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APPROXIMATION BY GENERALIZED FABER SERIES IN BERGMAN SPACES ON INFINITE DOMAINS WITH A QUASICONFORMAL BOUNDARY

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Abstract. Using an integral representation on infinite domains with a quasiconformal boundary the generalized Faber series for the functions in the Bergman space $A^2(G)$ are defined and their approximative properties are investigated.

1. Introduction and New Results

Let G be a simple connected domain in the complex plane $\mathbb{C}$ and let ω be a weight function given on G. For functions f analytic in G we set

$$A^2(G,\omega) := \left\{ f : \iint\limits_G |f(z)|^2 \, \omega(z) d\sigma_z < \infty \right\},$$

where $d\sigma_z$ denotes the Lebesgue measure in the complex plane $\mathbb{C}$.

If $\omega = 1$, we denote $A^2(G) := A^2(G,1)$. The space $A^2(G)$ is called the Bergman space on G. We refer to the spaces $A^2(G,\omega)$ as "weighted Bergman spaces". It becomes a normed spaces if we define

$$\|f\|_{A^2(G,\omega)} := \left(\iint\limits_G |f(z)|^2 \, \omega(z) d\sigma_z \right)^{1/2}.$$

Hereafter, we consider only the special weight $\omega(z) := 1/|z|^4$ in this work.

Now let L be a finite quasiconformal curve in the complex plane $\mathbb{C}$. We recall that L is called a quasiconformal curve if there exists a quasiconformal homeomorphism of the complex plane onto itself that maps a circle onto L. We denote by G_1 and G_2 the bounded and unbounded complements of $\mathbb{C} \setminus L$, respectively. It is clear that if $f \in A^2(G_2)$, then it has zero in ∞ at least second order. As in the bounded case [**7**, p. 5], $A^2(G_2)$ is a Hilbert space with the inner product

$$\langle f, g \rangle := \iint\limits_{G_2} f(z)\overline{g(z)} d\sigma_z,$$

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which can be easily verified. Moreover, the set of polynomials of $1/z$ are dense in $A^2(G_2)$ with respect to the norm

$$\|f\|_{A^2(G_2)} := (\langle f, f\rangle)^{1/2}.$$

Indeed, let $f \in A^2(G_2)$. If we substitute $z = 1/\zeta$ and define

$$f(z) = f\left(\frac{1}{\zeta}\right) =: f_*(\zeta),$$

then G_2 maps to a finite domain G_ζ, and $f_* \in A^2(G_\zeta)$, because

$$\iint\limits_{G_\zeta} |f_*(\zeta)|^2 \, d\sigma_\zeta = \iint\limits_{G_2} |f(z)|^2 \frac{d\sigma_z}{|z|^4} \leq c \iint\limits_{G_2} |f(z)|^2 \, d\sigma_z < \infty,$$

with some constant $c > 0$. Since f has zero in ∞ at least second order, the point $\zeta = 0$ is the zero of f_* at least second order and

$$\iint\limits_{G_\zeta} \left|\frac{f_*(\zeta)}{\zeta^2}\right|^2 d\sigma_\zeta = \iint\limits_{G_2} |f(z)|^2 \, d\sigma_z < \infty.$$

Hence $f_*(\zeta)/\zeta^2 \in A^2(G_\zeta)$. If $P_n(\varsigma)$ is a polynomial of ς, then we have

$$\begin{aligned} \iint\limits_{G_\zeta} \left|P_n(\zeta) - \frac{f_*(\zeta)}{\zeta^2}\right|^2 d\sigma_\zeta &= \iint\limits_{G_\zeta} \left|P_n(\zeta)\zeta^2 - f_*(\zeta)\right|^2 \frac{1}{|\zeta|^4} d\sigma_\zeta \\ &= \iint\limits_{G_2} \left|P_n\left(\frac{1}{z}\right)\frac{1}{z^2} - f(z)\right|^2 d\sigma_z. \end{aligned}$$

This implies that the set of polynomials of $1/z$ are dense in $A^2(G_2)$, since the set of polynomials $P_n(\zeta)$ are dense in $A^2(G_\zeta)$ with respect to the norm

$$\|f\|_{A^2(G_\zeta)} := (\langle f, f\rangle)^{1/2},$$

(see, for example: [**7**, Ch. 1]). Also, for $n = 1, 2, \ldots$ there exists a polynomial $P_n^*(1/z)$ of $1/z$, of degree $\leq n$, such that $E_n(f, G_2) = \|f - P_n^*\|_{A^2(G_2)}$ (see for example, [**6**, p. 59, Theorem 1.1.]), where

$$E_n(f, G_2) := \operatorname{Inf}\left\{\|f - P\|_{A^2(G_2)} : P \text{ is a polynomial of } 1/z, \quad \text{of degree} \quad \leq n\right\}$$

denotes the minimal error of approximation of f by polynomials of $1/z$ of degree at most n. The polynomial $P_n^*(1/z)$ is called the best approximant polynomial of $1/z$ to $f \in A^2(G_2)$.

Let D be the open unit disc and $w = \varphi(z)$ be the conformal mapping of G_1 onto $C\overline{D} := \mathbb{C} \setminus \overline{D}$, normalized by the conditions

$$\varphi(0) = \infty \quad \text{and} \quad \lim_{z \to 0} z\varphi(z) > 0,$$

and let ψ be the inverse of φ. In the neighborhood of the origin we have the expansion

$$\varphi(z) = \frac{\alpha}{z} + \alpha_0 + \alpha_1 z + ldots + \alpha_k z^k + \ldots\ .$$

Raising this function to the power m we obtain

$$[\varphi(z)]^m = F_m(1/z) + Q_m(z) \qquad \text{for} \quad z \in G_1, \tag{1.1}$$

where $F_m(1/z)$ denotes the polynomial of negative powers of z and the term $Q_m(z)$ contains non-negative powers of z and is analytic in the domain G_1. The polynomial $F_m(1/z)$ of negative powers of z is called the generalized Faber polynomial for the domain G_2. If $z \in G_2$, then integrating in the positive direction along L, we have

$$F_m\left(\frac{1}{z}\right) = -\frac{1}{2\pi i}\int_L \frac{[\varphi(\zeta)]^m}{\zeta - z} d\zeta = -\frac{1}{2\pi i}\int_{|w|=1} \frac{w^m \psi'(w)}{\psi(w) - z} dw.$$

This formula implies that the functions $F_m(1/z)$, $m = 1, 2, \ldots$ are the Laurent coefficients in the expansion of the function

$$\frac{\psi'(w)}{\psi(w) - z} \qquad z \in G_2, \ w \in C\overline{D}$$

in the neighborhood of the point $w = \infty$, i.e. the following expansion holds

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{m=1}^{\infty} F_m\left(\frac{1}{z}\right)\frac{1}{w^{m+1}} \qquad z \in G_2, \ w \in C\overline{D},$$

which converges absolutely and uniformly on compact subsets of $G_2 \times C\overline{D}$. Differentiation of this equality with respect to z gives

$$\frac{\psi'(w)}{(\psi(w) - z)^2} = \sum_{m=1}^{\infty} F_m'\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)\frac{1}{w^{m+1}}$$

or

$$\frac{z^2\psi'(w)}{(\psi(w) - z)^2} = \sum_{m=1}^{\infty} - F_m'\left(\frac{1}{z}\right)\frac{1}{w^{m+1}} \tag{1.2}$$

for every $(z, w) \in G_2 \times C\overline{D}$, where the series converges absolutely and uniformly on compact subsets of $G_2 \times C\overline{D}$. More information for Faber and generalized Faber polynomials can be found in [**12**, p. 255] and [**7**, p. 42].

In this work, for the first time, we obtain (Section 2, Lemma 2.1) an integral representation on the infinite domain G_2 with a quasiconformal boundary for a function $f \in A^2(G_2)$. By means of this integral representation in Section 2 we define a generalized Faber series of a function $f \in A^2(G_2)$ to be of the form

$$\sum_{m=1}^{\infty} a_m(f) F_m'\left(\frac{1}{z}\right),$$

with the generalized Faber coefficients $a_m(f)$, $m = 1, 2, \ldots$.

Our main results are presented in the following theorems, which are proved in Section 3.

Theorem 1.1. *Let $f \in A^2(G_2)$. If*

$$\sum_{m=1}^{\infty} a_m(f) F_m'\left(\frac{1}{z}\right) \tag{1.3}$$

is a generalized Faber series of f, then this series converges uniformly to f on the compact subsets of G_2.

Corollary 1.2. *Let $P_n(1/z)$ be a polynomial of degree n of $1/z$ and $P_n(1/z) \in A^2(G_2)$. If $a_m(P_n)$ are its generalized Faber coefficients, then $a_m(P_n) = 0$ for all $m \geq n+2$ and*

$$P_n\left(\frac{1}{z}\right) = \sum_{m=1}^{n+1} a_m(P_n) F_m'\left(\frac{1}{z}\right).$$

A uniqueness theorem for the series

$$\sum_{m=1}^{\infty} a_m(f) F_m'\left(\frac{1}{z}\right),$$

which converges to $f \in A^2(G_2)$ with respect to the norm $\|\cdot\|_{A^2(G_2)}$ is the following.

Theorem 1.3. *Let $\{a_m\}$ be a complex number sequence. If the series*

$$\sum_{m=1}^{\infty} a_m F_m'\left(\frac{1}{z}\right)$$

converges to a function $f \in A^2(G_2)$ in the norm $\|\cdot\|_{A^2(G_2)}$, then the a_m, $m = 1, 2, \ldots$, are the generalized Faber coefficients of f.

The following theorem estimates the error of the approximation of $f \in A^2(G_2)$ by the partial sums of the series (1.3) in the weighted norm $\|\cdot\|_{A^2(G_2,\omega)}$ for the special weight $\omega(z) := 1/|z|^4$, regarding to the minimal error $E_n(f, G_2)$.

Theorem 1.4. *If $f \in A^2(G_2)$, $\omega(z) := 1/|z|^4$ and*

$$S_n\left(f, \frac{1}{z}\right) = \sum_{m=1}^{n+1} a_m F_m'\left(\frac{1}{z}\right)$$

is the nth partial sum of its generalized Faber series

$$\sum_{m=1}^{\infty} a_m F_m'\left(\frac{1}{z}\right),$$

then

$$\|f - S_n(f, \cdot)\|_{A^2(G_2,\omega)} \leq \frac{c}{1-k^2}\sqrt{n} E_n(f, G_2),$$

for all natural numbers n and with a constant c independent of n.

Similar results for the bounded domains with a quasiconformal boundary were stated and proved in [**8**] and [**5**], respectively. These problems in the weghted cases were studied in [**9**] and [**10**].

We shall use $c, c_1, \ldots,$ to denote constants depending only on parameters that are not important for the questions of interest.

2. Definitions and Some Auxiliary Results

In [**4**], V.I. Belyi gave the following integral representation for the functions f analytic and bounded in the domain G_1

$$f(z) = -\frac{1}{\pi} \iint_{G_2} \frac{(f \circ y)(\zeta)}{(\zeta - z)^2} y_{\bar{\zeta}}(\zeta) d\sigma_\zeta, \qquad z \in G_1.$$

Here $y(z)$ is a K-quasiconformal reflection across the boundary L, i.e., a sense-reversing K-quasiconformal involution of the extended complex plane keeping every point of L fixed, such that $y(G_1) = G_2$, $y(G_2) = G_1$, $y(0) = \infty$ and $y(\infty) = 0$. Such a mapping of the plane does exist [**11**, p. 99]. As follows from Ahlfors theorem [**1**, p. 80] the reflection $y(z)$ can always be chosen canonical in the sense that it is differentiable on C almost everywhere, except possibly at the points of the curve L, and for any sufficiently small fixed $\delta > 0$ it satisfies the relations

$$\begin{aligned} |y_\varsigma| + |y_{\overline{\varsigma}}| &\leq c_1, &&\text{if}\quad \zeta \in \{\zeta \mid \quad \delta < |\varsigma| < 1/\delta, \quad \varsigma \notin L\} \\ |y_\varsigma| + |y_{\overline{\varsigma}}| &\leq c_2\, |\varsigma|^{-2}, &&\text{if}\quad |\varsigma| \geq 1/\delta \quad\text{or}\quad |\varsigma| \leq \delta, \end{aligned}$$

with some constants c_1 and c_2, independent of ζ.

Considering only the canonical quasiconformal reflections, I.M. Batchaev [**3**] generalized the integral representation above to functions $f \in A^2(G_1)$. The accurate proof of the Batchaev's result is given in [**2**, p. 110, Th. 4.4]. A similar integral representation can also be obtained for functions $f \in A^2(G_2)$. The following result holds.

Lemma 2.1. *Let $f \in A^2(G_2)$. If $y(z)$ is a canonical quasiconformal reflection with respect to L, then*

$$f(z) = -\frac{1}{\pi} \iint\limits_{G_1} \frac{(f \circ y)(\zeta) z^2}{(\zeta - z)^2 [y(\zeta)]^2} y_{\overline{\zeta}}(\zeta) d\sigma_\zeta, \qquad z \in G_2. \tag{2.1}$$

Proof. Let $y(z)$ a canonical quasiconformal reflection and $f \in A^2(G_2)$. If we substitute $\zeta = 1/u$ for $\zeta \in G_2$ and define

$$f(\zeta) = f\,(1/u) =: f_*(u),$$

then G_2 maps to a finite domain G_u and $f_* \in A^2(G_u)$. If $y^*(t)$ is a canonical quasiconformal reflection with respect to ∂G_u, then from the Batchaev's result we have

$$f_*(t) \;=\; -\frac{1}{\pi} \iint\limits_{C\overline{G_u}} \frac{(f_* \circ y^*)\,(u)}{(u - t)^2} y^*_{\overline{u}}(u) d\sigma_u, \qquad t \in G_u,$$

where $C\overline{G_u} := \mathbb{C} \setminus \overline{G_u}$. Substituting $u = 1/\zeta$ in this integral representation we get

$$\begin{aligned} f(z) \;&=\; f(1/t) = f_*(t) = -\frac{1}{\pi} \iint\limits_{G_1} \frac{(f_* \circ y^*)\,(1/\varsigma)}{(1/\varsigma - 1/z)^2} y^*_{\overline{u}}\,(1/\varsigma)\, J d\sigma_\zeta \\ &=\; \frac{1}{\pi} \iint\limits_{G_1} \frac{f[1/y^*\,(1/\zeta)] z^2}{(\zeta - z)^2} y^*_{\overline{\zeta}}\,(1/\varsigma)\, d\sigma_\zeta, \qquad z \in G_2. \end{aligned}$$

If we define

$$y(\zeta) \;:=\; \frac{1}{y^*\,(1/\varsigma)},$$

then $y(\zeta)$ becomes a canonical quasiconformal reflection with respect to L. Consequently, for $f \in A^2(G_2)$ we get

$$f(z) \;=\; -\frac{1}{\pi} \iint\limits_{G_1} \frac{(f \circ y)(\zeta) z^2}{(\zeta - z)^2 [y(\zeta)]^2} y_{\overline{\zeta}}(\zeta) d\sigma_\zeta, \qquad z \in G_2.$$

□

From now on, the reflection $y(z)$ will be a canonical K-quasiconformal reflection with respect to L.

Let $f \in A^2(G_2)$. Substituting $\zeta = \psi(w)$ in (2.1), we get

$$f(z) \;=\; -\frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{f\big(y(\psi(w))\big)\overline{\psi'}(w)y_{\overline{\zeta}}\big(\psi(w)\big)}{[y(\psi(w))]^2}\cdot\frac{z^2\psi'(w)}{\big(\psi(w)-z\big)^2}d\sigma_w, \qquad z \in G_2. \quad (2.2)$$

Thus, if we define the coefficients $a_m(f)$, $m = 1, 2, \ldots$, by

$$a_m(f) \;:=\; \frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{f\big(y(\psi(w))\big)\overline{\psi'}(w)}{w^{m+1}[y\big(\psi(w)\big)]^2}y_{\overline{\zeta}}\big(\psi(w)\big)d\sigma_w, \quad (2.3)$$

then, by (1.2) and (2.2), we can associate a formal series $\sum_{m=1}^{\infty} a_m(f)F'_m(1/z)$ with the function $f \in A^2(G_2)$, i.e.,

$$f(z) \;\sim\; \sum_{m=1}^{\infty} a_m(f)F'_m\,(1/z)\,.$$

We call this formal series a generalized Faber series of $f \in A^2(G_2)$, and the coefficients $a_m(f)$, $m = 1, 2, \ldots$, generalized Faber coefficients of f.

Lemma 2.2. *Let $\{F_m(1/z)\}$, $m = 1, 2, \ldots$, be the generalized Faber polynomials of $1/z$ for G_2. Then*

$$\sum_{m=1}^{n} \frac{\left\|F'_{m,z}\right\|^2_{A^2(G_2)}}{m} \;\leq\; n\pi.$$

Proof. Since $\overline{y}(\zeta)$ is a canonical K-quasiconformal mapping of the extended complex plane onto itself, we have $\left|\overline{y}_{\overline{\zeta}}\right| / \left|\overline{y}_{\zeta}\right| \leq k$ and $\left|\overline{y}_{\zeta}\right|^2 - \left|\overline{y}_{\overline{\zeta}}\right|^2 > 0$. Also, it is known that $\left|\overline{y}_{\overline{\zeta}}\right| = |y_\zeta|$ and $\left|\overline{y}_{\zeta}\right| = \left|y_{\overline{\zeta}}\right|$. Therefore, $|y_\zeta| / \left|y_{\overline{\zeta}}\right| \leq k$ and $\left|y_{\overline{\zeta}}\right|^2 - |y_\zeta|^2 > 0$. Hence

$$\begin{aligned}
&\iint\limits_{G_1} |(f\circ y)(\zeta)|^2 \left|y_{\overline{\zeta}}(\zeta)\right|^2 d\sigma_\zeta \\
&\quad= \iint\limits_{G_1} |(f\circ y)(\zeta)|^2 \left(1 - |y_\zeta|^2 / \left|y_{\overline{\zeta}}\right|^2\right)^{-1} \left(\left|y_{\overline{\zeta}}\right|^2 - |y_\zeta|^2\right) d\sigma_\zeta \\
&\quad\leq \frac{1}{1-k^2}\iint\limits_{G_1} |(f\circ y)(\zeta)|^2 \left(\left|y_{\overline{\zeta}}\right|^2 - |y_\zeta|^2\right) d\sigma_\zeta.
\end{aligned}$$

Since $\left(|y_\zeta|^2 - \left|y_{\overline{\zeta}}\right|^2\right)$ is the Jacobian of $y(\zeta)$, substituting ζ for $y(\zeta)$ on the right side of the last inequality we get

$$\iint\limits_{G_1} |(f\circ y)(\zeta)|^2 \left|y_{\overline{\zeta}}(\zeta)\right|^2 d\sigma_\zeta \;\leq\; \frac{\|f\|^2_{A^2(G_2)}}{1-k^2}.$$

□

3. Proofs of the New Results

Proof of Theorem 1.1. Let M be a compact subset of G_2 and $y(z)$ a canonical K-quasiconformal reflection with respect to L. Since by Lemma 2.1

$$\begin{aligned} f(z) &= -\frac{1}{\pi}\iint_{G_1} \frac{(f\circ y)(\zeta)z^2}{(\zeta-z)^2[y(\zeta)]^2} y_{\bar\zeta}(\zeta)d\sigma_\zeta \\ &= -\frac{1}{\pi}\iint_{C\overline{D}} \frac{f\big(y(\psi(w))\big)\overline{\psi'}(w)y_{\bar\zeta}\big(\psi(w)\big)}{[y\big(\psi(w)\big)]^2}\cdot\frac{z^2\psi'(w)}{\big(\psi(w)-z\big)^2}d\sigma_w, \qquad \text{for} \quad z\in M, \end{aligned}$$

by means of (2.3), Hölder's inequality and Lemma 4 we obtain

$$\begin{aligned} &\left| f(z) - \sum_{m=1}^{n} a_m(f)F'_m(1/z)\right| \\ &\qquad \le \frac{c_3\,\|f\|_{A^2(G_2)}}{\pi\sqrt{1-k^2}}\left(\iint_{C\overline{D}} \left|\frac{z^2\psi'(w)}{(\psi(w)-z)^2} + \sum_{m=1}^{n}\frac{F'_m(1/z)}{w^{m+1}}\right|^2 d\sigma_w\right)^{1/2}, \end{aligned} \tag{3.1}$$

for every $z\in M$, where the constant c_3 depends only on L.

Let $1<r<R<\infty$. In view of (1.2)

$$\begin{aligned} &\iint_{r<|w|<R}\left|\frac{z^2\psi'(w)}{\big(\psi(w)-z\big)^2} + \sum_{m=1}^{n}\frac{F'_m(1/z)}{w^{m+1}}\right|^2 d\sigma_w \\ &\qquad = \iint_{r<|w|<R}\left|\sum_{m=n+1}^{\infty}\frac{F'_m(1/z)}{w^{m+1}}\right|^2 d\sigma_w \\ &\qquad = \pi\sum_{m=n+1}^{\infty}\frac{1}{m}\left(\frac{1}{r^{2m}}-\frac{1}{R^{2m}}\right)\left|F'_m(1/z)\right|^2 \\ &\qquad \le 4\pi\sum_{m=n+1}^{\infty}\frac{\left|F'_m(1/z)\right|^2}{m+1} \end{aligned}$$

and by letting $r\to 1^+$ and $R\to\infty$ we get

$$\iint_{C\overline{D}}\left|\frac{z^2\psi'(w)}{\big(\psi(w)-z\big)^2} + \sum_{m=1}^{n}\frac{F'_m(1/z)}{w^{m+1}}\right|^2 d\sigma_w \le 4\pi\sum_{m=n+1}^{\infty}\frac{\left|F'_m(1/z)\right|^2}{m+1}. \tag{3.2}$$

Therefore, by (3.1), (3.2) and Lemma 3 we conclude that

$$\sum_{m=1}^{\infty} a_m(f)F'_m(1/z)$$

converges uniformly to f on M.

Proof of Corollary 1.2. Let $z\in G_2$. By Theorem 1.1 we have

$$P_n(1/z) = \sum_{m=1}^{\infty} a_m(P_n)F'_m(1/z).$$

On the other hand, $P_n(1/z)$ can be written in the form

$$P_n\left(1/z\right) \;=\; \sum_{k=1}^{n+1} A_k F_k'\left(1/z\right),$$

with the specific coefficients A_k, $k = 1, 2, \dots, n+1$. Let $y(z)$ be a canonical K-quasiconformal reflection relative to L. Since $y(z)$ is identical on L, by Green's formulae we get

$$\begin{aligned}
a_m(P_n) &= \frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{P_n\left[1/y\big(\psi(w)\big)\right]\overline{\psi'}(w)}{w^{m+1}[y\big(\psi(w)\big)]^2} y_{\overline{\zeta}}\big(\psi(w)\big) d\sigma_w \\
&= \sum_{k=1}^{n+1}\frac{A_k}{\pi}\iint\limits_{C\overline{D}} \frac{F_k'\left[1/y\big(\psi(w)\big)\right]\overline{\psi'}(w)}{w^{m+1}[y\big(\psi(w)\big)]^2} y_{\overline{\zeta}}(\psi(w)) d\sigma_w \\
&= \sum_{k=1}^{n+1}\frac{A_k}{\pi}\iint\limits_{C\overline{D}} -\frac{\partial}{\partial\overline{w}}\left(\frac{F_k\left[1/y(\psi(w))\right]}{w^{m+1}}\right) d\sigma_w \\
&= \sum_{k=1}^{n+1}\frac{A_k}{2\pi i}\int\limits_{|w|=1} \frac{F_k\left[1/\psi(w)\right]}{w^{m+1}} dw.
\end{aligned}$$

By (1.1)

$$F_m\left[1/\psi(w)\right] \;=\; w^m - Q_m\big(\psi(w)\big),$$

where $Q_m\big(\psi(w)\big)$ is analytic in $C\overline{D}$, and then

$$\frac{1}{2\pi i}\int\limits_{|w|=1} \frac{F_k\left[1/\psi(w)\right]}{w^{m+1}} dw = QATOPD \quad \begin{cases} 1, & \text{if} \quad k = m, \\ 0, & \text{if} \quad k \neq m, \end{cases} \tag{3.3}$$

which implies that $a_m(P_n) = A_m$, for $m = 1, \dots, n+1$, and $a_m(P_n) = 0$ for all $m \geq n+2$. Hence

$$P_n\left(1/z\right) \;=\; \sum_{m=1}^{n+1} a_m\left(P_n\right) F_m'\left(1/z\right).$$

Proof of Theorem 1.3. Let $y(z)$ be a canonical K-quasiconformal reflection relative to L and

$$S_n\left(f, 1/z\right) \;:=\; \sum_{m=1}^{n+1} a_m F_m'\left(1/z\right)$$

be the nth partial sum of

$$\sum_{m=1}^{\infty} a_m F_m'\left(1/z\right).$$

Using (3.3) it can be shown that

$$\lim_{n\to\infty}\frac{1}{\pi}\iint\limits_{C\overline{D}} \frac{S_n\left[1/y\big(\psi(w)\big)\right]\overline{\psi'}(w)}{w^{m+1}[y\big(\psi(w)\big)]^2} y_{\overline{\zeta}}\big(\psi(w)\big) d\sigma_w = a_m, \qquad m = 1, 2, \dots . \tag{3.4}$$

If m and n are natural numbers, then by using Hölder's inequality and Lemma 4 we get

$$|a_m(f) - a_m| \leq \frac{1}{\pi} \left| \iint\limits_{C\overline{D}} \frac{f(y(\psi(w))) - S_n\left[1/y(\psi(w))\right] \overline{\psi'}(w)}{w^{m+1}[y(\psi(w))]^2} y_{\overline{\zeta}}(\psi(w)) d\sigma_w \right|$$

$$+ \left| \frac{1}{\pi} \iint\limits_{C\overline{D}} \frac{S_n\left[1/y\big(\psi(w)\big)\right] \overline{\psi'}(w)}{w^{m+1}[y\big(\psi(w)\big)]^2} y_{\overline{\zeta}}\big(\psi(w)\big) d\sigma_w - a_m \right| \leq \frac{1}{\pi} \left(\iint\limits_{C\overline{D}} \frac{d\sigma_w}{|w|^{2m+2}} \right)^{1/2}$$

$$\times \left(\iint\limits_{C\overline{D}} \frac{\left| f\big(y(\psi(w))\big) - S_n\left[1/y\big(\psi(w)\big)\right] \right|^2 |\psi'(w)|^2 \left| y_{\overline{\zeta}}\big(\psi(w)\big) \right|^2}{\left| y\big(\psi(w)\big) \right|^4} d\sigma_w \right)^{1/2}$$

$$+ \left| \frac{1}{\pi} \iint\limits_{C\overline{D}} \frac{S_n\left[1/y\big(\psi(w)\big)\right] \overline{\psi'}(w)}{w^{m+1}[y\big(\psi(w)\big)]^2} y_{\overline{\zeta}}\big(\psi(w)\big) d\sigma_w - a_m \right|$$

$$\leq \frac{c_4}{\sqrt{m\pi}} \left(\iint\limits_{G_1} \left| \big((f - S_n) \circ y\big)(\zeta) \right|^2 \left| y_{\overline{\zeta}}(\zeta) \right|^2 d\sigma_\zeta \right)^{1/2}$$

$$+ \left| \frac{1}{\pi} \iint\limits_{C\overline{D}} \frac{S_n\left[1/y\big(\psi(w)\big)\right] \overline{\psi'}(w)}{w^{m+1}[y\big(\psi(w)\big)]^2} y_{\overline{\zeta}}\big(\psi(w)\big) d\sigma_w - a_m \right|$$

$$\leq \frac{c_4 \left\| f - S_n \right\|_{A^2(G_2)}}{\sqrt{m\pi(1-k^2)}}$$

$$+ \left| \frac{1}{\pi} \iint\limits_{C\overline{D}} \frac{S_n\left[1/y\big(\psi(w)\big)\right] \overline{\psi'}(w)}{w^{m+1}[y\big(\psi(w)\big)]^2} y_{\overline{\zeta}}\big(\psi(w)\big) d\sigma_w - a_m \right|. \tag{3.5}$$

Since $\lim\limits_{n\to\infty} \|f - S_n\|_{A^2(G_2)} = 0$, (3.4) and (3.5) show that $a_m(f) = a_m$, $m = 1, 2, \ldots$.

Proof of Theorem 1.4. Let $y(z)$ be a canonical K-quasiconformal reflection with respect to L, and $P_n^*(1/z)$ the best approximant polynomial to $f \in A^2(G_2)$ in the norm $\|\cdot\|_{A^2(G_2)}$. For $z \in G_2$, by means of Hölder's inequality, Lemma 4 and

Corollary 1.2 we obtain

$$
\begin{aligned}
|f(z) - S_n(f, 1/z)| &\le |f(z) - P_n^*(1/z)| + |P_n^*(1/z) - S_n(f, 1/z)| \\
&\le |f(z) - P_n^*(1/z)| + \left| \sum_{m=1}^{n+1} \big(a_m(P_n^*) - a_m(f)\big) F_m'(1/z) \right| \\
&\le |f(z) - P_n^*(1/z)| \\
&+ \frac{1}{\pi} \left| \iint_{C\overline{D}} \frac{(f \circ y - P_n^* \circ y)\big(\psi(w)\big)\overline{\psi'}(w) y_{\overline{\zeta}}\big(\psi(w)\big)}{[y\big(\psi(w)\big)]^2} \sum_{m=1}^{n+1} \frac{F_m'(1/z)}{w^{m+1}} d\sigma_w \right| \\
&\le |f(z) - P_n^*(1/z)| \\
&+ \frac{1}{\pi} \left(\iint_{C\overline{D}} \frac{\left|(f \circ y - P_n^* \circ y)\big(\psi(w)\big)\right|^2 |\psi'(w)|^2 \left| y_{\overline{\zeta}}\big(\psi(w)\big)\right|^2}{\left| y\big(\psi(w)\big)\right|^4} d\sigma_w \right)^{1/2} \\
&\times \left(\iint_{C\overline{D}} \left| \sum_{m=1}^{n+1} \frac{F_m'(1/z)}{w^{m+1}} \right|^2 d\sigma_w \right)^{1/2} \\
&\le |f(z) - P_n^*(1/z)| + \frac{c_5}{\pi} \left(\iint_{G_1} |(f \circ y - P_n^* \circ y)(\zeta)|^2 \left| y_{\overline{\zeta}}(\zeta) \right|^2 d\sigma_\zeta \right)^{1/2} \\
&\times \left(\pi \sum_{m=1}^{n+1} \frac{|F_m'(1/z)|^2}{m} \right)^{1/2} \\
&\le |f(z) - P_n^*(1/z)| + \frac{c_5}{\sqrt{\pi(1-k^2)}} \|f - P_n^*\|_{A^2(G_2)} \left(\sum_{m=1}^{n+1} \frac{|F_m'(1/z)|^2}{m} \right)^{1/2} \\
&= |f(z) - P_n^*(1/z)| + \frac{c_5}{\sqrt{\pi(1-k^2)}} E_n(f, G_2) \left(\sum_{m=1}^{n+1} \frac{|F_m'(1/z)|^2}{m} \right)^{1/2}
\end{aligned}
$$

for all natural numbers n. This shows that

$$
|f(z) - S_n(f, 1/z)|^2 \le 2\,|f(z) - P_n^*(1/z)|^2 + \frac{2c_5}{\pi(1-k^2)} E_n^2(f, G_2) \sum_{m=1}^{n+1} \frac{|F_m'(1/z)|^2}{m}.
$$

Multiplying both sides by $1/|z|^4$ and take into account that $1/|z|^4 \le c_6$ for $z \in G_2$ and with a constant c_6, we get

$$|f(z) - S_n(f, 1/z)|^2 \frac{1}{|z|^4} \le c_7 |f(z) - P_n^*(1/z)|^2 + \frac{c_8}{\pi(1-k^2)} E_n^2(f, G_2) \sum_{m=1}^{n+1} \frac{|F'_{m,z}(1/z)|^2}{m}.$$

Now, by integrating both sides over G_2 and by virtue of Lemma 2.2 we get

$$\begin{aligned} \|f(z) - S_n(f,\cdot)\|^2_{A^2(G_2,\omega)} &\le c_7 E_n^2(f, G_2) + \frac{c_8}{\pi(1-k^2)} E_n^2(f, G_2) \sum_{m=1}^{n+1} \frac{\|F'_{m,z}\|^2_{A^2(G_2)}}{m} \\ &\le \left(c_7 + \frac{c_8(n+1)}{1-k^2}\right) E_n^2(f, G_2) \\ &\le \frac{c_9 n}{1-k^2} E_n^2(f, G_2), \end{aligned}$$

i.e.,

$$\|f(z) - S_n(f,\cdot)\|_{A^2(G_2,\omega)} \le \frac{c}{1-k^2} \sqrt{n} E_n(f, G_2)$$

for all natural numbers n.

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