

ABEL'S METHOD ON SUMMATION BY PARTS AND BALANCED q -SERIES IDENTITIES

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The Abel method on summation by parts is reformulated to present new and elementary proofs of several classical identities of terminating balanced basic hypergeometric series. The examples strengthen our conviction that as traditional analytical instrument, the revised Abel method on summation by parts is indeed a very natural choice for working with basic hypergeometric series.

1. INTRODUCTION

For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and \triangle , respectively, by

$$(1) \quad \nabla\tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \triangle\tau_k = \tau_k - \tau_{k+1}$$

where \triangle is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign.

Then **Abel's lemma** on summation by parts may be reformulated as

$$\sum_{k=0}^{\infty} B_k \nabla A_k = [AB]_{\infty} - A_{-1} B_0 + \sum_{k=0}^{\infty} A_k \triangle B_k$$

provided that the limit $[AB]_{\infty} := \lim_{m \rightarrow \infty} A_m B_{m+1}$ exists and one of the nonterminating series just displayed is convergent.

In fact, according to the definition of the backward difference, we have

$$\sum_{k=0}^m B_k \nabla A_k = \sum_{k=0}^m B_k \{A_k - A_{k-1}\} = \sum_{k=0}^m A_k B_k - \sum_{k=0}^m A_{k-1} B_k.$$

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Replacing k by $1 + k$ for the last sum, we derive the following expression:

$$\begin{aligned} \sum_{k=0}^m B_k \nabla A_k &= A_m B_{m+1} - A_{-1} B_0 + \sum_{k=0}^m A_k \{B_k - B_{k+1}\} \\ &= A_m B_{m+1} - A_{-1} B_0 + \sum_{k=0}^m A_k \triangle B_k. \end{aligned}$$

Letting $m \rightarrow \infty$, we get the desired formula.

Recently, the author [4, 5] has systematically reviewed several fundamental basic hypergeometric series identities through the Abel lemma on summation by parts. The approach can briefly be described as follows:

- Applying Abel's lemma on summation by parts to a given q -series Ω , the machinery establishes a recurrence relation.
- Iterating the recursive equation derives a transformation formula on the Ω -series involving a new free integer parameter m .
- Truncating the Ω -series by specifying one of the parameters in the transformation yields a terminating series identity.
- Finally, the limiting case $m \rightarrow \infty$ of the transformation (if exists, of course) leads to a nonterminating series identity.

The objective of the present work is to explore the applications of Abel's lemma on summation by parts to terminating balanced q -series identities that, as a common underlying structure, involve q -shifted factorials with bases q and q^2 . Several q -series identities will be exemplified in a unified manner by means of Abel's lemma on summation by parts such as the q -analogues of the second Gauss summation theorem and the Bailey formula on ${}_2F_1(1/2)$ -series due to ANDREWS [1]; the terminating q -analogues of the Watson and Whipple formulae on ${}_3F_2$ -series, discovered respectively by ANDREWS [2] and JAIN [8]. Several new transformation formulae will also be established. They will show again that as classical analytic weapon, Abel's lemma on summation by parts is indeed a very natural and powerful method in dealing with basic hypergeometric series identities.

In order to facilitate the readability of the paper, we reproduce the notations of q -shifted factorial and basic hypergeometric series.

For two indeterminate q and x , the shifted-factorial with base q is defined by

$$(2) \quad (x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k) \quad \text{for } n = 1, 2, \dots$$

The product and fraction of shifted factorials are abbreviated respectively to

$$(3) \quad [\alpha, \beta, \dots, \gamma; q]_n = (\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n,$$

$$(4) \quad \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \middle| q \right]_n = \frac{(\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n}{(A; q)_n (B; q)_n \cdots (C; q)_n}.$$

Following BAILEY [3], GASPER-RAHMAN [6] and SLATER [9], the basic hypergeometric series is defined by

$$(5) \quad {}_{1+\lambda}\phi_\lambda \left[\begin{matrix} a_0, a_1, \dots, a_\lambda \\ b_1, \dots, b_\lambda \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} \left[\begin{matrix} a_0, a_1, \dots, a_\lambda \\ q, b_1, \dots, b_\lambda \end{matrix} \middle| q \right]_n z^n$$

where the base q will be restricted to $|q| < 1$ for nonterminating q -series.

When $qa_0a_1 \cdots a_\lambda = b_1b_2 \cdots b_\lambda$, the series just defined is called *balanced*, which will be the main subject of the present paper.

2. q -ANALOGUE OF GAUSS' SECOND SUMMATION THEOREM

Define the \mathfrak{G} -function by

$$(6) \quad \mathfrak{G}(a, b) := \sum_{k \geq 0} \frac{(a; q)_k (b; q)_k}{(q; q)_k (qab; q^2)_k} q^{\binom{k+1}{2}}.$$

For the two sequences defined by

$$A_k = \left[\begin{matrix} qa, & qb \\ q, & qab \end{matrix} \middle| q \right]_k \quad \text{and} \quad B_k = \frac{(qab; q)_k}{(qab; q^2)_k} q^{\binom{k}{2}}$$

it is almost trivial to compute the limiting relation

$$A_{-1}B_0 = [AB]_\infty = 0$$

as well as the finite differences

$$\nabla A_k = q^k \left[\begin{matrix} a, & b \\ q, & qab \end{matrix} \middle| q \right]_k \quad \text{and} \quad \Delta B_k = \frac{1 - q^k}{1 - qab} \frac{(qab; q)_k}{(q^3ab; q^2)_k} q^{\binom{k}{2}}.$$

Then we can manipulate the \mathfrak{G} -series by Abel's lemma on summation by parts as follows:

$$\begin{aligned} \mathfrak{G}(a, b) &= \sum_{k \geq 0} \frac{(a; q)_k (b; q)_k}{(q; q)_k (qab; q^2)_k} q^{\binom{k+1}{2}} \\ &= \sum_{k \geq 0} B_k \nabla A_k = \sum_{k \geq 0} A_k \Delta B_k \\ &= \sum_{k \geq 0} \frac{1 - q^k}{1 - qab} \frac{(qa; q)_k (qb; q)_k}{(q; q)_k (q^3ab; q^2)_k} q^{\binom{k}{2}} \\ &= \frac{(1 - qa)(1 - qb)}{(1 - qab)(1 - q^3ab)} \sum_{k \geq 0} \frac{(q^2a; q)_k (q^2b; q)_k}{(q; q)_k (q^5ab; q^2)_k} q^{\binom{k+1}{2}} \end{aligned}$$

where the last passage has been justified by shifting the summation index $k \rightarrow 1+k$.
Therefore, we have established the following recurrence relation

$$(7) \quad \mathfrak{G}(a, b) = \mathfrak{G}(q^2a, q^2b) \frac{(1-qa)(1-qb)}{(1-qab)(1-q^3ab)}.$$

Iterating this relation m -times, we get the following functional equation.

Lemma 1. (Recurrence relation on nonterminating series)

$$\mathfrak{G}(a, b) = \mathfrak{G}(q^{2m}a, q^{2m}b) \frac{(qa; q^2)_m (qb; q^2)_m}{(qab; q^2)_{2m}}.$$

Letting $m \rightarrow \infty$ and then appealing to Euler's q -exponential function (cf. [6, II.2])

$$\sum_{n \geq 0} \frac{q^{\binom{n}{2}} (-x)^n}{(q; q)_n} = (x; q)_\infty$$

we get the following limiting relation

$$\lim_{m \rightarrow \infty} \mathfrak{G}(q^m a, q^m b) = \sum_{k \geq 0} \frac{q^{\binom{k+1}{2}}}{(q; q)_k} = (-q; q)_\infty = \frac{1}{(q; q^2)_\infty}$$

which leads to the q -analogue of Gauss second theorem (cf. BAILEY [3, §2.4]).

Corollary 2. (ANDREWS [1, Eq. 1.8])

$$\sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k}{(q; q)_k (qab; q^2)_k} q^{\binom{k+1}{2}} = \frac{(qa; q^2)_\infty (qb; q^2)_\infty}{(q; q^2)_\infty (qab; q^2)_\infty}.$$

3. q -ANALOGUE OF BAILEY'S SUMMATION THEOREM

Define the \mathfrak{H} -function by

$$(8) \quad \mathfrak{H}(a, c) := \sum_{k \geq 0} \frac{(a; q)_k (q/a; q)_k}{(q^2; q^2)_k (c; q)_k} q^{\binom{k}{2}} c^k.$$

For the two sequences defined by

$$\mathcal{A}_k = \frac{(q/c; q)_k}{(q^2; q^2)_k} q^{\binom{k+1}{2}} c^k \quad \text{and} \quad \mathcal{B}_k = \left[\begin{array}{c} a, \quad q/a \\ 1/c, \quad qc \end{array} \middle| q \right]_k$$

it is not hard to check the limiting relation

$$\mathcal{A}_{-1}\mathcal{B}_0 = [\mathcal{A}\mathcal{B}]_\infty = 0$$

as well as the finite differences

$$\nabla\mathcal{A}_k = \frac{1 - q^k c}{1 - c} \frac{(1/c; q)_k}{(q^2; q^2)_k} q^{\binom{k}{2}} c^k \quad \text{and} \quad \Delta\mathcal{B}_k = \frac{(1 - ac)(1 - qc/a)}{(1 - c)(1 - qc)} \left[\begin{matrix} a, q/a \\ q/c, q^2 c \end{matrix} \middle| q \right]_k q^k.$$

Then we can reformulate the \mathfrak{H} -series by Abel's lemma on summation by parts as follows:

$$\begin{aligned} \mathfrak{H}(a, c) &= \sum_{k \geq 0} \frac{(a; q)_k (q/a; q)_k}{(q^2; q^2)_k (c; q)_k} q^{\binom{k}{2}} c^k = \sum_{k \geq 0} \mathcal{B}_k \nabla\mathcal{A}_k = \sum_{k \geq 0} \mathcal{A}_k \Delta\mathcal{B}_k \\ &= \frac{(1 - ac)(1 - qc/a)}{(1 - c)(1 - qc)} \sum_{k \geq 0} \frac{(a; q)_k (q/a; q)_k}{(q^2; q^2)_k (q^2 c; q)_k} q^{\binom{k}{2}} (q^2 c)^k. \end{aligned}$$

Therefore, we have established the following recurrence relation

$$(9) \quad \mathfrak{H}(a, c) = \mathfrak{H}(a, q^2 c) \frac{(1 - ac)(1 - qc/a)}{(1 - c)(1 - qc)}.$$

Iterating this relation m -times, we get the following functional equation.

Lemma 3. (Recurrence relation on nonterminating series)

$$\mathfrak{H}(a, c) = \mathfrak{H}(a, q^{2m} c) \frac{(ac; q^2)_m (qc/a; q^2)_m}{(c; q)_{2m}}.$$

Letting $m \rightarrow \infty$ and then noting that $\mathfrak{H}(a, 0) = 1$, we derive the following q -analogue of Bailey's summation theorem (cf. BAILEY [3, §2.4]).

Corollary 4. (ANDREWS [1, Eq. 1.9])

$$\sum_{k=0}^{\infty} \frac{(a; q)_k (q/a; q)_k}{(q^2; q^2)_k (c; q)_k} q^{\binom{k}{2}} c^k = \frac{(ac; q^2)_\infty (qc/a; q^2)_\infty}{(c; q)_\infty}.$$

4. q -WATSON SUMMATION FORMULAE DUE TO ANDREWS AND JAIN

Define the \mathfrak{W} -function by

$$(10) \quad \mathfrak{W}(a, b, c) := {}_4\phi_3 \left[\begin{matrix} a, b, \sqrt{c}, -\sqrt{c} \\ c, \sqrt{qab}, -\sqrt{qab} \end{matrix} \middle| q; q \right].$$

For the two sequences defined by

$$C_k = \left[\begin{matrix} qa, & qb \\ q, & qab \end{matrix} \middle| q \right]_k \quad \text{and} \quad D_k = \frac{(qab; q)_k (c; q^2)_k}{(qab; q^2)_k (c; q)_k}$$

it is not difficult to compute the limiting relation

$$C_{-1}D_0 = 0 \quad \text{and} \quad [CD]_\infty = \frac{(c; q^2)_\infty}{(qab; q^2)_\infty} \left[\begin{matrix} qa, & qb \\ q, & c \end{matrix} \middle| q \right]_\infty$$

as well as the finite differences

$$\nabla C_k = q^k \left[\begin{matrix} a, & b \\ q, & qab \end{matrix} \middle| q \right]_k \quad \text{and} \quad \Delta D_k = \frac{(1-q^k)(1-qab/c)}{(1-qab)(1-1/c)} \frac{(qab; q)_k (c; q^2)_k}{(q^3ab; q^2)_k (qc; q)_k} q^k.$$

Then the \mathfrak{W} -series can be manipulated through Abel's lemma on summation by parts as follows:

$$\begin{aligned} \mathfrak{W}(a, b, c) &= \sum_{k \geq 0} \frac{(a; q)_k (b; q)_k (c; q^2)_k}{(q; q)_k (c; q)_k (qab; q^2)_k} q^k = \sum_{k \geq 0} D_k \nabla C_k = [CD]_\infty + \sum_{k \geq 0} C_k \Delta D_k \\ &= \frac{(c; q^2)_\infty}{(qab; q^2)_\infty} \left[\begin{matrix} qa, & qb \\ q, & c \end{matrix} \middle| q \right]_\infty + \frac{(1-qab/c)}{(1-qab)(1-1/c)} \sum_{k \geq 0} (1-q^k) \frac{(qa; q)_k (qb; q)_k (c; q^2)_k}{(q; q)_k (qc; q)_k (q^3ab; q^2)_k} q^k \\ &= \frac{(c; q^2)_\infty}{(qab; q^2)_\infty} \left[\begin{matrix} qa, & qb \\ q, & c \end{matrix} \middle| q \right]_\infty + \frac{(1-qa)(1-qb)(1-qab/c)}{(1-qab)(1-q^3ab)(1-1/qc)} \sum_{k \geq 0} \frac{(q^2a; q)_k (q^2b; q)_k (q^2c; q^2)_k}{(q; q)_k (q^2c; q)_k (q^5ab; q^2)_k} q^k \end{aligned}$$

where the last passage has been justified by shifting the summation index $k \rightarrow 1+k$.

Therefore, we have established the following recurrence relation

$$(11) \quad \mathfrak{W}(a, b, c) = \mathfrak{W}(q^2a, q^2b, q^2c) \frac{(1-qa)(1-qb)(1-qab/c)}{(1-qab)(1-q^3ab)(1-1/qc)} + \frac{(c; q^2)_\infty}{(qab; q^2)_\infty} \left[\begin{matrix} qa, & qb \\ q, & c \end{matrix} \middle| q \right]_\infty.$$

Iterating this relation m -times leads us to the following partial sum expression

$$\begin{aligned} \mathfrak{W}(a, b, c) &= \mathfrak{W}(q^{2m}a, q^{2m}b, q^{2m}c) \frac{[qa, qb, qab/c; q^2]_m}{(qab; q^2)_{2m} (qc; q^2)_m} q^{m^2} (-c)^m \\ &+ \sum_{k=0}^{m-1} q^{k^2} (-c)^k \frac{[qa, qb, qab/c; q^2]_k}{(qab; q^2)_{2k} (qc; q^2)_k} \frac{(q^{2k}c; q^2)_\infty}{(q^{1+4k}ab; q^2)_\infty} \left[\begin{matrix} q^{1+2k}a, & q^{1+2k}b \\ q, & q^{2k}c \end{matrix} \middle| q \right]_\infty \end{aligned}$$

which can further be simplified to the following transformation theorem.

Theorem 5. (Transformation on balanced ${}_4\phi_3$ -series)

$$\begin{aligned} \mathfrak{W}(a, b, c) &= \mathfrak{W}(q^{2m}a, q^{2m}b, q^{2m}c) \frac{[qa, qb, qab/c; q^2]_m}{(qab; q^2)_{2m} (qc; q^2)_m} q^{m^2} (-c)^m \\ &+ \frac{(c; q^2)_\infty}{(qab; q^2)_\infty} \left[\begin{matrix} qa, & qb \\ q, & c \end{matrix} \middle| q \right]_\infty \sum_{k=0}^{m-1} \left[\begin{matrix} qab/c \\ q^2a, q^2b \end{matrix} \middle| q^2 \right]_k q^{k^2} (-c)^k. \end{aligned}$$

Taking $c = q^{-2m}$ in this theorem, we recover directly Jain's q -analogue of Watson's summation formula (cf. BAILEY [3, §3.3]).

Corollary 6. (Jain's terminating ${}_4\phi_3$ -series identity [8, Eq. 3.17] see also [7, Eq. 1.5])

$$\sum_{k=0}^n \frac{(a; q)_k (b; q)_k (q^{-2n}; q^2)_k}{(q; q)_k (qab; q^2)_k (q^{-2n}; q)_k} q^k = \frac{(qa; q^2)_n (qb; q^2)_n}{(q; q^2)_n (qab; q^2)_n}.$$

Instead, if we let $b = q^{-\delta-2m}$ with $\delta = 0, 1$ in Theorem 5, then we will obtain another terminating q -analogue of the above mentioned WATSON'S summation formula.

Corollary 7. (Andrews' terminating ${}_4\phi_3$ -series identity [2, Thm 1]; see also [7, Eq. 1.4] and [10, Eq. 1.1]) For $b = q^{-n}$ with n being a nonnegative integer, there holds the following identity.

$${}_4\phi_3 \left[\begin{matrix} a, b, \sqrt{c}, -\sqrt{c} \\ c, \sqrt{qab}, -\sqrt{qab} \end{matrix} \middle| q; q \right] = \begin{cases} \left[\begin{matrix} q, qc/a \\ q/a, qc \end{matrix} \middle| q^2 \right]_{n/2}, & n\text{-even;} \\ 0, & n\text{-odd.} \end{cases}$$

In addition, the limiting case $m \rightarrow \infty$ of Theorem 5 yields the following surprising transformation formula on nonterminating series.

Proposition 8. (Transformation on nonterminating balanced series)

$${}_4\phi_3 \left[\begin{matrix} a, b, \sqrt{c}, -\sqrt{c} \\ c, \sqrt{qab}, -\sqrt{qab} \end{matrix} \middle| q; q \right] = \frac{(c; q^2)_\infty}{(qab; q^2)_\infty} \left[\begin{matrix} qa, qb \\ q, c \end{matrix} \middle| q \right]_\infty \sum_{k=0}^{\infty} \left[\begin{matrix} qab/c \\ q^2 a, q^2 b \end{matrix} \middle| q^2 \right]_k q^{k^2} (-c)^k.$$

5. q -WHIPPLE SUMMATION FORMULAE

Define the \mathfrak{M} -function by

$$(12) \quad \mathfrak{M}(a, c, e) := {}_4\phi_3 \left[\begin{matrix} a, q/a, \sqrt{c}, -\sqrt{c} \\ -q, e, qc/e \end{matrix} \middle| q; q \right].$$

For the two sequences defined by

$$\mathcal{C}_k = \frac{(qa; q)_k (q^2 c; q^2)_k}{(qac; q)_k (q^2; q^2)_k} \quad \text{and} \quad \mathcal{D}_k = \left[\begin{matrix} 1/a, qac \\ e, qc/e \end{matrix} \middle| q \right]_k$$

it is routine to verify the limiting relation

$$\mathcal{C}_{-1} \mathcal{D}_0 = 0 \quad \text{and} \quad [\mathcal{CD}]_\infty = \frac{(q^2 c; q^2)_\infty}{(q^2; q^2)_\infty} \left[\begin{matrix} qa, 1/a \\ e, qc/e \end{matrix} \middle| q \right]_\infty$$

as well as the finite differences

$$\begin{aligned}\nabla\mathcal{C}_k &= \frac{1-q^k/a}{1-1/a} \frac{(a;q)_k}{(qac;q)_k} \frac{(c;q^2)_k}{(q^2;q^2)_k} q^k, \\ \Delta\mathcal{D}_k &= \frac{(1-ae)(1-qac/e)}{a(1-e)(1-qc/e)} \left[\begin{matrix} 1/a, & qac \\ qe, & q^2c/e \end{matrix} \middle| q \right]_k q^k.\end{aligned}$$

Then the \mathfrak{M} -series can be reformulated through Abel's lemma on summation by parts as follows:

$$\begin{aligned}\mathfrak{M}(a, c, e) &= \sum_{k \geq 0} \frac{(a;q)_k (q/a;q)_k (c;q^2)_k}{(q^2;q^2)_k (e;q)_k (qc/e;q)_k} q^k = \sum_{k \geq 0} \mathcal{D}_k \nabla\mathcal{C}_k = [\mathcal{CD}]_\infty + \sum_{k \geq 0} \mathcal{C}_k \Delta\mathcal{D}_k \\ &= \frac{(q^2c;q^2)_\infty}{(q^2;q^2)_\infty} \left[\begin{matrix} qa, & 1/a \\ e, & qc/e \end{matrix} \middle| q \right]_\infty + \frac{(1-ae)(1-qac/e)}{a(1-e)(1-qc/e)} \sum_{k \geq 0} \frac{(qa;q)_k (1/a;q)_k (q^2c;q^2)_k}{(q^2;q^2)_k (qe;q)_k (q^2c/e;q)_k} q^k.\end{aligned}$$

Therefore, we have established the following recurrence relation

$$(13) \quad \mathfrak{M}(a, c, e) = \mathfrak{M}(qa, q^2c, qe) \frac{(1-ae)(1-qac/e)}{a(1-e)(1-qc/e)} + \frac{(q^2c;q^2)_\infty}{(q^2;q^2)_\infty} \left[\begin{matrix} qa, & 1/a \\ e, & qc/e \end{matrix} \middle| q \right]_\infty.$$

Iterating this relation m -times, we get the following partial sum expression

$$\begin{aligned}\mathfrak{M}(a, c, e) &= \mathfrak{M}(q^m a, q^{2m} c, q^m e) \frac{(ae;q^2)_m (qac/e;q^2)_m}{(e;q)_m (qc/e;q)_m} q^{-\binom{m}{2}} a^{-m} \\ &+ \sum_{k=0}^{m-1} q^{-\binom{k}{2}} a^{-k} \frac{(ae;q^2)_k (qac/e;q^2)_k}{(e;q)_k (qc/e;q)_k} \left[\begin{matrix} q^{k+1}a, & q^{-k}/a \\ q^k e, & q^{k+1}c/e \end{matrix} \middle| q \right]_\infty \frac{(q^{2+2k}c;q^2)_\infty}{(q^2;q^2)_\infty}.\end{aligned}$$

Simplifying the last partial sum, we derive the following transformation theorem.

Theorem 9. (Transformation on balanced $4\phi_3$ -series)

$$\begin{aligned}\mathfrak{M}(a, c, e) &= \mathfrak{M}(q^m a, q^{2m} c, q^m e) \frac{(ae;q^2)_m (qac/e;q^2)_m}{(e;q)_m (qc/e;q)_m} q^{-\binom{m}{2}} a^{-m} \\ &+ \frac{(q^2c;q^2)_\infty}{(q^2;q^2)_\infty} \left[\begin{matrix} qa, & 1/a \\ e, & qc/e \end{matrix} \middle| q \right]_\infty \sum_{k=0}^{m-1} (-1)^k \left[\begin{matrix} ae, & qac/e \\ q^2c \end{matrix} \middle| q^2 \right]_k q^{-k^2} a^{-2k}.\end{aligned}$$

Taking $a = q^{-m}$ in this theorem, we recover directly the following terminating q -analogue of Whipple's summation formula (cf. BAILEY [3, §3.4]).

Corollary 10. (Andrews' terminating $4\phi_3$ -series identity [2, Thm. 2]; see also [10, Eq. 1.2]) *For $a = q^{-m}$ with m being a nonnegative integer, there holds the following identity.*

$${}_4\phi_3 \left[\begin{matrix} a, & q/a, & \sqrt{c}, & -\sqrt{c} \\ -q, & e, & qc/e \end{matrix} \middle| q; q \right] = q^{\binom{m+1}{2}} \frac{(ae;q^2)_m (qac/e;q^2)_m}{(e;q)_m (qc/e;q)_m}.$$

Instead, if we let $c = q^{-2m}$ in Theorem 9, then we will obtain another terminating q -analogue of the above mentioned Whipple's summation formula.

Corollary 11. (Jain's terminating ${}_4\phi_3$ -series identity [7, Eq. 3.19])

$${}_4\phi_3 \left[\begin{matrix} a, q/a, q^{-m}, -q^{-m} \\ -q, e, q^{1-2m}/e \end{matrix} \middle| q; q \right] = \frac{(ae; q^2)_m (qe/a; q^2)_m}{(e; q)_{2m}}.$$

6. TRANSFORMATION ON NONTERMINATING q -WHIPPLE SUMS

This section will further investigate transformations on the \mathfrak{M} -series defined in the last section. It will finally be expressed in terms of ${}_2\phi_1$ -series.

6.1. For the two sequences defined by

$$E_k = \frac{(q^2c; q^2)_k (e/c; q)_k}{(q^2; q^2)_k (e; q)_k} \quad \text{and} \quad F_k = \left[\begin{matrix} a, q/a \\ q^2c/e, q^{-1}e/c \end{matrix} \middle| q \right]_k$$

we have no difficulty to confirm the limiting relation

$$E_{-1}F_0 = 0 \quad \text{and} \quad [EF]_\infty = -qc/e \left[\begin{matrix} a, q/a \\ e, qc/e \end{matrix} \middle| q \right]_\infty \frac{(q^2c; q^2)_\infty}{(q^2; q^2)_\infty}$$

as well as the finite differences

$$\begin{aligned} \nabla E_k &= \frac{1 - q^{k+1}c/e}{1 - qc/e} \frac{(c; q^2)_k (q^{-1}e/c; q)_k}{(q^2; q^2)_k (e; q)_k} q^k, \\ \Delta F_k &= \frac{(1 - qac/e)(1 - q^2c/ae)}{(1 - qc/e)(1 - q^2c/e)} \left[\begin{matrix} a, q/a \\ q^3c/e, e/c \end{matrix} \middle| q \right]_k q^k. \end{aligned}$$

Then Abel's lemma on summation by parts can be used to manipulate the \mathfrak{M} -series defined in the last section as follows:

$$\begin{aligned} \mathfrak{M}(a, c, e) &= \sum_{k \geq 0} \frac{(a; q)_k (q/a; q)_k (c; q^2)_k}{(q^2; q^2)_k (e; q)_k (qc/e; q)_k} q^k = \sum_{k \geq 0} F_k \nabla E_k = [EF]_\infty + \sum_{k \geq 0} E_k \Delta F_k \\ &= \frac{-qc}{e} \left[\begin{matrix} a, q/a \\ e, qc/e \end{matrix} \middle| q \right]_\infty \frac{(q^2c; q^2)_\infty}{(q^2; q^2)_\infty} + \frac{(1 - qac/e)(1 - q^2c/ae)}{(1 - qc/e)(1 - q^2c/e)} \sum_{k \geq 0} \frac{(a; q)_k (q/a; q)_k (q^2c; q^2)_k}{(q^2; q^2)_k (e; q)_k (q^3c/e; q)_k} q^k. \end{aligned}$$

Therefore, we have established another recurrence relation

$$(14) \quad \mathfrak{M}(a, c, e) = \mathfrak{M}(a, q^2c, e) \frac{(1 - qac/e)(1 - q^2c/ae)}{(1 - qc/e)(1 - q^2c/e)} - \frac{qc}{e} \left[\begin{matrix} a, q/a \\ e, qc/e \end{matrix} \middle| q \right]_\infty \frac{(q^2c; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Iterating this relation m -times, we get the following partial sum expression

$$\begin{aligned} \mathfrak{M}(a, c, e) &= \mathfrak{M}(a, q^{2m}c, e) \frac{(qac/e; q^2)_m (q^2c/ae; q^2)_m}{(qc/e; q)_{2m}} \\ &\quad - \frac{qc}{e} \sum_{k=0}^{m-1} q^{2k} \left[\begin{matrix} qac/e, q^2c/ae \\ qc/e, q^2c/e \end{matrix} \middle| q^2 \right]_k \frac{(q^{2+2k}c; q^2)_\infty}{(q^2; q^2)_\infty} \left[\begin{matrix} a, q/a \\ e, q^{1+2k}c/e \end{matrix} \middle| q \right]_\infty. \end{aligned}$$

Simplifying the last partial sum, we derive the following transformation theorem.

Theorem 12. (Transformation on balanced ${}_4\phi_3$ -series)

$$\begin{aligned} \mathfrak{M}(a, c, e) &= \mathfrak{M}(a, q^{2m}c, e) \frac{(qac/e; q^2)_m (q^2c/ae; q^2)_m}{(qc/e; q)_{2m}} \\ &\quad - \frac{qc}{e} \frac{(q^2c; q^2)_\infty}{(q^2; q^2)_\infty} \left[\begin{matrix} a, q/a \\ e, qc/e \end{matrix} \middle| q \right]_\infty \sum_{k=0}^{m-1} \left[\begin{matrix} qac/e, q^2c/ae \\ q^2c \end{matrix} \middle| q^2 \right]_k q^{2k}. \end{aligned}$$

Taking $c = q^{-2m}$ in this theorem, we recover the terminating balanced series identity stated in Corollary 11. If letting $m \rightarrow \infty$ in Theorem 12, then we derive the following interesting transformation.

Corollary 13. (Nonterminating transformation formula)

$$\begin{aligned} {}_4\phi_3 \left[\begin{matrix} a, q/a, \sqrt{c}, -\sqrt{c} \\ -q, e, qc/e \end{matrix} \middle| q; q \right] &= {}_3\phi_2 \left[\begin{matrix} 0, a, q/a \\ -q, e \end{matrix} \middle| q; q \right] \frac{(qac/e; q^2)_\infty (q^2c/ae; q^2)_\infty}{(qc/e; q)_\infty} \\ &\quad - \frac{qc}{e} \frac{(q^2c; q^2)_\infty}{(q^2; q^2)_\infty} \left[\begin{matrix} a, q/a \\ e, qc/e \end{matrix} \middle| q \right]_\infty \sum_{k=0}^{\infty} \left[\begin{matrix} qac/e, q^2c/ae \\ q^2c \end{matrix} \middle| q^2 \right]_k q^{2k}. \end{aligned}$$

6.2. Denote the last ${}_3\phi_2$ -series by

$$(15) \quad \mathfrak{S}(a, e) := {}_3\phi_2 \left[\begin{matrix} 0, a, q/a \\ -q, e \end{matrix} \middle| q; q \right].$$

For the two sequences defined by

$$\mathcal{E}_k = \frac{(q/e; q)_k}{(q^2; q^2)_k} \quad \text{and} \quad \mathcal{F}_k = \left[\begin{matrix} a, q/a \\ 1/e, qe \end{matrix} \middle| q \right]_k$$

we can compute without difficulty the limiting relation

$$\mathcal{E}_{-1}\mathcal{F}_0 = 0 \quad \text{and} \quad [\mathcal{E}\mathcal{F}]_\infty = \frac{-e}{(q^2; q^2)_\infty} \left[\begin{matrix} a, q/a \\ e \end{matrix} \middle| q \right]_\infty$$

as well as the finite differences

$$\nabla \mathcal{E}_k = \frac{1 - q^k e (1/e; q)_k}{1 - e} q^k \quad \text{and} \quad \Delta \mathcal{F}_k = \frac{(1 - ae)(1 - qe/a)}{(1 - e)(1 - qe)} \left[\begin{matrix} a, q/a \\ q/e, q^2e \end{matrix} \middle| q \right]_k q^k.$$

Then Abel's lemma on summation by parts can be utilized to reformulate the \mathfrak{S} -series as follows:

$$\begin{aligned}\mathfrak{S}(a, e) &= \sum_{k \geq 0} \frac{(a; q)_k (q/a; q)_k}{(q^2; q^2)_k (e; q)_k} q^k = \sum_{k \geq 0} \mathcal{F}_k \nabla \mathcal{E}_k = [\mathcal{E}\mathcal{F}]_\infty + \sum_{k \geq 0} \mathcal{E}_k \Delta \mathcal{F}_k \\ &= \frac{-e}{(q^2; q^2)_\infty} \left[\begin{matrix} a, q/a \\ e \end{matrix} \middle| q \right]_\infty + \frac{(1-ae)(1-qe/a)}{(1-e)(1-qe)} \sum_{k \geq 0} \frac{(a; q)_k (q/a; q)_k}{(q^2; q^2)_k (q^2e; q)_k} q^k.\end{aligned}$$

Therefore, we have established the following recurrence relation

$$(16) \quad \mathfrak{S}(a, e) = \mathfrak{S}(a, q^2e) \frac{(1-ae)(1-qe/a)}{(1-e)(1-qe)} - \frac{e}{(q^2; q^2)_\infty} \left[\begin{matrix} a, q/a \\ e \end{matrix} \middle| q \right]_\infty.$$

Iterating this relation m -times, we get the following partial sum expression

$$\mathfrak{S}(a, e) = \mathfrak{S}(a, q^{2m}e) \frac{(ae; q^2)_m (qe/a; q^2)_m}{(e; q)_{2m}} - \sum_{k=0}^{m-1} \frac{(ae; q^2)_k (qe/a; q^2)_k}{(e; q)_{2k}} \left[\begin{matrix} a, q/a \\ q^{2k}e \end{matrix} \middle| q \right]_\infty \frac{q^{2k}e}{(q^2; q^2)_\infty}.$$

Simplifying the last partial sum, we derive the following transformation theorem.

Theorem 14. (Transformation on nonterminating ${}_3\phi_2$ -series)

$$\mathfrak{S}(a, e) = \mathfrak{S}(a, q^{2m}e) \frac{(ae; q^2)_m (qe/a; q^2)_m}{(e; q)_{2m}} - \frac{e}{(q^2; q^2)_\infty} \left[\begin{matrix} a, q/a \\ e \end{matrix} \middle| q \right]_\infty \sum_{k=0}^{m-1} q^{2k} [ae, qe/a; q^2]_k.$$

Letting $m \rightarrow \infty$ in this theorem, we find the following very strange transformation.

Corollary 15. (Transformation on nonterminating ${}_3\phi_2$ -series)

$$\begin{aligned}{}_3\phi_2 \left[\begin{matrix} 0, a, q/a \\ -q, e \end{matrix} \middle| q; q \right] &= {}_2\phi_1 \left[\begin{matrix} a, q/a \\ -q \end{matrix} \middle| q; q \right] \times \frac{(ae; q^2)_\infty (qe/a; q^2)_\infty}{(e; q)_\infty} \\ &\quad - \frac{e}{(q^2; q^2)_\infty} \left[\begin{matrix} a, q/a \\ e \end{matrix} \middle| q \right]_\infty \sum_{k=0}^{\infty} q^{2k} [ae, qe/a; q^2]_k.\end{aligned}$$

Substituting this transformation into Corollary 13, we derive the following less elegant transformation formula.

Corollary 16. (Transformation formula on nonterminating balanced ${}_4\phi_3$ -series)

$$\begin{aligned}{}_4\phi_3 \left[\begin{matrix} a, q/a, \sqrt{c}, -\sqrt{c} \\ -q, e, qc/e \end{matrix} \middle| q; q \right] &= {}_2\phi_1 \left[\begin{matrix} a, q/a \\ -q \end{matrix} \middle| q; q \right] \frac{[ae, qe/a, qac/e, q^2c/ae; q^2]_\infty}{[e, qc/e; q]_\infty} \\ &\quad - e \left[\begin{matrix} qac/e, q^2c/ae \\ q^2 \end{matrix} \middle| q^2 \right]_\infty \left[\begin{matrix} a, q/a \\ e, qc/e \end{matrix} \middle| q \right]_\infty \sum_{k=0}^{\infty} q^{2k} [ae, qe/a; q^2]_k \\ &\quad - \frac{qc (q^2c; q^2)_\infty}{e (q^2; q^2)_\infty} \left[\begin{matrix} a, q/a \\ e, qc/e \end{matrix} \middle| q \right]_\infty \sum_{k=0}^{\infty} \left[\begin{matrix} qac/e, q^2c/ae \\ q^2c \end{matrix} \middle| q^2 \right]_k q^{2k}.\end{aligned}$$

Taking $a = q^{-m}$ in this Corollary and then applying the q -Chu-Vandermonde-Gauss summation theorem [6, II.6], we recover the identity stated in Corollary 10.

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