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Sliding mode control of periodic review perishable inventories with multiple suppliers and transportation losses

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Abstract. The purpose of this paper is to develop robust and computationally efficient supply chain management strategy ensuring fast reaction to the demand variations for periodic review perishable inventory systems. For that purpose, we apply a sliding mode approach and we propose a new discrete time warehouse management strategy. The strategy employs the sliding hyperplane appropriately designed to ensure a dead-beat performance of the closed loop system. Our strategy not only explicitly takes into account decay of goods stored in the warehouse (perishing inventories) but it also accounts for transportation losses which take place on the way from suppliers to the warehouse. The proposed strategy ensures full customers' demand satisfaction, minimizes the on-hand inventory volume and prevents from exceeding the warehouse capacity. This reflects the need of simultaneous minimization of the lost sales costs and inventory holding costs. Furthermore, the strategy ensures that the ordered quantities of goods are always non-negative and upper bounded. These favourable properties of the proposed strategy are formally stated as a lemma and three theorems and proved in the paper.

Key words: discrete time sliding mode control, sliding surface design, inventory control.

1. Introduction

The control theoretic approach to the management of logistic processes, and in particular to the problem of supply chain management has recently become an important research subject. A good overview of the techniques used in the field and the obtained results can be found in [1–6]. The first application of the control theory methods to the management of logistic processes was reported in the early 1950s when Simon [7] applied the servomechanism control algorithm to find an efficient strategy of goods replenishment in continuous time, single product inventory control systems. A few years later the discrete time servomechanism control algorithm for the purpose of efficient goods replenishment has been proposed [8]. The next landmark in this field was the work of Forrester [9], who analyzed the amplification of demand fluctuations when moving upstream in the supply chain, later called bullwhip effect. First block diagram representation of inventory and order based production control system model and its dynamic analysis was presented by Towill [10]. A number of other control theory attributes, which were used to model and analyze inventory-production systems can be found in literature. Important contribution to this scope of study with respect to control of bullwhip effect has been presented in [11–15]. The authors of those papers aimed at smoothing an ordering policy and inventory levels and proved that utilizing control theory methods, it is possible to successfully prevent the bullwhip effect. Over the last two decades numerous innovative solutions have been presented, and therefore, further in this section we are able to mention only a few, arbitrarily selected examples. In [12] and [16] autoregressive moving average (ARMA) system structure has been applied in order to model uncertain demand. Then in [6, 17–19] model predictive control of supply chain has been proposed and in [20] a robust controller for the continuous-time system with uncertain processing time and delay has been designed by minimising H_∞-norm. However, practical implementation of the strategy described in [20] requires application of numerical methods in order to obtain the control law parameters, which limits its analytical tractability. Also estimation techniques have been used in inventory management literature. The recursive least squares method and Kalman filter, were applied in [17] and [21] for lead time identification and in [12] and [16] for demand forecasting respectively. Several other methods, including convex programming [22], genetic algorithms [23], heuristic techniques [24] and simulations [25] have also been applied to improve warehouse operation.

In [26] state space representation of supply chains is proposed and lead-time delay is explicitly taken into account by the introduction of additional state variables. This approach results in the optimal controller designed by minimisation of quadratic performance index. A similar approach is applied in [27] where an LQ optimal sliding mode controller is designed. However, both papers [26] and [27] are concerned with conventional, non-deteriorating inventories only. An extension of the results presented in [26] to the case of perishable inventories is given in [28], an LQ optimal sliding mode controller for supply chains with deteriorating stock is proposed in [29] and a dead-beat sliding mode controller for a single supplier logistic system is designed in [30]. However, none of the papers [26–30] takes into account transportation losses (or in other words goods decay during the order procurement time). Therefore, in this paper we consider perishable inventories replenished by multiple suppliers and

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we explicitly account for the ordered goods losses during the non-negligible lead time.

In this paper we consider a periodic-review inventory system with perishable goods replenished from multiple supply sources. However, in contrast to the previously published results we consider not only losses which take place when the commodity is stored in the warehouse, but also those which happen during the supply process, i.e. the losses on the way from the supplier to the warehouse. We propose a discrete time representation of the supply chain dynamics and we apply discrete time sliding mode methodology [31–38] to design the controller for the considered system. The controller design objective is on one hand to fully satisfy the imposed demand, and on the other, to minimize at the same time the on-hand inventory volume. This reflects the need of simultaneous minimization of the lost sales costs and inventory holding costs. Since the demand may vary quite rapidly, we determine the sliding hyperplane so that the proposed discrete-time sliding-mode controller ensures the dead-beat system performance. Hence the closed-loop system is stabilized and its error converges to zero in the shortest possible time. This approach results in good dynamics of the closed loop system and its fast reaction to the unpredictable variations of demand. Moreover, the sliding mode controller proposed further in this paper leads to chattering free system operation. The controller is determined analytically in a closed form, which allows us to state and formally prove important properties of proposed inventory policy. First, we prove that the designed management policy always generates strictly positive and upper bounded order quantities, which is an important issue from the practical point of view. Next, we define the warehouse capacity which provides enough space for all incoming shipments. Finally, we state and prove conditions ensuring that all the imposed demand is fully satisfied and 100% service level is achieved. In many practical applications this is a desirable property directly related to customer satisfaction and it becomes indispensable when it comes to strategic supplies.

2. Problem statement

In this paper we consider a periodic review production-inventory system, where a distribution centre, replenished from multiple supply sources, provides products for customers or another production stage. The analysed inventory system is subject to an *a priori* unknown, bounded, time-varying demand. The flow of goods and information in the considered system (with transportation losses and the on-hand stock deterioration) is presented in Fig. 1.

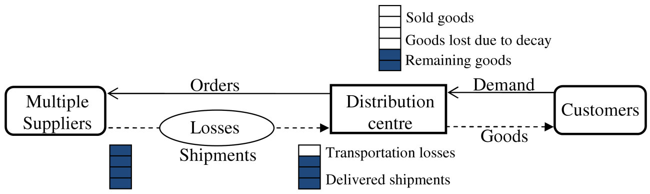

Fig. 1. Flow of goods and information in the considered production-inventory system

The main objective of this work is to design a stable supply policy, which will maximize demand satisfaction from the resources available at the distribution centre. The design procedure proposed in the paper not only explicitly takes into account the delay (lead-time) between placing of an order at the suppliers and goods arrival at the distribution centre, but it also directly accounts for the on-hand stock deterioration and commodity losses in the supply process, i.e. the losses which take place during the lead-time. The model of the analysed production-inventory system is illustrated in Fig. 2.

The stock replenishment orders $u(kT)$ are placed at regular time instants kT, where T is the review period of the considered process and $k = 0, 1, 2, \ldots$. The particular value of each order is calculated on the basis of the current stock level $y(kT)$, the stock reference level y_{ref} and the order history. We assume that replenishment order $u(kT)$ can be split among r supply options. As a consequence, in each time instant, β_i of the total order is placed at supplier i ($i = 1, \ldots, r$), where β_i is a fraction from interval [0,1] satisfying

$$\sum_{i=1}^{r} \beta_i = 1. \tag{1}$$

Each nonzero order placed at the supplier is realized with lead-time delay L_i, which is a multiple of the review period. Thus, $L_i = n_i T$, where n_i is a positive integer. Without the loss of generality, we can order the supply alternatives according to their lead time as follows

$$L_1 \le L_2 \le \ldots \le L_r. \tag{2}$$

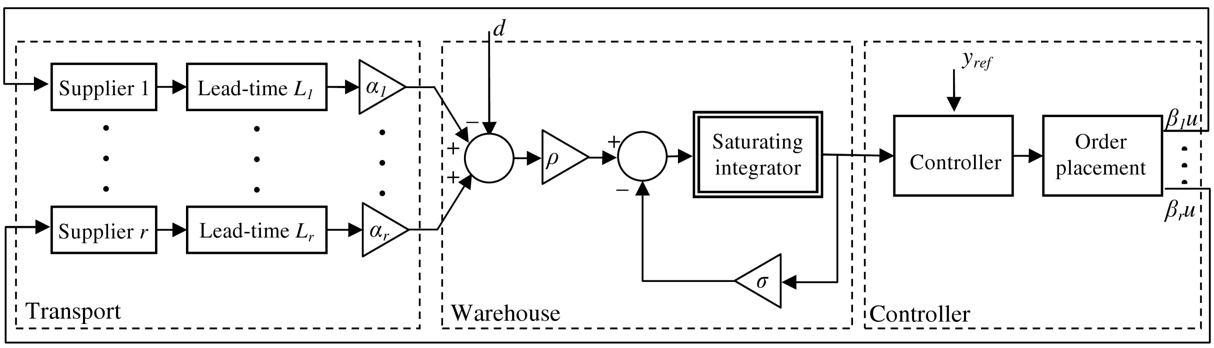

Fig. 2. System model

The imposed demand (the amount of goods requested from inventory in period k) is modelled as an unknown, bounded function of time $0 \leq d(kT) \leq d_{\max}$. In contrast to many other approaches proposed earlier in the literature, we neither expect any particular distribution of the demand nor, we assume correlation or autocorrelation in the demand. The only assumption required in our work is that the demand is non-negative and upper bounded by a known, possibly very big constant value $d_{\max}$. Therefore, this definition of the demand is quite general and makes the presented approach fairly universal. According to this definition the following two situations might occur:

- If there is a sufficient amount of goods in the warehouse, then the imposed demand is fully satisfied.
- If the imposed demand is greater than the amount of goods available at the on-hand stock and in arriving shipments, then only some part of the demand is satisfied. Hence, additional demand is lost, as we assume that the sales are not backordered.

Let $h(kT)$ denote the amount of goods sold to customers or sent to retailers in the distribution network at time instant kT. Then

$$0 \leq h(kT) \leq d(kT) \leq d_{\max}. \tag{3}$$

The stock balance equation for the considered system with perishable inventory has the following form

$$y\left[(k+1)T\right] = \rho\left[y(kT) + u_R(kT) - h(kT)\right], \tag{4}$$

where $u_R(kT)$ is the order received at time kT. The fraction of perishable stock which remains in the warehouse after each review period is represented by $\rho = 1 - \sigma$. We assume that incoming shipments also deteriorate during transportation process. Consequently, the fractions of ordered goods which arrive at the warehouse are represented by $\alpha_i (i = 1, \ldots, r)$, where

$$0 < \alpha_i \leq 1. \tag{5}$$

Thus, the order received at time kT is expressed by

$$u_R(kT) = \sum_{i=1}^{r} \alpha_i \beta_i u\left[(k - n_i)T\right]. \tag{6}$$

Furthermore, we assume that the warehouse is initially empty, i.e. $y(kT) = 0$ for $k < 0$, and the first order is placed at the time instant $kT = 0$. Due to the lead-time delay, the first order arrives at the warehouse at the time instant n_1, and $y(kT) = 0$ for any $k \leq n_1$. Taking into account our assumptions, initial conditions and (6), the stock level for any $k > 0$ can be expressed as

$$\begin{aligned} y(kT) &= \sum_{i=1}^{r} \alpha_i \beta_i \sum_{j=0}^{k-1} \rho^{k-j} u\left[(j - n_i)T\right] - \sum_{j=0}^{k-1} \rho^{k-j} h(jT) \\ &= \sum_{i=1}^{r} \alpha_i \beta_i \sum_{j=0}^{k-n_i-1} \rho^{k-n_i-j} u(jT) - \sum_{j=0}^{k-1} \rho^{k-j} h(jT). \end{aligned} \tag{7}$$

In order to make our notation as concise as possible, in the remainder of the paper we will use k as the independent variable in place of kT. Let us consider the following discrete time state space representation of the analysed inventory system

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{b}u(k) + \mathbf{v}h(k), \\ y(k) &= \mathbf{q}^T \mathbf{x}(k), \end{aligned} \tag{8}$$

where $\mathbf{x}(k) = [x_1(k)\ x_2(k)\ \ldots\ x_n(k)]^T$ is the state vector, $x_1(k) = y(k)$ is the on-hand stock level at time instant k and $x_j(k) = u(k - n + j - 1)$ for any $j = 2, \ldots, n$ represents delayed input signal u. Furthermore, $\mathbf{A}$ is $n \times n$ state matrix, $\mathbf{b}$, $\mathbf{v}$, and $\mathbf{q}$ are $n \times 1$ vectors

$$\mathbf{A} = \begin{bmatrix} \rho & \rho a_2 & \rho a_3 & \cdots & \rho a_n \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -\rho \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

The system order is equal to $n = n_r + 1 = L_r/T + 1$ and it depends on the review period and lead-time of the last supplier L_r. The elements a_i $(i = 2, \ldots, n)$ in the first row of the state matrix $\mathbf{A}$, are calculated as follows

$$a_i = \sum\nolimits_{j: n_j = n-i+1} \alpha_j \beta_j, \tag{10}$$

i.e. for any $i = 2, \ldots, n$; a_i is the sum of those products $\alpha_j \beta_j$ for which $n_j = n - i + 1$.

The desired system state vector is defined as

$$\begin{aligned} \mathbf{x}_\mathbf{d}^T &= \begin{bmatrix} 1 & (1-\rho)/(\rho\Omega) & \ldots & (1-\rho)/(\rho\Omega) \end{bmatrix} y_{ref} \\ &= \begin{bmatrix} 1 & \sigma/(\rho\Omega) & \ldots & \sigma/(\rho\Omega) \end{bmatrix} y_{ref}, \end{aligned} \tag{11}$$

where y_{ref} denotes the reference stock level, and

$$\Omega = \sum_{j=1}^{r} \alpha_j \beta_j = \sum_{j=2}^{n} a_j. \tag{12}$$

Since the main objective of the controller design procedure is to obtain stable supply policy, it is necessary to stabilize the first state variable at the reference level. Therefore, when choosing the desired state vector, it is necessary to take into account that the commodities perish at the rate $1 - \rho$ when kept in the warehouse as well as the fact that they decay during the transportation process at the rate proportional to Ω. Substituting the proposed desired system state vector into the state equation, one can verify that in the steady state when the demand is equal to zero, the on-hand stock is refilled by incoming shipments at the rate equal to $y_{ref}(1-\rho)/(\rho\Omega) = y_{ref}\sigma/(\rho\Omega)$.

3. Proposed inventory management policy

In this section we present the controller design procedure for the considered multi-supplier inventory system (8)–(9) with perishable goods and transportation losses. The procedure is based on the discrete time sliding mode approach. In the first part of this section, the choice of the sliding hyperplane ensuring dead-beat performance is described. Next, in the latter part we formulate and prove the most important properties of the proposed inventory management policy.

3.1. Dead-beat sliding-mode controller design. For the sliding mode controller design purpose we introduce a sliding hyperplane described by

$$s(k) = \mathbf{c}^T \mathbf{e}(k) = 0, \tag{13}$$

where $\mathbf{c} = [c_1 \quad c_2 \quad \ldots \quad c_n]^T$ is such a vector that $\mathbf{c}^T \mathbf{b} \neq 0$. Parameters $c_1, c_2, \ldots, c_n$ will be determined further in this section. The closed-loop system error may be expressed as $\mathbf{e}(k) = \mathbf{x}_d - \mathbf{x}(k)$. Substituting (8) into equation $\mathbf{c}^T \mathbf{e}(k+1) = 0$ we obtain the following control law

$$u(k) = \left(\mathbf{c}^T \mathbf{b}\right)^{-1} \mathbf{c}^T \left[\mathbf{x_d} - \mathbf{A}\mathbf{x}(k)\right], \tag{14}$$

One may easily notice from (14), that the controller performance depends on the choice of the sliding plane parameters $c_1, c_2, \ldots, c_n$. Using (9) we can rewrite (14) as

$$u(k) = c_n^{-1} y_{ref} \left[c_1 + (1-\rho)/\rho\Omega \sum_{j=2}^{n} c_j\right] +$$
$$-c_n^{-1} \left\{c_1 \rho x_1(k) + c_1 \rho a_2 x_2(k) + \sum_{j=3}^{n} (c_1 \rho a_j + c_{j-1}) x_j(k)\right\}, \tag{15}$$

In order to find parameters of the hyperplane which will ensure that the system error is eliminated in finite (and the smallest possible) number of control steps, we analyse coefficients of the characteristic polynomial of the closed-loop system state matrix $\mathbf{A}_c = [\mathbf{I}_n - \mathbf{b}(\mathbf{c}^T\mathbf{b})^{-1}\mathbf{c}^T]\mathbf{A}$. The polynomial $\det(z\mathbf{I}_n - \mathbf{A}_c)$ can be expressed as follows

$$\begin{aligned}\det(zI_n - \mathbf{A}_c) &= z^n + \frac{c_1 \rho a_n + c_{n-1} - \rho c_n}{c_n} z^{n-1} \\ &+ \frac{c_1 \rho a_{n-1} + c_{n-2} - \rho c_{n-1}}{c_n} z^{n-2} \\ &+ \cdots + \frac{c_1 \rho a_2 - \rho c_2}{c_n} z.\end{aligned} \tag{16}$$

A discrete-time system is asymptotically stable if and only if all its eigenvalues are located inside the unit circle on the z plane. Additionally, for the dead-beat performance, the characteristic polynomial of the closed-loop system should have the following form

$$\det(z\mathbf{I_n} - \mathbf{A_c}) = z^n \tag{17}$$

which is satisfied when

$$c_1 = c_n \Psi \rho^{-1} \quad \text{and} \quad c_j = c_n \Psi \sum_{i=1}^{j-1} \rho^{i-j} a_{i+1} \quad \text{for} \quad j \geq 2, \tag{18}$$

where

$$\Psi = \rho^{n-1} \Big/ \sum_{i=1}^{n-1} \rho^{i-1} a_{i+1}. \tag{19}$$

Using (10), we can rewrite (19) as follows

$$\Psi = 1 \Big/ \sum_{i=1}^{r} \rho^{-n_i} \alpha_i \beta_i. \tag{20}$$

Hence, vector $\mathbf{c}$ describing the parameters of the sliding hyperplane has the following form

$$\begin{aligned}\mathbf{c}^T = \big[\Psi\rho^{-1} \quad \Psi\rho^{-1}a_2 \quad \Psi\left(\rho^{-2}a_2 + \rho^{-1}a_3\right) \\ \cdots \quad \Psi \sum_{i=1}^{n-2} \rho^{i-n+1} a_{i+1} \quad 1\big] c_n,\end{aligned} \tag{21}$$

which guarantees that the closed-loop system has all its eigenvalues located at the origin of the z plane.

Substituting (21) into (15) we get

$$c_1 + (1-\rho)/(\rho\Omega) \sum_{j=2}^{n} c_j = c_n / (\rho\Omega). \tag{22}$$

The control law can be expressed in the following way

$$u(k) = \frac{y_{ref}}{\rho\Omega} - \Psi x_1(k) - \Psi \sum_{j=2}^{n} a_j \sum_{i=j}^{n} \rho^{i-j} x_i(k). \tag{23}$$

According to the state space representation of inventory system (9), the first state variable denotes the on-hand stock level $x_1(k) = y(k)$, and the other state variables are equal to the delayed control signals generated at the previous $n-1$ review periods $x_j(k) = u(k-n+j-1)$. Therefore, taking into account that system order is equal to $n = n_r + 1$ we obtain

$$u(k) = \frac{y_{ref}}{\rho\Omega} - \Psi y(k) - \Psi \sum_{i=1}^{r} \alpha_i \beta_i \sum_{j=k-n_i}^{k-1} \rho^{k-n_i-j} u(j). \tag{24}$$

The obtained control signal incorporates reference stock level, current stock level and open orders modified with respect to transportation losses and goods decay in the warehouse.

3.2. Properties of the proposed controller. One of the fundamental issues in practical implementation of each inventory policy is to ensure that the quantity of goods shipped to the warehouse is always nonnegative and upper bounded. Therefore, now we introduce a lemma and a theorem which show that the proposed policy indeed ensures these two highly desirable properties. First of all, it can be easily noticed from (24) that $u(0) = y_{ref}/(\rho\Omega)$. Furthermore, for any $k \geq 1$ the following lemma holds.

Lemma. If the proposed inventory policy is applied, then for any $k \geq 1$

$$u(k) = (1-\rho) \frac{y_{ref}}{\rho\Omega} + \Psi\rho h(k-1). \tag{25}$$

Proof. Substituting (7) into (24), we get

$$
\begin{aligned}
u(k) &= \frac{y_{ref}}{\rho\Omega} - \Psi\sum_{i=1}^{r}\alpha_i\beta_i\sum_{j=0}^{k-n_i-1}\rho^{k-n_i-j}u(j) \\
&+\Psi\sum_{j=0}^{k-1}\rho^{k-j}h(j) - \Psi\sum_{i=1}^{r}\alpha_i\beta_i\sum_{j=k-n_i}^{k-1}\rho^{k-n_i-j}u(j) \\
&= \frac{y_{ref}}{\rho\Omega} + \Psi\sum_{i=1}^{r}\alpha_i\beta_i\rho^{-n_i}\sum_{j=0}^{k-1}\rho^{k-j}u(j) \\
&+\Psi\sum_{j=0}^{k-1}\rho^{k-j}h(j).
\end{aligned}
\tag{26}
$$

Then applying (20) to (26) we obtain

$$
u(k) = \frac{y_{ref}}{\rho\Omega} - \sum_{j=0}^{k-1}\rho^{k-j}u(j) + \Psi\sum_{j=0}^{k-1}\rho^{k-j}h(j). \tag{27}
$$

For $k = 1$, it follows immediately from (27) that

$$
\begin{aligned}
u(1) &= \frac{y_{ref}}{\rho\Omega} - \rho u(0) + \Psi\rho h(0) \\
&= \frac{y_{ref}}{\rho\Omega} - \rho\frac{y_{ref}}{\rho\Omega} + \Psi\rho h(0) \\
&= (1-\rho)\frac{y_{ref}}{\rho\Omega} + \Psi\rho h(0)
\end{aligned}
\tag{28}
$$

which shows that the lemma is indeed satisfied for $k = 1$. Now let us assume that (25) is true for all integers up to some $l > 1$. Using this assumption and (27), the order quantity generated at time instant $l + 1$ can be expressed in the following form

$$
\begin{aligned}
u(l+1) &= \frac{y_{ref}}{\rho\Omega} - \sum_{j=0}^{l}\rho^{l+1-j}u(j) \\
&+\Psi\sum_{j=0}^{l}\rho^{l+1-j}h(j) = \frac{y_{ref}}{\rho\Omega} + \rho\frac{y_{ref}}{\rho\Omega} - \rho\frac{y_{ref}}{\rho\Omega} \\
&-\rho\sum_{j=0}^{l-1}\rho^{l-j}u(j) - \rho u(l) + \Psi\rho\sum_{j=0}^{l-1}\rho^{l-j}h(j) \\
&+\Psi\rho h(l) = \frac{y_{ref}}{\rho\Omega} - \rho\frac{y_{ref}}{\rho\Omega} \\
&+\rho\left[\frac{y_{ref}}{\rho\Omega} - \sum_{j=0}^{l-1}\rho^{l-j}u(j) + \Psi\sum_{j=0}^{l-1}\rho^{l-j}h(j)\right] \\
&-\rho u(l) + \Psi\rho h(l) \\
&= (1-\rho)\frac{y_{ref}}{\rho\Omega} + \Psi\rho h(l).
\end{aligned}
\tag{29}
$$

Since l is an arbitrary positive integer, it follows from the principle of mathematical induction that (25) is true for any integer $k \geq 1$. This concludes the proof of the lemma.

Theorem 1. If the proposed warehouse management policy is applied, then for any $k \geq 0$ the control signal satisfies the following inequalities

$$
\begin{aligned}
&(1-\rho)\frac{y_{ref}}{\rho\Omega} \leq u(k) \\
&\leq \max\left[\frac{y_{ref}}{\rho\Omega}, (1-\rho)\frac{y_{ref}}{\rho\Omega} + \Psi\rho d_{\max}\right].
\end{aligned}
\tag{30}
$$

Proof. It follows directly from (24) that $u(0) = y_{ref}/(\rho\Omega)$, and this implies that the theorem is satisfied for $k = 0$. Moreover, since the demand is always bounded as stated by inequalities (3), then for any $k > 0$, from the lemma proved above, we obtain

$$
(1-\rho)\frac{y_{ref}}{\rho\Omega} \leq u(k) \leq (1-\rho)\frac{y_{ref}}{\rho\Omega} + \Psi\rho d_{\max} \tag{31}
$$

which ends the proof of Theorem 1.

The next theorem states another important property of the proposed policy, namely it shows that the inventory level never exceeds its reference value. This theorem shows that if the warehouse capacity is selected at least equal to y_{ref}, then enough storage space at the distribution centre for all incoming shipments will always be provided.

Theorem 2. If the proposed inventory management policy is applied, then for any $k \geq 0$ the stock level is always upper bounded by y_{ref}, i.e.

$$
y(k) \leq y_{ref}. \tag{32}
$$

Proof. Due to initial conditions and lead-time delay, the considered warehouse is empty for any $k \leq n_1$. Hence, we need to show that inequality (32) holds for any $k \geq n_1$.

Let us assume that for some integer $l \geq n_1$, $y(l) \leq y_{ref}$. Then, we demonstrate that this inequality is also satisfied for $l + 1$. The stock level at the time instant $l + 1$, based on the inventory balance equation, can be expressed as

$$
y(l+1) = \rho\left[y(l) + \sum_{i=1}^{r}\alpha_i\beta_i u(l-n_i) - h(l)\right]. \tag{33}
$$

Substituting (7) and (27) into (33), we obtain

$$
\begin{aligned}
y(l+1) &= \rho y(l) + \rho\sum_{i=1}^{r}\alpha_i\beta_i\frac{y_{ref}}{\rho\Omega} \\
&-\rho\sum_{i=1}^{r}\alpha_i\beta_i\sum_{j=0}^{l-n_i-1}\rho^{l-n_i-j}u(j) \\
&+\rho\Psi\sum_{i=1}^{r}\alpha_i\beta_i\sum_{j=0}^{l-n_i-1}\rho^{l-n_i-j}h(j) - \rho h(l) \\
&= \rho y(l) + y_{ref} - \rho y(l) \\
&-\rho\Psi\sum_{i=1}^{r}\alpha_i\beta_i\sum_{j=l-n_i}^{l-1}\rho^{l-n_i-j}h(j) - \rho h(l) \\
&= y_{ref} - \rho\Psi\sum_{i=1}^{r}\alpha_i\beta_i\rho^{-n_i}\sum_{j=l-n_i}^{l}\rho^{l-j}h(j).
\end{aligned}
\tag{34}
$$

Since $h(k)$ is always nonnegative, then $y(l+1) \leq y_{ref}$. Using the principle of the mathematical induction we conclude that the theorem is satisfied for any $k \geq 0$.

Now we formulate and prove the last theorem, which shows how to select the reference stock level, so that full demand satisfaction is guaranteed. In other words, this theorem demonstrates how big warehouse capacity is needed, to ensure that all sales are realized from the readily available resources.

Theorem 3. If the proposed inventory policy is applied, and the target stock level satisfies the following inequality

$$y_{ref} > \Psi d_{\max} \sum_{i=1}^{r} \alpha_i \beta_i \rho^{-n_i} \sum_{j=0}^{n_i} \rho^{j+1} \tag{35}$$

then for any $k \geq n$ the stock level is strictly positive.

Proof: Assumption (3) implies that the realized demand is always upper bounded. Hence, taking into account (28), (34) and (35), for any $k \geq n$, we obtain

$$\begin{aligned} y(k) &= y_{ref} - \Psi \sum_{i=1}^{r} \alpha_i \beta_i \rho^{-n_i} \sum_{j=l-1-n_i}^{l-1} \rho^{l-j} h(j) \\ &\geq y_{ref} - \Psi d_{\max} \sum_{i=1}^{r} \alpha_i \beta_i \rho^{-n_i} \sum_{j=l-1-n_i}^{l-1} \rho^{l-j} > 0. \end{aligned} \tag{36}$$

This concludes the proof.

4. Numerical example

In order to verify the effectiveness of the proposed supply policy we performed a number of simulation tests. The system parameters considered in the tests are: review period $T = 1$ day, inventory deterioration rate $\sigma = 0.12$, which implies that $\rho = 1 - 0.12 = 0.88$, and the maximum daily demand at the distribution centre $d_{\max} = 70$ items. The warehouse is replenished from 4 supply sources. The lead-time delays of the sources ($L_i = n_i T$), transportation channel loss coefficients α_i and order partitioning coefficients β_i, $(i = 1, 2, 3, 4)$ are presented in Table 1. Furthermore, the elements a_i $(i = 2, \ldots, 8)$ of the first row of state matrix **A**, calculated according to (10), are given in Table 2.

Table 1
Supply chain parameters

i	1	2	3	4
L_i [days]	1	3	4	7
α_i [–]	0.95	0.92	0.89	0.84
β_i [–]	0.35	0.25	0.25	0.15

Table 2
Elements of the first row of state matrix

i	2	3	4	5	6	7	8
a_i	0.126	0	0	0.2225	0.23	0	0.3325

Further in this section, we demonstrate results of the simulation tests performed for two different reference values of the stock level y_{ref}. In the first test we verify the performance of our strategy with the reference value of the stock level calculated according to Theorem 3. It follows from the theorem that in order to obtain full demand satisfaction, the reference stock level should be greater than 218.26 items. Thus, we set $y_{ref} = 225$ items. In the second simulation scenario we analyze effectiveness of the proposed control strategy in the case of the reference stock level reduced by 20% and equal to 180 items.

The actual demand for both tests is depicted in Fig. 3. It can be seen from the figure the demand changes instantly between its extreme values and contains some stochastic component which fades after 45 review periods. Rapid changes of the demand determine the most unfavourable simulation scenario and enable us to verify the system performance even in the very adverse market conditions.

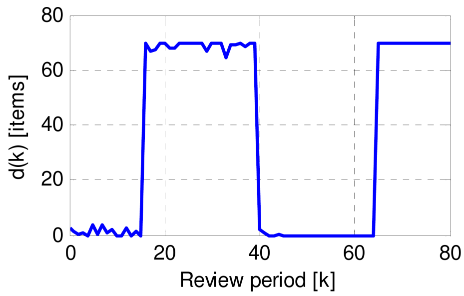

Fig. 3. Demand at the distribution center

The orders generated by the proposed policies are shown in Fig. 4. It can be easily seen from the figure that in both simulation scenarios the control signal is always nonnegative and bounded. Moreover, the system quickly reacts to sudden changes in the demand without undesirable oscillations. However, since the initial values of the control signal are relatively big, application of a reaching law approach based control scheme to decrease the values may be a feasible option [30, 33–35].

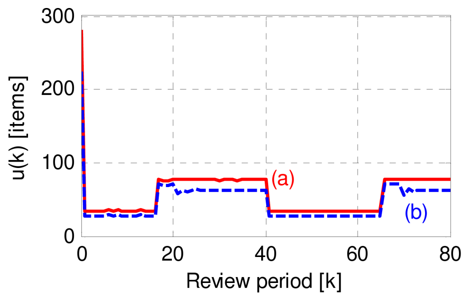

Fig. 4. Order quantities for: a) $y_{ref} = 225$ items, b) $y_{ref} = 180$ items

Figure 5 presents the on-hand stock. The figure shows that for the controller with $y_{ref} = 225$ items after the initial phase of the control process our supply policy always guarantees full satisfaction of the market demand. Furthermore, it can be seen from the figure that the proposed strategy with the reference stock value decreased by 20% also works properly and ensures smaller stock volume at the expense of moderate sales loss. This observation proves that the proposed strategy may also be applied when limited storage capacity is available. In those circumstances customer demand is no longer fully satisfied and the service level slightly decreases, but holding costs are essentially reduced. Finally, Fig. 6 shows that the representative point of the system reaches the sliding hyperplane in one step and then remains in the vicinity of the plane, i.e. $s \in \langle 0,\ c_n \Psi d_{\max} \rangle = \langle 0,\ 50.19 \rangle$. This is a direct result of the demand acting in the considered system as a unidirectional unmatched disturbance.

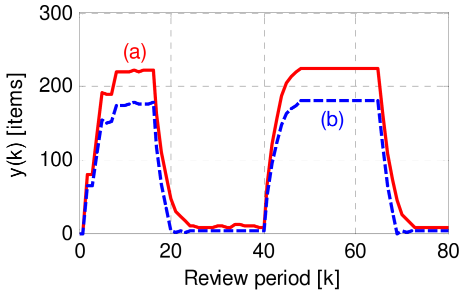

Fig. 5. On-hand stock: a) $y_{ref} = 225$ items, b) $y_{ref} = 180$ items

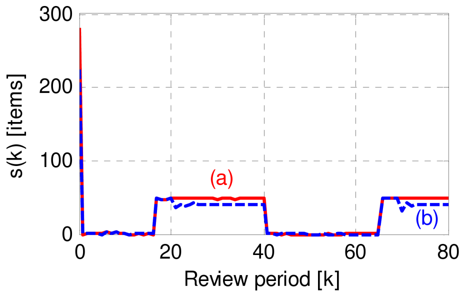

Fig. 6. Sliding variable $s(k)$: a) $y_{ref} = 225$ items, b) $y_{ref} = 180$ items

5. Conclusions

In this paper a new discrete time, chattering free sliding mode control strategy for periodic review supply chain management has been proposed. The strategy takes into account perishable inventories with transportation losses, i.e. not only it explicitly concerns goods decay in the warehouse, but it also accounts for the losses which take place during the delivery process. In order to ensure fast reaction of the controlled system to the unpredictable changes of demand, the sliding hyperplane is selected so that the dead-beat performance of the closed loop system is achieved. The proposed strategy ensures full demand satisfaction, eliminates the risk of warehouse overflow and always generates non-negative and bounded orders. These favourable properties have been formulated as theorems, formally proved and verified in a simulation example. The supply chain management strategy proposed in this paper is computationally efficient and straightforward in software implementation.

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