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ERGODIC MEASURES FOR SECTIONAL-ANOSOV FLOWS

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Abstract. Let X be a sectional-Anosov flow. We prove that if the ergodic measures of X are dense in the set of the invariant ones, then the closure of the periodic orbits of X is a homoclinic class. In particular, every Venice mask exhibits an invariant measure which cannot be approximated by ergodic ones.

1. Introduction

The *sectional-Anosov flows* [**19**] were introduced as a generalization of the Anosov flows to include the *geometric* and *multidimensional Lorenz attractors* [**2**], [**14**], [**11**] (and the examples in [**20**]). These flows have been widely studied in the recent literature [**3**], [**4**], [**5**], [**6**], [**7**], [**8**], [**10**].

In this paper we will study those sectional-Anosov flows for which the ergodic measures are dense in the set of the invariant ones. This hypothesis is verified in many sectional-Anosov flows, although they have specification only in the Anosov case [**27**]. Under such a hypothesis we will prove that the closure of the periodic orbits is a homoclinic class. Consequently, a Venice mask ([**16**], [**17**]) has at least one invariant measure which cannot be approximated by ergodic ones. Let us state our results in a precise way.

Hereafter the term *flow* will refer to a C^1 vector field X defined on a compact connected Riemannian manifold M inwardly transverse to the boundary ∂M (if nonempty). The semiflow induced by X will be denoted by X_t. We say that $\Lambda \subset M$ is *invariant* with respect to a flow X if $X_t(\Lambda) = \Lambda$ for every $t \geq 0$. Every invariant set is clearly contained in the *maximal invariant set* defined by

$$M(X) = \bigcap_{t \geq 0} X_t(M).$$

A compact invariant set Λ of X is *transitive* if $\Lambda = \omega(x)$ for some $x \in \Lambda$, where $\omega(x) = \{y \in M : y = \lim_{n\to\infty} X_{t_n}(x)$ for some sequence $t_n \to \infty\}$ denotes the omega-limit set of x. A point $x \in M$ is a *singularity* or *periodic* if $X(x) = 0$ or if there exists a minimal $t > 0$ such that $X_t(x) = x$. Denote by $Per(X)$ the set of periodic orbits of X. We say that X is *transitive* or *has dense periodic orbits* depending on whether $M(X)$ is transitive or the closure of the periodic points.

By a *Borel probability measure* of M we mean a σ-additive measure μ with $\mu(M) = 1$ defined in the Borelians of M. The set of Borel probability measures of M is denoted by $\mathcal{P}$. This set is a compact metric space if endowed with the *weak* topology*, i.e., the topology defined by the convergence $\mu_n \to \mu$ if and only if

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$\int \phi d\mu_n \to \int \phi d\mu$ for every continuous map $\phi : M \to \mathbb{R}$. A Borel probability measure μ is *invariant* with respect to the flow X if $\mu(X_t(A)) = A$ for every measurable set A and every $t \geq 0$. It turns out that the space $\mathcal{P}(X)$ of invariant measures of X is a compact subset of $\mathcal{P}$. We say that $\mu \in \mathcal{P}(X)$ is *ergodic* if $\mu(A) \in \{0, 1\}$ for every measurable invariant set A of X.

On the other hand, a compact invariant set Λ of X *has a dominated splitting with respect to the tangent flow* if there are a continuous invariant splitting $T_\Lambda M = E \oplus F$ and positive numbers K, λ such that

$$\|DX_t(x)e_x\| \cdot \|f_x\| \leq Ke^{-\lambda t}\|DX_t(x)f_x\| \cdot \|e_x\|,$$

for all $x \in \Lambda, t \geq 0, (e_x, f_x) \in E_x \times F_x$. A compact invariant set Λ is *partially hyperbolic* if it has a *partially hyperbolic splitting*, i.e., a dominated splitting $T_\Lambda M = E \oplus F$ with respect to the tangent flow whose dominated subbundle E is *contracting*, i.e., $\|DX_t(x)v^s_x\| \leq Ke^{-\lambda t}\|v^s_x\|$ for every $x \in \Lambda$, $v^s_x \in E^s_x$ and $t \geq 0$.

The Riemannian metric $\langle \cdot, \cdot \rangle$ of M induces what is called 2-*Riemannian metric* (c.f. [**22**]) $\langle \cdot, \cdot / \cdot \rangle$ defined by

$$\langle u, v/w \rangle_p = \langle u, v \rangle_p \cdot \langle w, w \rangle_p - \langle u, w \rangle_p \cdot \langle v, w \rangle_p, \quad \forall p \in M, \forall u, v, w \in T_pM.$$

Such a 2-Riemannian metric induces a 2-*norm* [**13**] defined by

$$\|u, v\| = \sqrt{\langle u, u/v \rangle_p}, \quad \forall p \in M, \forall u, v \in T_pM.$$

It is the area of the paralellogram generated by u and v in T_pM.

We say that the central subbundle F of a dominated splitting $T_\Lambda M = E \oplus F$ is *sectionally expanding* if

$$\|DX_t(x)u, DX_t(x)v\| \geq K^{-1}e^{\lambda t}\|u, v\|, \qquad \forall x \in \Lambda, u, v \in F_x, t \geq 0.$$

A singularity x of X is hyperbolic if $DX(x)$ has no purely imaginary eigenvalues.

Following [**18**], we say that a partially hyperbolic set Λ is *sectional-hyperbolic* if its singularities are hyperbolic and if its central subbundle is sectionally expanding. We say that a flow is *sectional-Anosov* if its maximal invariant set is sectional-hyperbolic [**19**].

Let x be a periodic point of X. Denote by $\pi(x)$ the minimal positive number satisfying $X_{\pi(x)}(x) = x$. Clearly 1 is an eigenvalue of $DX_{\pi(x)}(x)$ with eigenvector $X(x)$. If the remainder eigenvalues of $DX_{\pi(x)}(x)$ have modulus different from 1, then we say that the orbit $O(x) = \{X_t(x) : t \in \mathbb{R}\}$ (or the point x) is a *hyperbolic periodic orbit* (resp. *hyperbolic periodic point*) of X. A flow is *star* if it cannot be approximated in the C^1 topology by flows with nonhyperbolic periodic points or singularities.

The Invariant Manifold Theory [**15**] asserts that through any hyperbolic periodic point x of a flow X there passes a pair of invariant manifolds, the so-called strong stable and unstable manifolds $W^{ss}(x)$ and $W^{uu}(x)$, tangent at x to the eigenspaces corresponding to the eigenvalue of modulus less and bigger than 1 respectively. Saturating them with the flow we obtain the stable and unstable manifolds $W^s(x)$ and $W^u(x)$ respectively. We say that O is a *homoclinic orbit* (associated to a periodic saddle x) if $O \subset W^s(x) \cap W^u(x) \setminus O(x)$. If, additionally, $\dim(T_qW^s(x) \cap T_qW^u(x)) = 1$, then we say that O is a *transverse homoclinic orbit* (associated to x). We define the *homoclinic class* $H(x)$ associated to x as the closure of the transverse homoclinic orbits associated to x. A compact invariant set is a homoclinic class if it coincides with the homoclinic class associated to some periodic saddle.

With these definitions we can state our main result.

Theorem 1.1. *Let X be a sectional-Anosov flow. If the ergodic measures of X are dense in $\mathcal{P}(X)$, then $Cl(Per(X))$ is a homoclinic class.*

From this result we obtain that certain sectional-Anosov flows exhibit an invariant measure far from the ergodic ones. Recall that a *Venice mask* is a sectional-Anosov flow which has dense periodic orbits but is not transitive. Examples of Venice masks with only one singularity were first exhibited in [**9**]. Examples with two or more singularities were discovered recently by Lopez and Sanchez [**16**], [**17**]. It is known that the maximal invariant set of any three-dimensional Venice mask with only one singularity is the union of two homoclinic classes which intersect along the unstable manifold of a singularity [**21**]. Hence every three-dimensional Venice mask with only one singularity exhibits an invariant measure which cannot be approximated by ergodic ones. Here we will prove that the latter property is true not only for the three-dimensional Venice masks with only one singularity but also for every Venice mask. More precisely, we will prove the following corollary.

Corollary 1.2. *Every Venice mask exhibits an invariant measure which cannot be approximated by ergodic ones.*

These results motivate the search for necessary and sufficient conditions for the denseness of the ergodic measures (of a sectional-Anosov flow) in the set of the invariant ones.

2. Proofs

The following lemma is a flow-version of a result due to Parthasarathy [**23**]. For any flow X denote by $\mathcal{P}_{erg}(X)$ the ergodic members of $\mathcal{P}(X)$. A *residual subset* of a metric space is a subset containing the intersection of countably many open and dense subsets.

Lemma 2.1. *For every flow X, $\mathcal{P}_{erg}(X)$ is dense in $\mathcal{P}(X)$ if and only if $\mathcal{P}_{erg}(X)$ is residual in $\mathcal{P}(X)$.*

Proof. It suffices to prove that if $\mathcal{P}_{erg}(X)$ is dense in $\mathcal{P}(X)$, then $\mathcal{P}_{erg}(X)$ is residual in $\mathcal{P}(X)$. For this we follow the proof of Proposition 5.1 in [**1**].

Assume that $\mathcal{P}_{erg}(X)$ is dense in $\mathcal{P}(X)$. Given a continuous function $\psi : M \to \mathbb{R}$ we define

$$\mathcal{P}_\psi(X) = \bigcap_{l\in\mathbb{N}^+}\bigcup_{t\geq 1}\left\{\mu\in\mathcal{P}(X) : \int\left|\frac{1}{t}\int_0^t \psi(X_s(z))ds - \int\psi(x)\mu(x)\right| d\mu(z) < \frac{1}{l}\right\}.$$

We claim every $\mu \in \mathcal{P}(X)$ satisfying the limit

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \psi(X_s(z))ds = \int\psi(x)d\mu(x) \qquad \text{for } \mu\text{-a.e. } z\in M \tag{2.1}$$

belongs to $P_\psi(X)$. Indeed, take $\mu \in \mathcal{P}(X)$ satisfying this limit and define the family of measurable maps $\psi_t : N \to \mathbb{R}$ by

$$\psi_t(z) = \frac{1}{t}\int_0^t \psi(X_s(z))ds.$$

By (2.1) and Birkhoff's Theorem we have that ψ_t converges pointwise to the constant function $\int \psi(x)d\mu(x)$. On the other hand, N is compact and ψ is continuous so there is $C > 0$ such that $|\psi(z)| \leq C$ for every $z \in N$. Then, $|\psi_t(z)| \leq C$ for all $(t, z) \in \mathbb{R}^+ \times N$ and so

$$\lim_{t\to\infty} \int \left| \psi_t(z) - \int \psi(x)d\mu(x) \right| d\mu(z) = 0$$

by Lebesgue's Dominated Convergence Theorem. From this we get $\mu \in P_\psi(X)$ proving the claim.

Next we observe that for any given $l \in \mathbb{N}^+$ the set

$$O_l = \bigcup_{t\geq 1} \left\{ \mu \in P(X) : \int \left| \frac{1}{t} \int_0^t \psi(X_s(z))ds - \int \psi(x)d\mu(x) \right| d\mu(z) < \frac{1}{l} \right\}$$

is open in $\mathcal{P}(X)$. Moreover, every ergodic measure of X satisfies (2.1) and so belongs to $\mathcal{P}_\psi(X)$ by the previous claim. Then, since the set of ergodic measures is dense by hypothesis, we obtain that $\mathcal{P}_\psi(X)$ is also dense in $\mathcal{P}(X)$. We conclude that $\mathcal{P}_\psi(X)$ is residual in $\mathcal{P}(X)$, for every continuous map $\psi : N \to \mathbb{R}$.

Now let $C^0(N)$ be the space of continuous maps from N to $\mathbb{R}$ and let ψ_k be a countable dense subset of $C^0(N)$. We have just proved that each $\mathcal{P}_{\psi_k}(X)$ is residual in $\mathcal{P}(X)$ and so $\bigcap_{k\in\mathbb{N}} \mathcal{P}_{\psi_k}(X)$ is also residual in $\mathcal{P}(X)$. Using standard approximation arguments one can show $\bigcap_{k\in\mathbb{N}} \mathcal{P}_{\psi_k}(X) \subset \mathcal{P}_{erg}(X)$. Since $\bigcap_{k\in\mathbb{N}} \mathcal{P}_{\psi_k}(X)$ is residual in $\mathcal{P}(X)$, we are done. □

The second lemma is a general fact about *star flows*, i.e., flows which cannot be approximated in the C^1 topology by ones with nonhyperbolic periodic points or singularities.

Define the support $supp(\mu)$ of a measure μ as the set of points x for which $\mu(U) > 0$ for any neighborhood U of x.

The *measure center* of a flow X (c.f. [**28**]) the union of the supports of its invariant measures, i.e.,

$$A(X) = \bigcup_{\nu\in\mathcal{P}(X)} supp(\nu).$$

Notice that $A(X) \subset M(X)$ and $A(X) = M(X)$ if and only if $M(X)$ is the support of an invariant measure (by the results in [**1**]). We also have that $A(X)$ is contained in the nonwandering set $\Omega(X)$ but, in general, $A(X) \neq \Omega(X)$ even for sectional-Anosov flows (for an example see [**7**]).

With these definitions we can state the following result.

Lemma 2.2. *The measure center of a star flow for which the ergodic measures are dense in the set of the invariant ones is either a singularity or a homoclinic class.*

Proof. Let X be a star flow. By Proposition 5.4 in [**1**] there is a residual subset of invariant measures whose support is $A(X)$. Now assume that the ergodic measures are dense in the set of invariant measures. Then, the set of ergodic measures of X is residual in $\mathcal{P}(X)$ by Lemma 2.1. Since the intersection of residual subsets of a compact metric space (like $\mathcal{P}(X)$) is residual (and so nonempty), we can choose an ergodic measure μ satisfying $supp(\mu) = A(X)$. Let $Sing(X)$ denote the set of singularities of X. If $\mu(Sing(X)) > 0$, then μ is the Dirac measure supported on a singularity σ. In this case we obtain $A(X) = \{\sigma\}$ and we are done. Otherwise,

$\mu(Sing(X)) = 0$ and then, by Theorem 5.4 in [**26**], there is homoclinic class H of X such that $supp(\mu) \subset H$. Since the periodic orbits are dense on any homoclinic class, and every periodic orbit is the support of an invariant measure, we also have $H \subset A(X)$. Consequently, $A(X) = H$ and we are done. $\square$

There are star flows for which both the ergodic measures are dense in the set of the invariant ones and the measure center is a singularity [**25**].

Now we can prove our results.

Proof of Theorem 1.1. Let X be a sectional-Anosov flow for which the ergodic measures are dense in the set of the invariant ones. Since every sectional-Anosov flow is star, we can apply Lemma 2.2 to obtain that its the measure center is either a singularity or a homoclinic class. Since every sectional-Anosov flow has homoclinic (and hence periodic) points [**3**], we have that the former alternative cannot occur. Then, the measure center is a homoclinic class. On the other hand, since each periodic orbit is the support of an invariant measure, we obtain that $Cl(Per(X))$ is contained in the measure center. Since the periodic orbits are dense in any homoclinic class, we conclude that $Cl(Per(X))$ coincides with the measure center and so it is a homoclinic class. This ends the proof. $\square$

Proof of Corollary 1.2. Suppose by contradiction that there is a Venice mask X for which every invariant measure can be approximated by ergodic ones. Then, the set of ergodic measures of X is dense in $\mathcal{P}(X)$ and so $Cl(Per(X))$ is a homoclinic class by Theorem 1.1. Since X has dense periodic orbits, $M(X) = Cl(Per(X))$ and so $M(X)$ is a homoclinic class. Since homoclinic classes are transitive sets, we conclude that $M(X)$ (and so X) are transitive which is absurd. This concludes the proof. $\square$

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