
\end{aligned}$$

Grouping the first two summands together and applying the induction hypothesis to the result we get the right hand side of (9).

Lemma 4. *For integers k, n, r such that $1 < k \leq n$ and $r \geq 0$ it holds that*

$$\sum_{\substack{j_1,\ldots,j_k \geq 0 \\ j_1+\cdots+j_k=r+1}} n^{j_1}(n-1)^{j_2}\cdots(n-k+1)^{j_k} = \sum_{i=0}^{r+1} \binom{n-k}{i} \frac{(i+k-1)!}{(k-1)!} \left\{ {i+r+1 \atop i+k} \right\}. \tag{10}$$

Proof. The proof is done using the strong induction on r. For any k and n and $r = 0$ we have

$$n + (n-1) + \cdots + (n-k+1) = \left\{ {k+1 \atop k} \right\} + (n-k)k,$$

which is obviously true since $\left\{ {k+1 \atop k} \right\} = \frac{k(k+1)}{2}$. Suppose now that (10) is satisfied up to $r-1$. Thus it remains to prove that it is satisfied for r.

Splitting the sum on the left hand side of (10) into two parts: for $j_1 = 0$ and $j_1 \geq 0$, we get

$$\begin{aligned} \sum_{\substack{j_1,\ldots,j_k \geq 0 \\ j_1+\cdots+j_k=r+1}} & n^{j_1}(n-1)^{j_2}\cdots(n-k+1)^{j_k} \\ &= \sum_{\substack{j_1,\ldots,j_k \geq 0 \\ j_1+\cdots+j_k=r+1}} (n-1)^{j_2}\cdots(n-k+1)^{j_k} \\ &+ \sum_{j_1=1}^{r+1} n^{j_1} \sum_{\substack{j_2,\ldots,j_k \geq 0 \\ j_2+\cdots+j_k=r+1-j_1}} (n-1)^{j_2}\cdots(n-k+1)^{j_k}. \end{aligned} \tag{11}$$

The sum of indices in the inner sum of the second summand of (11) is smaller than $r+1$ so the induction hypothesis is applicable (in this case for $n-1$, $k-1$ and $r+1-j_1$). The first summand can be recursively split in the same manner until all indices except the last one are equal to zero. After this process (including the application of induction hypothesis) we obtain

$$\begin{aligned} \sum_{\substack{j_1,\ldots,j_k \geq 0 \\ j_1+\cdots+j_k=r+1}} & n^{j_1}(n-1)^{j_2}\cdots(n-k+1)^{j_k} = (n-k+1)^{r+1} \\ &+ \sum_{\ell=1}^{k-1}\sum_{j=1}^{r+1}(n-\ell+1)^j \sum_{i=0}^{r+1-j} \binom{n-k}{i} \frac{(i+k-\ell-1)!}{(k-\ell-1)!} \left\{ {k-\ell+r+1-j \atop i+k-\ell} \right\}. \end{aligned} \tag{12}$$

Substituting the index j with $m = k+r-1-\ell-j$ and, subsequently, the index ℓ with $s = k-\ell+1$, as well as applying the identity (4) to $(n-k+1)^{r+1}$, (12) becomes

$$\begin{aligned}
&\sum_{s=2}^{k}\sum_{i=0}^{r}\sum_{m=i+s-2}^{r+s-2}\binom{n-k}{i}(n-k+s)^{r+s-1-m}\frac{(i+s-2)!}{(s-2)!}\left\{{m+1 \atop i+s-1}\right\}\\
&\qquad+\sum_{i=0}^{r+1}\frac{(n-k+1)!}{(n-k+1-i)!}\left\{{r+1 \atop i}\right\}\\
&\quad=\sum_{s=2}^{k}\frac{(n-k+s)}{(s-1)!}\sum_{i=0}^{r}\binom{n-k}{i}\sum_{m=i+s-2}^{r+s-2}\Bigg((n-k+s)^{r+s-2-m}(s-1)\\
&\qquad\qquad\times(i+s-2)!\left\{{m+1 \atop i+s-1}\right\}\Bigg)\\
&\qquad+(n-k+1)\sum_{i=1}^{r+1}\binom{n-k}{i-1}(i-1)!\left\{{r+1 \atop i}\right\}.
\end{aligned}$$

Applying Lemma 1 to the two inner sums and grouping the summands we get

$$\sum_{s=1}^{k}\sum_{i=0}^{r}(n-k+s)\binom{n-k}{i}\frac{(i+s-1)!}{(s-1)!}\left\{{r+s \atop i+s}\right\}.$$

Applying now Lemma 3 we obtain the right hand side of (10). Hence the proof is completed.

Remark 1. It is easy to prove that the Lemmas 3 and 4 are also true for $k=1$.

Now we state and prove two lemmas that will play an important role in the proofs of the theorems. They are similar to those from [**6**].

Let $\mathcal{F}$ be a class of continuous distribution functions F such that $F(0)=0$ and whose density function f allows Maclaurin expansion for all $x>0$.

Lemma 5. *Let F be a distribution function that belongs to $\mathcal{F}$. If for all natural q it holds that*

$$f^{(q)}(0)=(-1)^q f^{q+1}(0), \tag{13}$$

then $f(x)=\lambda e^{-\lambda x}$ for some $\lambda>0$.

Proof. The Maclaurin expansion of f for positive values of x gives

$$f(x)=\sum_{q=0}^{\infty}f^{(q)}(0)\frac{x^q}{q!}=\sum_{q=0}^{\infty}(-1)^q f^{q+1}(0)\frac{x^q}{q!}=f(0)e^{-f(0)x}. \tag{14}$$

For f to be a density, $f(0)$ must be positive, so it follows that $f(0)>0$ so $f(x)$ is the density of the exponential distribution with $\lambda=f(0)$.

Lemma 6. *Let F be a distribution function that belongs to the class $\mathcal{F}$. Denote $A_m(x) = F^m(x)f(x)$. If the condition (13) is satisfied for all $0 \le q \le r-m$, $r > m$, then*

$$A_m^{(r)}(0) = (-1)^{r-m} f^{r+1}(0) \left\{ {r+1 \atop m+1} \right\} m!. \tag{15}$$

Proof. The r th derivative of $A_m(x)$ is

$$A_m^{(r)}(x) = \sum_{\substack{j_1,\dots,j_{m+1}\ge 0\\ j_1+\dots+j_{m+1}=r}} \binom{r}{j_1,\dots,j_{m+1}} F^{(j_1)}(x)\cdots F^{(j_m)}(x) f^{(j_{m+1})}(x).$$

Using the fact that $F(0)=0$ we get

$$A_m^{(r)}(0) = \sum_{\substack{j_1,\dots,j_m\ge 1,\ j_{m+1}\ge 0\\ j_1+\dots+j_{m+1}=r}} \binom{r}{j_1,\dots,j_{m+1}} f^{(j_1-1)}(0)\cdots f^{(j_m-1)}(0) f^{(j_{m+1})}(0). \tag{16}$$

Since all derivatives are of orders smaller or equal to $r-m$, using (13) we obtain

$$\begin{aligned}
A_m^{(r)}(0) &= \sum_{\substack{j_1,\dots,j_m\ge 1,\ j_{m+1}\ge 0\\ j_1+\dots+j_{m+1}=r}} \binom{r}{j_1,\dots,j_{m+1}} (-1)^{r-m} f^{r+1}(0)\\
&= (-1)^{r-m} f^{r+1}(0) \sum_{\substack{j_1,\dots,j_m,j_{m+1}\ge 1\\ j_1+\dots+j_{m+1}=r}} \binom{r}{j_1,\dots,j_{m+1}}\\
&\qquad + (-1)^{r-m} f^{r+1}(0) \sum_{\substack{j_1,\dots,j_m\ge 1,\ j_{m+1}=0\\ j_1+\dots+j_{m+1}=r}} \binom{r}{j_1,\dots,j_m}\\
&= (-1)^{r-m} f^{r+1}(0) \left(\left\{ {r \atop m+1} \right\} (m+1)! + \left\{ {r \atop m} \right\} m! \right)\\
&= (-1)^{r-m} f^{r+1}(0) m! \left\{ {r+1 \atop m+1} \right\}.
\end{aligned}$$

In the last line we used the identity (1).

Remark 2. For $r \le m$ Lemma 6 is valid without any condition imposed on derivatives of f.

3. MAIN RESULTS

Let $X_{(k;n)}$ be the k-th order statistic from the sample of size n. We now state and prove the characterization theorems.

Theorem 1. *Let $X_1, \ldots, X_n$ be a random sample from the distribution F that belongs to $\mathcal{F}$. Let k be a fixed number such that $1 < k \leq n$. If*

$$X_{(k-1;n-1)} + \frac{1}{n} X_n \stackrel{d}{=} X_{(k;n)} \tag{17}$$

then $X \sim \mathcal{E}(\lambda), \lambda > 0$.

Proof. Equalizing the densities from (17) we get

$$\begin{aligned}\int_0^x \frac{(n-1)!}{(k-2)!(n-k)!} & F^{k-2}(x-y)(1-F(x-y))^{n-k} f(x-y) n f(ny) \mathrm{d}y \\ &= \frac{n!}{(k-1)!(n-k)!} F^{k-1}(x)(1-F(x))^{n-k} f(x),\end{aligned}$$

or

$$\begin{aligned}(k-1) \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} & \int_0^x A_{i+k-2}(x-y) f(ny) \mathrm{d}y \\ &= f(x) \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} \int_0^x A_{i+k-2}(y) \mathrm{d}y.\end{aligned} \tag{18}$$

Using induction we prove that (13) holds for every natural q which by Lemma 5 implies that $f(x)$ is exponential density.

Differentiating the integral equation (18) k times we get

$$\begin{aligned}(k-1) \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} & \Bigg(\sum_{j=0}^{k-1} n^{k-1-j} f^{(k-1-j)}(nx) A_{i+k-2}^{(j)}(0) \\ & + \int_0^x A_{i+k-2}^{(k)}(x-y) f(ny) \mathrm{d}y \Bigg) \\ = \sum_{i=0}^{n-k} (-1)^i (i+k-1) \binom{n-k}{i} & \Bigg(\sum_{j=1}^{k} \binom{k}{j} f^{(k-j)}(x) A_{i+k-2}^{(j-1)}(0) \\ & + f^{(k)}(x) \int_0^x A_{i+k-2}^{(k)}(y) \mathrm{d}y \Bigg).\end{aligned}$$

Letting $x = 0$ and eliminating zero terms we get

$$\begin{aligned}&(k-1)\big(nf'(0)(k-2)!f^{k-1}(0) + f(0)A_{k-2}^{(k-1)}(0) - (n-k)(k-1)!f^{k+1}(0)\big) \\ &= (k-1)f(0)A_{k-2}^{(k-1)}(0) + (k-1)f'(0)(k-2)!f^{k-1}(0)k - k(n-k)(k-1)!f^{k+1}(0),\end{aligned}$$

from which we get $f'(0) = -f^2(0)$, which means that (13) holds for $q = 1$. Suppose now that (13) is satisfied for all $q \leq r$. We prove that it holds for $q = r + 1$.

Differentiating the integral equation (18) $k+r$ times, letting $x=0$ and eliminating zero terms we get

$$\begin{aligned}(k-1)&\sum_{i=0}^{r+1}(-1)^i\binom{n-k}{i}\sum_{j=i+k-2}^{k+r-1}n^{k+r-1-j}f^{(k+r-1-j)}(0)A_{i+k-2}^{(j)}(0)\\&=\sum_{i=0}^{r+1}(-1)^i(i+k-1)\binom{n-k}{i}\sum_{j=i+k-2}^{k+r-1}\binom{k+r}{j+1}f^{(k+r-1-j)}(0)A_{i+k-2}^{(j)}(0).\end{aligned}$$

Without loss of generality, here and in similar situations afterwards, we put the upper limit in the sums to be $r+1$. This is possible since for $r+1>n-k$ the summands for $i>n-k$ are equal to zero.

The terms for $i=0$ and $j=k+r-1$ are equal and hence they cancel out. Splitting the summation into two parts for $i=0$ and $i>0$ we get

$$\begin{aligned}(k-1)&\Big(n^{r+1}f^{(r+1)}(0)A_{k-2}^{(k-2)}(0)+\sum_{j=k-1}^{k+r-2}n^{k+r-1-j}f^{(k+r-1-j)}(0)A_{k-2}^{(j)}\Big)\\&+(k-1)\sum_{i=1}^{r+1}\sum_{j=i+k-2}^{k+r-1}(-1)^i\binom{n-k}{i}n^{k+r-1-j}f^{(k+r-1-j)}(0)A_{i+k-2}^{(j)}(0)\\=(k-1)&\binom{k+r}{k-1}f^{(r+1)}(0)A_{k-2}^{(k-2)}(0)+(k-1)\sum_{k-1}^{k+r-2}\binom{k+r}{j+1}f^{(k+r-1-j)}(0)A_{k-2}^{(j)}(0)\\&+\sum_{i=1}^{r+1}\sum_{j=i+k-2}^{k+r-1}(-1)^i\binom{n-k}{i}(i+k-1)\binom{k+r}{j+1}f^{(k+r-1-j)}(0)A_{i+k-2}^{(j)}(0).\end{aligned}$$

Applying the induction hypothesis to the derivatives of functions f and, consequently, via Lemma 6, to A_{i+k-2} and grouping the summands we obtain

$$\begin{aligned}f^{(r+1)}&(0)f^{k-1}(0)(k-1)!\Big(n^{r+1}-\binom{k+r}{k-1}\Big)\\&=(-1)^{r+1}f^{k+r+1}(0)\Big((k-1)!\sum_{j=k-1}^{k+r-2}\Big(\binom{k+r}{j+1}-n^{k+r-1-j}\Big)\left\{{j+1\atop k-1}\right\}\\&\quad+\sum_{i=1}^{r+1}\sum_{j=i+k-2}^{k+r-1}\binom{n-k}{i}\Big((i+k-1)\binom{k+r}{j+1}\\&\quad-(k-1)n^{k+r-1-j}\Big)(i+k-2)!\left\{{j+1\atop i+k-1}\right\}\Big).\end{aligned}$$

To prove the induction step it remains to show that

$$(k-1)!\left(n^{r+1}-\binom{k+r}{k-1}+\sum_{j=k-1}^{k+r-2}\left(n^{k+r-1-j}-\binom{k+r}{j+1}\right)\left\{{j+1 \atop k-1}\right\}\right)$$
$$=\sum_{i=1}^{r+1}\sum_{j=i+k-2}^{k+r-1}\binom{n-k}{i}\Big((i+k-1)\binom{k+r}{j+1}$$
$$-(k-1)n^{k+r-1-j}\Big)(i+k-2)!\left\{{j+1 \atop i+k-1}\right\}.$$

Joining the summation for $i=0$ and $i>0$ back together we get

$$\sum_{i=0}^{r+1}\sum_{j=i+k-2}^{k+r-1}\binom{n-k}{i}(k-1)n^{k+r-1-j}(i+k-2)!\left\{{j+1 \atop i+k-1}\right\}$$
$$=\sum_{i=0}^{r+1}\binom{n-k}{i}(i+k-1)!\sum_{j=i+k-2}^{k+r-1}\binom{k+r}{j+1}\left\{{j+1 \atop i+k-1}\right\}.$$

Using identity (2) and Lemma 1 we complete the proof.

Theorem 2. *Let $X_1,\dots,X_n$ be a random sample from the distribution F that belongs to $\mathcal{F}$ and let X_0 be a random variable independent of the sample that follows the same distribution. Let k be a fixed number such that $1<k\le n$. If*

$$X_{(k-1;n)}+\frac{1}{n-k+1}X_0\overset{d}{=}X_{(k;n)} \tag{19}$$

then $X\sim\mathcal{E}(\lambda),\lambda>0$.

We omit the proof since it follows a completely analogous procedure to the proof of Theorem 1 with the application of Lemma 2 in the last step.

Theorem 3. *Let $X_1,\dots,X_n$ be a random sample from the distribution F that belongs to $\mathcal{F}$. Let k be a fixed number such that $1\le k\le n$. If*

$$\frac{1}{n}X_1+\frac{1}{n-1}X_2+\cdots+\frac{1}{n-k+1}X_k\overset{d}{=}X_{(k;n)} \tag{20}$$

then $X\sim\mathcal{E}(\lambda),\lambda>0$.

Proof. Let $k\ge 2$. Equalizing the respective densities as in the previous proof we get

$$\begin{aligned}&\int_0^x f(n(x-y_2))\cdots\int_0^{y_{k-1}}f\big((n-k+2)(y_{k-1}-y_k)\big)f\big((n-k+1)y_k\big)\mathrm{d}y_2\cdots\mathrm{d}y_k\\&\quad=\frac{1}{(k-1)!}f(x)\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\int_0^\infty A_{k-2+i}(x)\mathrm{d}x.\end{aligned} \tag{21}$$

Denote the left hand side of (21) by $J_{k,n}(x)$. Obviously, it can be expressed as

$$J_{k,n}(x) = \int_0^x f(n(x-y_2))J_{k-1,n-1}(y_2)\mathrm{d}y_2,$$
$$J_{1,1}(x) = f((n-k+1)x).$$

The $(k+r)$-th derivative of $J_{k,n}$ is

$$J_{k,n}^{(k+r)}(x) = \sum_{j=0}^{k+r-1} n^j f^{(j)}(0) J_{k-1,n-1}^{(k+r-j-1)}(x) + \int_0^x f^{(k+1)}(n(x-y_2))n^{k+r} J_{k-1,n-1}^{(r+1)}(y_2)\mathrm{d}y_2.$$

Letting $x=0$ we get

$$\begin{aligned} J_{k,n}^{(k+r)}(0) &= \sum_{j=0}^{k+r-1} n^j f^{(j)}(0) J_{k-1,n-1}^{(k+r-j-1)}(0), \\ J_{1,1}^{(s)}(0) &= (n-k+1)^s f^{(s)}(0), \text{ for every } s \geq 0. \end{aligned} \tag{22}$$

Applying the recurrence relation (22) $k-1$ times we obtain

$$J_{k,n}^{(k+1)}(0) = \sum_{j_1=0}^{k+r-1} n^{j_1} f^{(j_1)}(0) \sum_{j_2=0}^{k+r-2-j_1} (n-1)^{j_2} f^{(j_2)}(0) \cdots \sum_{j_{k-1}=0}^{r+1-\sum_{\ell=1}^{k-2} j_\ell} (n-k+2)^{j_{k-1}} f^{(j_{k-1})}(0)(n-k+1)^{r+1-\sum_{\ell=1}^{k-1} j_\ell} f^{(r+1-\sum_{\ell=1}^{k-1} j_\ell)}(0).$$

Then the $(k+r)$-th derivative of the left hand side of (21) becomes

$$\sum_{\substack{j_1,\ldots,j_k \geq 0 \\ j_1+\cdots+j_k=r+1}} n^{j_1}(n-1)^{j_2}\cdots(n-k+1)^{j_k} f^{(j_1)}(0) f^{(j_2)}(0)\cdots f^{(j_k)}(0).$$

As before we prove by induction that (13) holds for every q. For $r=0$, the kth derivative of (21) at $x=0$ is

$$\begin{aligned} &(n+n-1+\cdots+n-k+1)f'(0)f^{k-1}(0) \\ &\qquad = \frac{1}{(k-2)!} f(0) A_{k-2}^{(k-1)}(0) + f'(0)f^{k-1}(0)k - k(n-k)f^{k+1}(0). \end{aligned} \tag{23}$$

From (16) we get

$$A_{k-2}^{(k-1)}(0) = f^{k-2}(0)f'(0)(k-1)! + (k-2)f'(0)f^{k-2}(0)\frac{(k-1)!}{2}.$$

Substituting this in (23) we get $f'(0) = -f^2(0)$ which means (13) holds for $q = 1$. Suppose now that (13) is satisfied for all $q \leq r$. We shall prove that it holds for $q = r + 1$. The $(k + r)$th derivative of (21) at $x = 0$ is

$$\begin{aligned}
&\sum_{\substack{j_1,\dots,j_k \geq 0 \\ j_1+\cdots+j_k=r+1}} n^{j_1}(n-1)^{j_2}\cdots(n-k+1)^{j_k} f^{(j_1)}(0)f^{(j_2)}(0)\cdots f^{(j_k)}(0) \\
&\quad = \sum_{i=0}^{r+1}(-1)^i \frac{(i+k-1)}{(k-1)!}\binom{n-k}{i}\sum_{j=i+k-2}^{k+r-1}\binom{k+r}{j+1} f^{(k+r-1-j)}(0)A^{(j)}_{i+k-2}(0).
\end{aligned} \tag{24}$$

Applying the induction hypothesis, the left hand side of (24) becomes

$$\begin{aligned}
&f^{k-1}(0)f^{(r+1)}(n^{r+1} + \cdots + (n-k+1)^{r+1}) \\
&\qquad + \sum_{\substack{0 \leq j_1,\dots,j_k < r+1 \\ j_1+\cdots+j_k=r+1}} n^{j_1}(n-1)^{j_2}\cdots(n-k+1)^{j_k}(-1)^{r+1}f^{r+1+k}(0),
\end{aligned}$$

while the right hand side of (24) can be expressed as

$$\begin{aligned}
&\sum_{i=1}^{r+1}\binom{n-k}{i}\sum_{j=i+k-2}^{k+r-1}\binom{k+r}{j+1}(-1)^{r+1}f^{k+r+1}\frac{(i+k-1)!}{(k-1)!}\left\{ {j+1 \atop i+k-1} \right\} \\
&\qquad + \sum_{j=k-1}^{k+r-2} f^{k+r+1}(0)(-1)^{r+1}\left\{ {j+1 \atop k-1} \right\} + \binom{k+r}{k-1}f^{(r+1)}(0) \\
&\qquad + \frac{1}{(k-2)!}f(0)A^{(k+r-1)}_{k-2}(0).
\end{aligned}$$

The term $A^{(k+r-1)}_{k-2}(0)$ can be evaluated using (15) and (16) as

$$\begin{aligned}
A^{k+r-1}_{k-2}(0) = {} & \frac{(k+r-1)!}{(r+1)!}f^{k-2}(0)f^{(r+1)}(0) \\
& + \frac{(k+r-1)!}{(r+2)!}(k-2)f^{k-2}(0)f^{(r+1)}(0) \\
& + \sum_{\substack{1 \leq j_1,\dots,j_{k-2} < r+2 \\ 0 \leq j_{k-1} < r+1 \\ j_1+\cdots+j_k=k+r-1}} (-1)^{r+1}f^{r+k}(0)\frac{(k+r-1)!}{j_1!\cdots j_{k-1}!}.
\end{aligned}$$

After transformations given above and grouping the summands, (24) becomes

$$\begin{aligned}
&f^{(r+1)}(0)\left(n^{r+1} + \cdots + (n-k+1)^{r+1} - \binom{k+r}{k-1} - \binom{k+r-1}{k-2} - \binom{k+r-1}{k-3}\right) \\
&\quad = (-1)^{r+1}f^{r+2}(0)\left(\sum_{j=k-1}^{k+r-2}\binom{k+r}{j+1}\left\{ {j+1 \atop k-1} \right\}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{r+1}\binom{n-k}{i}\sum_{j=i+k-2}^{k+r-1}\binom{k+r}{j+1}\frac{(i+k-1)!}{(k-1)!}\left\{ j+1 \atop i+k-1 \right\}\\
&+\frac{1}{(k-2)!}\sum_{\substack{1\le j_1,\dots,j_{k-2}<r+2\\ 0\le j_{k-1}<r+1\\ j_1+\cdots+j_k=k+r-1}}\frac{(k+r-1)!}{j_1!\cdots j_{k-1}!}\\
&-\sum_{\substack{0\le j_1,\dots,j_k<r+1\\ j_1+\cdots+j_k=r+1}} n^{j_1}(n-1)^{j_2}\cdots(n-k+1)^{j_k}\Bigg).
\end{aligned}
$$

To prove the induction step it remains to show that the expressions in the brackets on both sides are equal. Joining the summands back together and applying the identity (2) we obtain

$$
\sum_{\substack{j_1,\dots,j_k\ge 0\\ j_1+\cdots+j_k=r+1}} n^{j_1}(n-1)^{j_2}\cdots(n-k+1)^{j_k}=\sum_{i=0}^{r+1}\binom{n-k}{i}\frac{(i+k-1)!}{(k-1)!}\left\{ i+r+1 \atop i+k \right\},
$$

which follows from Lemma 4. Hence, the proof for $k\ge 2$ is completed.

In case of $k=1$ the proof is done in analogously, but it is much simpler, so we omit it here. □

The following corollary, which follows directly from Theorem 3, is a conjecture stated in [**4**].

Corollary 1. *Let $X_1,\dots,X_n$ be a random sample from the distribution F that belongs to $\mathcal{F}$. If*

$$
X_1+\frac{1}{2}X_2+\cdots+\frac{1}{n}X_n\stackrel{d}{=}X_{(n;n)},
$$

then $X\sim\mathcal{E}(\lambda),\lambda>0$.

4. DISCUSSION

In this paper we presented three new characterizations of the exponential distribution. Besides its contribution to the area of characterizations, these theorems also have applications in the field of goodness-of-fit testing. Some tests based on these characterizations have been considered in [**11**] and [**12**].

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