NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 39 (2009), 215-239

ASYMPTOTIC PROPERTIES OF COMPOSITION OPERATORS ON THE BERGMAN SPACE

Namita Das and Pabitra Kumar Jena

(Received December 2009)

Abstract. In this paper we derive certain asymptotic properties of composition operators involving the shift operators, Berezin transform and a class of unitary operators defined on the Bergman space. We also discuss about some intertwining properties of composition operators which leads to obtain many ergodicity properties of composition operators.

1. Introduction

Let $dA(z)$ denote the Lebesgue area measure on the open unit disk $\mathbb{D}$, normalized so that the measure of the disk $\mathbb{D}$ equals 1. The Bergman space $L_a^2(\mathbb{D})$ is the Hilbert space consisting of analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $K_z \in L_a^2(\mathbb{D})$ such that $f(z) = \langle f, K_z \rangle$ for every $f \in L_a^2(\mathbb{D})$. The normalized reproducing kernel k_z is the function $\frac{K_z}{\|K_z\|_2}$. Here the norm $\| \cdot \|_2$ and the inner product $\langle , \rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1} z^n$. Then $\{e_n\}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$. Let $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1 - z\overline{w})^2} = \sum_{n=0}^{\infty} e_n(z) \overline{e_n(w)}$. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $L_a^2(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$; here P is the orthogonal projection from $L^2(D, dA)$ onto $L_a^2(\mathbb{D})$. The Hankel operator $H_\phi : L_a^2(\mathbb{D}) \to (L_a^2(\mathbb{D}))^\perp$ with symbol $\phi \in L^\infty(\mathbb{D})$ is defined by $H_\phi f = (I - P)(\phi f)$. The little Hankel operator $S_\phi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ is defined by $S_\phi f = PJ(\phi f)$ where $J : L^2(\mathbb{D}, dA) \to L^2(\mathbb{D}, dA)$ is defined as $Jf(z) = f(\overline{z})$. Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$ and $h^\infty(\mathbb{D})$ be the space of bounded harmonic functions on $\mathbb{D}$. Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $Aut(\mathbb{D})$ such that
(i) $(\phi_a \ o \ \phi_a)(z) \equiv z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

2010 *Mathematics Subject Classification* : 47B38, 47B33, 47B35.

Keywords : composition operators, Bergman space, Berezin transform.

In fact, $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is $J_{\phi_a(z)} = |k_a(z)|^2 = \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}$. Given $\lambda \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_\lambda f$ on $\mathbb{D}$ by $U_\lambda f(z) = k_\lambda(z) f(\phi_\lambda(z))$. Notice that U_λ is a bounded linear operator on $L^2(\mathbb{D}, dA)$ and $L^2_a(\mathbb{D})$ for all $\lambda \in \mathbb{D}$. Further, it can be verified that $U^2_\lambda = I$, the identity operator, $U^*_\lambda = U_\lambda, U_\lambda(L^2_a(\mathbb{D})) \subset L^2_a(\mathbb{D})$ and $U_\lambda((L^2_a(\mathbb{D}))^\perp) \subset (L^2_a(\mathbb{D}))^\perp$ for all $\lambda \in \mathbb{D}$. Thus $U_\lambda P = PU_\lambda$ for all $\lambda \in \mathbb{D}$. Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic. Define the composition operator C_ϕ from $L^2_a(\mathbb{D})$ into itself by $C_\phi f = f o \phi$. The operator C_ϕ is a bounded linear operator on $L^2_a(\mathbb{D})$ and $\|C_\phi\| \leq \frac{1+|\phi(0)|}{1-|\phi(0)|}$. Given $a \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define the function $C_a f = f o \phi_a$, where $\phi_a \in Aut(\mathbb{D})$. The map C_a is a composition operator on $L^2_a(\mathbb{D})$. Let $L^2_h(\mathbb{D}) = L^2_a(\mathbb{D}) \bigoplus (\overline{L^2_a(\mathbb{D})})_0$ where $(\overline{L^2_a(\mathbb{D})})_0 = \{\bar{f} : f \in L^2_a(\mathbb{D}), f(0) = 0\}$ and L^1_h be the space of harmonic functions in $L^1(\mathbb{D}, dA)$. Let $H(\mathbb{D})$ be the space of holomorphic functions from $\mathbb{D}$ into itself.

In this paper we discuss about some asymptotic properties and ergodicity properties of composition operators defined on the Bergman space. In section 2, we derive certain asymptotic properties of composition operators involving the shift operators, the Berezin transform and the class of unitary operators $U_\lambda, \lambda \in \mathbb{D}$. In section 3, we established certain intertwining properties of composition operators and exploited these to obtain some ergodicity properties of composition operators. The applications of these results are discussed in section 4.

Let $\mathcal{L}(L^2_a(\mathbb{D}))$ be the space of all bounded linear operators from $L^2_a(\mathbb{D})$ into itself and $\mathcal{LC}(L^2_a(\mathbb{D}))$ be the set of all compact operators in $\mathcal{L}(L^2_a(\mathbb{D}))$. For a bounded linear operator S on $L^2_a(\mathbb{D})$, the Berezin transform of S is the function $B(S)$ on $\mathbb{D}$ defined by $B(S)(z) = \langle Sk_z, k_z \rangle = \widetilde{S}(z)$. The Berezin transform $B(\phi)(z)$ of a function $\phi \in L^\infty(\mathbb{D}, dA)$ is defined to be the Berezin transform of the Toeplitz operator T_ϕ. In other words,

$$B(\phi)(z) = B(T_\phi)(z) = \int_{\mathbb{D}} \phi\left(\frac{z-w}{1-\bar{z}w}\right) dA(w) = \widetilde{\phi}(z).$$

The last equality follows from a change of variable in the definition of the Berezin transform. The above integral formula extends the Berezin transform to $L^1(\mathbb{D}, dA)$ and clearly gives $(B\phi)(z) = \phi(z)$ for any harmonic function $\phi \in L^1(\mathbb{D}, dA)$. For $\phi \in L^1(\mathbb{D}, dA)$, $B\phi = \widetilde{\phi}$ is an infinitely differentiable function on $\mathbb{D}$ and if ϕ is bounded then so is $B\phi = \widetilde{\phi}$ and $\|\widetilde{\phi}\|_\infty \leq \|\phi\|_\infty$. If $\phi \geq \psi$, then $\widetilde{\phi} \geq \widetilde{\psi}$. On $\mathbb{D}$, the only measure left invariant by all Mobius transformations is the pseudo-hyperbolic measure $d\eta(z) = \frac{dA(z)}{(1-|z|^2)^2}$. It turns out that the Berezin transform behaves well with respect to the invariant measures. The mapping $B : f \to \widetilde{f}$ is a contractive linear operator on each of the spaces $L^p(\mathbb{D}, d\eta), 1 \leq p \leq \infty$. Further, the Berezin transform B is a bounded operator on the spaces $L^p(\mathbb{D}, dA), 1 < p < \infty$ but B is not a bounded operator on $L^1(\mathbb{D}, dA)$. For details see [12].

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $L^2(\mathbb{T})$ be the space of square integrable, measurable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure on $\mathbb{T}$. The sequence $\{e^{int}\}_{n=-\infty}^{\infty}$ form an orthonorml basis for $L^2(\mathbb{T})$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of f are $a_n(f) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta, n \in \mathbb{Z}$ where $\mathbb{Z}$ is the set of all integers. The Hardy space $H^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ consisting of functions f with $a_n(f) = 0$ for all negative integers n. Since $a_n(f)$ is a bounded linear functional on $L^2(\mathbb{T})$ for any fixed n, and $H^2 = \displaystyle\bigcap_{n<0} \ker\, a_n$, it follows that $H^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$ and therefore a Hilbert space. Let $\mathcal{P}$ denote the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Let $L^\infty(\mathbb{T})$ be the space of all essentially bounded measurable functions on $\mathbb{T}$. For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator L_φ from $H^2(\mathbb{T})$ into itself is defined by $L_\varphi f = \mathcal{P}(\varphi f)$ and the Hankel operator D_φ from $H^2(\mathbb{T})$ into itself is defined by $D_\varphi f = \mathcal{P}(\mathcal{J}(\varphi f))$. Here $\mathcal{J}$ is the mapping from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$ defined by $\mathcal{J}f(e^{it}) = f(e^{-it})$. Let $H^\infty(\mathbb{T}) = \{f \in L^\infty(\mathbb{T}) : a_n(f) = 0 \quad \text{for } n < 0, n \in \mathbb{Z}\}$. A function $q \in H^\infty(\mathbb{T})$ is said to be an inner function if $|q(e^{it})| = 1$ almost everywhere on $\mathbb{T}$.

Let H and K be nonzero complex Hilbert spaces. The tensor product of $x \in H$ and $y \in K$ is a conjugate bilinear functional $x \otimes y : H \times K \longrightarrow \mathbb{C}$ defined by $(x \otimes y)(u, v) = \langle x, u\rangle\langle y, v\rangle$ for every $(u, v) \in H \times K$. The collection of all (finite) sums of tensors $x_i \otimes y_i$ with $x_i \in H$ and $y_i \in K$, denoted by $H \otimes K$, is a complex linear space equipped with an inner product $\langle , \rangle : (H \otimes K) \times (H \otimes K) \longrightarrow \mathbb{C}$ defined, for arbitrary $\displaystyle\sum_{i=1}^{N} x_i \otimes y_i$ and $\displaystyle\sum_{j=1}^{M} w_j \otimes z_j$ in $H \otimes K$, by $\left\langle \displaystyle\sum_{i=1}^{N} x_i \otimes y_i, \sum_{j=1}^{M} w_j \otimes z_j \right\rangle = \displaystyle\sum_{i=1}^{N}\sum_{j=1}^{M} \langle x_i, w_j\rangle \langle y_i, z_j\rangle$ (the same notation for the inner products on H, K and $H \otimes K$). The tensor product on $H \otimes K$ of two operators T in $\mathcal{L}(H)$ and S in $\mathcal{L}(K)$ is the operator $T \otimes S : H \otimes K \longrightarrow H \otimes K$ defined by $(T \otimes S)\displaystyle\sum_{i=1}^{N} x_i \otimes y_i = \sum_{i=1}^{N} Tx_i \otimes Sy_i$ for every $\displaystyle\sum_{i=1}^{N} x_i \otimes y_i \in H \otimes K$, which lies in $\mathcal{L}(H \times K)$. The completion of the inner product space $H \otimes K$, denoted by $H\hat{\otimes}K$, is the tensor product space of H and K . The extension of $T \otimes S$ over the Hilbert space $H\hat{\otimes}K$ denoted by $T\hat{\otimes}S$, is the tensor product of T and S on the tensor product space, which lies in $\mathcal{L}(H\hat{\otimes}K)$.

2. Asymptotic Properties Involving Shift Operators and Berezin Transform

Let $R : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ be such that $(Rf)(z) = \frac{f(z)-f(0)}{z}$. Let $S = T_z$, the Toeplitz operator on $L^2_a(\mathbb{D})$ with symbol z which is called the Bergman shift operator. Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then $Sf = \sum_{k=0}^{\infty} a_k z^{k+1}$, and $S^* f = \sum_{k=1}^{\infty} \frac{k}{k+1} a_k z^{k-1}$. Notice that $RS = I, R1 = 0$ and $Rz^j = z^{j-1}, j \geq 1$. Moreover,

$$T_{\bar{z}} z^j = P(\bar{z} z^j) = P(|z|^2 z^{j-1}) = \begin{cases} \frac{j}{j+1} z^{j-1}, & \text{if } j \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $R = T_{\bar{z}} + K$ where K is a compact operator on $L^2_a(\mathbb{D})$ and if $\{e_j\}_{j\geq 0} = \{\sqrt{j+1} z^j\}_{j \geq 0}$ is the standard orthonormal basis for $L^2_a(\mathbb{D})$, then

$$\langle Re_j, e_i \rangle = \begin{cases} \sqrt{\frac{j+1}{i+1}}, & \text{if } j = i+1; \\ 0, & \text{otherwise.} \end{cases}$$

Barria and Halmos [3] introduced the concept of asymptotic Toeplitz operators on the Hardy space as follows. An operator $T \in \mathcal{L}(H^2(\mathbb{T}))$ is asymptotic Toeplitz if $\{L^{*n} T L^n\}_{n=0}^{\infty}$ converges in the strong operator topology in $\mathcal{L}(H^2(\mathbb{T}))$ where L is the unilateral shift on $H^2(\mathbb{T})$. Shapiro in [27] obtained conditions on ϕ such that the composition operator C_ϕ on $H^2(\mathbb{T})$ is asymptotic Toeplitz. In this section we obtained conditions on ϕ such that $\{S^{*n} C_\phi S^n\}$ converges strongly as $n \to \infty$. We have shown that if $\phi : \mathbb{D} \to \mathbb{D}$ is holomorphic, then the sequence $\{S^{*n} C_\phi S^n\}$ and $\{S^{*n} C^*_\phi S^n\}$ converges in SOT. Further, we have shown that if ϕ fixes the origin but is not a rotation then the sequence $\{S^{*n} C_\phi S^n\}$ converges to 0 in the SOT.

Theorem 2.1 If $\phi : \mathbb{D} \to \mathbb{D}$ is holomorphic, then $S^{*n} C^*_\phi S^n \to 0$ strongly and $S^{*n} C_\phi S^n \to 0$ strongly.

Proof : Fix f and n. Then choose $g \in L^2_a(\mathbb{D}), \|g\| = 1$ such that $\|S^{*n} C^*_\phi S^n f\| = \left\langle S^{*n} C^*_\phi S^n f, g \right\rangle$. Thus

$$\begin{aligned} \|S^{*n} C^*_\phi S^n f\| &= \langle S^n f, C_\phi S^n g \rangle \\ &= \langle z^n f, \phi^n (go\phi) \rangle \\ &\leq \int_{\mathbb{D}} |f||go\phi||\phi|^n dA \\ &\leq \left(\int_{\mathbb{D}} |f|^2 |\phi|^{2n} dA \right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |go\phi|^2 dA \right)^{\frac{1}{2}} \\ &\leq \|C_\phi\| \left(\int_{\mathbb{D}} |f|^2 |\phi|^{2n} dA \right)^{\frac{1}{2}}. \end{aligned}$$

Since $|\phi| < 1$ on $\mathbb{D}$, the last integral above converges to 0 as $n \to \infty$ which establishes that $S^{*n} C_\phi^* S^n \to 0$ strongly. Now we shall show $S^{*n} C_\phi S^n \longrightarrow 0$ strongly. Fix $f \in L_a^2(\mathbb{D})$ and observe that because $\|S^*\| = 1, \|S^{*n} C_\phi S^n f\|^2 \le \|C_\phi S^n f\|^2 = \|\phi^n (f o \phi)\|^2 = \int_{\mathbb{D}} |\phi|^{2n} |f o \phi|^2 dA$. Since $|\phi| < 1$, we have $|\phi|^n \to 0$ a.e. on $\mathbb{D}$. Now because $f o \phi \in L_a^2(\mathbb{D})$, we see that $\|S^{*n} C_\phi S^n f\|^2 \to 0$ by the Lebesgue dominated convergence theorem. □

Theorem 2.2 If ϕ fixes the origin but is not a rotation, then the sequence $\{S^{*n} C_\phi^* S^n\}$ converges to 0 in the strong operator topology provided we assume $0.\infty = 0$.

Proof : The reproducing kernels $\{K_z : z \in \mathbb{D}\}$ have linear span dense in $L_a^2(\mathbb{D})$ and the operator norms $\|S^{*n} C_\phi^* S^n\|$ are uniformly bounded, so it suffices to prove that $\lim_{n\to\infty} \|S^{*n} C_\phi^* S^n K_z\| = 0$ for all $z \in \mathbb{D}$ where $S = T_w$. Notice that

$$
\begin{aligned}
(S^n K_z)(w) = w^n K_z(w) &= \frac{w^n}{(1-\overline{z}w)^2} \\
&= \frac{1}{\overline{z}^n} \left[K_z(w) - \frac{P_{n-1}(w)}{1-\overline{z}w} \right]
\end{aligned}
$$

where $P_{n-1}(w) = \sum_{k=0}^{n-1} (\overline{z}w)^k$. Thus $(C_\phi^* S^n K_z)(w) = \frac{1}{\overline{z}^n} \left[K_{\phi(z)}(w) - C_\phi^* \frac{P_{n-1}(w)}{1-\overline{z}w} \right]$. Since $\phi(0) = 0$, the operator C_ϕ has lower triangular matrix with respect to the orthonormal basis $\{e_n\}_{n=0}^\infty$ for $L_a^2(\mathbb{D})$. Thus C_ϕ^* has upper triangular matrix, that is, $C_\phi^* z^k$ is a polynomial of degree $\le k$ for each nonnegative integer k. Now $C_\phi^* \left(\frac{1}{1-\overline{z}w} \right) = \frac{1}{1-\overline{\phi(z)}w}$ and $S^{*n} \frac{1}{\overline{z}^n} C_\phi^* \left(\frac{1}{1-\overline{z}w} \right) = \frac{1}{\overline{z}^n} S^{*n} \left(\frac{1}{1-\overline{\phi(z)}w} \right) = \left(\frac{\overline{\phi(z)}}{\overline{z}} \right)^n \frac{1}{1-\overline{\phi(z)}w}$. Further,

$$
\begin{aligned}
S^{*n} \frac{1}{\overline{z}^n} C_\phi^* \left(\frac{\overline{z}w}{1-\overline{z}w} \right) &= S^{*n} \frac{1}{\overline{z}^n} C_\phi^* \left[\frac{1}{1-\overline{z}w} - 1 \right] \\
&= S^{*n} \frac{1}{\overline{z}^n} \left[C_\phi^* \left(\frac{1}{1-\overline{z}w} \right) - C_\phi^* 1 \right] \\
&= S^{*n} \frac{1}{\overline{z}^n} \left[\frac{1}{1-\overline{\phi(z)}w} - 1 \right] \\
&= \left(\frac{\overline{\phi(z)}}{\overline{z}} \right)^n \frac{1}{1-\overline{\phi(z)}w}
\end{aligned}
$$

and

$$\begin{aligned}
S^{*n}\frac{1}{\overline{z}^n}C_\phi^*\left(\frac{\overline{z}^2w^2}{1-\overline{z}w}\right) &= S^{*n}\frac{1}{\overline{z}^n}C_\phi^*\left[\frac{1}{1-\overline{z}w}-(1+\overline{z}w)\right]\\
&= S^{*n}\frac{1}{\overline{z}^n}\left[C_\phi^*\left(\frac{1}{1-\overline{z}w}\right)-C_\phi^*(1+\overline{z}w)\right]\\
&= S^{*n}\frac{1}{\overline{z}^n}\left[\frac{1}{1-\overline{\phi(z)}w}-1-C_\phi^*(\overline{z}w)\right]\\
&= \left(\frac{\overline{\phi(z)}}{\overline{z}}\right)^n\frac{1}{1-\overline{\phi(z)}w}-0=\left(\frac{\overline{\phi(z)}}{\overline{z}}\right)^n\frac{1}{1-\overline{\phi(z)}w}.
\end{aligned}$$

Similarly we can calculate

$$S^{*n}\frac{1}{\overline{z}^n}C_\phi^*\left(\frac{\overline{z}^kw^k}{1-\overline{z}w}\right)=\left(\frac{\overline{\phi(z)}}{\overline{z}}\right)^n\frac{1}{1-\overline{\phi(z)}w}\text{ for all } k=0,1,2,........,n-1.$$

Thus

$$\begin{aligned}
\left(S^{*n}C_\phi^*S^nK_z\right)(w) &= S^{*n}\frac{1}{\overline{z}^n}K_{\phi(z)}(w)-S^{*n}\frac{1}{\overline{z}^n}C_\phi^*\frac{P_{n-1}(w)}{1-\overline{z}w}\\
&= \left(\frac{\overline{\phi(z)}}{\overline{z}}\right)^n K_{\phi(z)}(w)-\sum_{k=0}^{n-1}\left(\frac{\overline{\phi(z)}}{\overline{z}}\right)^n\frac{1}{1-\overline{\phi(z)}w}\\
&= \left(\frac{\overline{\phi(z)}}{\overline{z}}\right)^n\left[K_{\phi(z)}(w)-\sum_{k=0}^{n-1}\frac{1}{1-\overline{\phi(z)}w}\right]\\
&= \left(\frac{\overline{\phi(z)}}{\overline{z}}\right)^n\left[K_{\phi(z)}(w)-\frac{n}{1-\overline{\phi(z)}w}\right].
\end{aligned}$$

Hence $\|S^{*n}C_\phi^*S^nK_z\|\le\left|\frac{\phi(z)}{z}\right|^n\left[\|K_{\phi(z)}\|+\|\frac{n}{1-\overline{\phi(z)}w}\|\right]$. Now since

$$\frac{1}{1-\overline{\phi(z)}w}=\sum_{k=0}^{\infty}\left(\overline{\phi(z)}\right)^k w^k,$$

we obtain

$$\begin{aligned}
\int_{\mathbb{D}}\frac{1}{\left|1-\overline{\phi(z)}w\right|^2}dA(w) &= \sum_{k=0}^{\infty}|\phi(z)|^{2k}\int_{\mathbb{D}}|w|^{2k}dA(w)\\
&= \sum_{k=0}^{\infty}\frac{|\phi(z)|^{2k}}{k+1}\\
&= \log\frac{1}{1-|\phi(z)|^2}
\end{aligned}$$

and $n\|\frac{1}{1-\overline{\phi(z)}w}\|=\log\frac{1}{(1-|\phi(z)|^2)^n}$ tends to ∞ as $n\to\infty$. Thus

$$\|S^{*n}C_\phi^*S^nK_z\|\le\left|\frac{\phi(z)}{z}\right|^n\left[\|K_{\phi(z)}\|+n\log\frac{1}{(1-|\phi(z)|^2)}\right]$$

for all $n \geq 0$ and $z \in \mathbb{D}$. Since ϕ fixes the origin and is not a rotation, the Schwarz lemma guarantees that $|\phi(z)| < |z|$ for all $z \in \mathbb{D}$ and $\left|\frac{\phi(z)}{z}\right|^n \|K_{\phi(z)}\| \longrightarrow 0$ as $n \to \infty$ for each $z \in \mathbb{D}$ and $\left|\frac{\phi(z)}{z}\right|^n n \log \frac{1}{(1-|\phi(z)|^2)} \longrightarrow 0$ as $n \longrightarrow \infty$ provided we assume $0.\infty = 0.\square$

Englis in [13] has shown that the set of all Toeplitz operators $T_\phi, \phi \in L^\infty(\mathbb{D})$ is dense in $\mathcal{L}(L^2_a(\mathbb{D}))$ in strong operator topology. In fact, the result remains in force if we consider Toeplitz operators with symbols in any w^* dense subset of $L^\infty(\mathbb{D})$. If $\mathcal{D}(\mathbb{D})$ is the set of all infinitely differentiable functions on $\mathbb{D}$ whose support is a compact subset of $\mathbb{D}$ then the set $\{T_\phi : \phi \in \mathcal{D}(\mathbb{D})\}$ is strong operator topology dense in $\mathcal{L}(L^2_a(\mathbb{D}))$. There is a natural intermediate function space between $\mathcal{D}(\mathbb{D})$ and $L^\infty(\mathbb{D})$, namely $C(\overline{\mathbb{D}})$, the functions continuous on the closed unit disk $\overline{\mathbb{D}}$. Englis also proved that [13], the norm closure of $\{T_\phi : \phi \in C(\overline{\mathbb{D}})\}$ coincides with $\{T_\phi : \phi \in C(\overline{\mathbb{D}})\} + \mathcal{LC}(L^2_a(\mathbb{D}))$. We shall show below that if $\psi \in C(\overline{\mathbb{D}})$ is harmonic and if C_ϕ is neither the identity nor a compact operator on $L^2_a(\mathbb{D})$ then $T_\psi - C_\phi$ cannot be compact. Now fix a composition operator C_ϕ, which is neither the identity nor compact and a Toeplitz operator T_ψ where $\psi \in C(\overline{\mathbb{D}})$ is harmonic and set $\Delta = T_\psi - C_\phi$. We claim Δ is not compact. Assume ψ is not a.e. zero on $\mathbb{T}$, since otherwise T_ψ is the zero operator and then $\Delta = -C_\phi$, which we are assuming non compact. It is enough to show that the adjoint operator $\Delta^* = T^*_\psi - C^*_\phi$ is not compact. Since $k_a \to 0$ weakly as $|a| \to 1^-$, to show Δ^* is not compact, it suffices to show that $\{\|\Delta^* k_a\|\}$ does not converge to zero as $|a| \to 1^-$. In what follows we shall prove $\{\langle \Delta^* k_a, k_a\rangle\}$ does not converge to zero. Now

$$
\begin{aligned}
\langle \Delta^* k_a, k_a\rangle &= \left\langle T_{\overline{\psi}} k_a, k_a \right\rangle - \left\langle C^*_\phi k_a, k_a \right\rangle \\
&= \left\langle \overline{\psi} k_a, k_a \right\rangle - \left(1 - |a|^2\right)^2 \left\langle K_{\phi(a)}, K_a \right\rangle \\
&= \widetilde{\overline{\psi}}(a) - \left(1 - |a|^2\right)^2 K_{\phi(a)}(a) \\
&= \widetilde{\overline{\psi}}(a) - \frac{\left(1 - |a|^2\right)^2}{\left(1 - \overline{\phi(a)} a\right)^2}
\end{aligned}
$$

where $\widetilde{\overline{\psi}}$ denotes the Berezin transform of $\overline{\psi}$. Since C_ϕ is not the identity operator on $L^2_a(\mathbb{D})$, the map ϕ is not the identity on $\mathbb{D}$. By the “ boundary uniqueness ” property of bounded holomorphic functions [10], the set $E = \{\xi \in \mathbb{T} : \phi(\xi) \neq \xi\}$ therefore has full measure : $m(E) = 1$ where dm denotes the Lebesgue measure on $\mathbb{T}$ and $\phi(\xi)$ denotes the radial limit of ϕ at ξ (which exists at m a.e. point of $\mathbb{T}$, see [10]). Since we have assumed ψ is not a.e. zero on $\mathbb{T}$, the set $F = \{\xi \in \mathbb{T} : \psi(\xi) \neq 0\}$ has positive measure. Thus $E \cap F$ has positive measure and in particular is not empty. Fix a point $\xi \in E \cap F$. We have as $r \to 1^-, \widetilde{\overline{\psi}}(r\xi) \to \overline{\psi}(\xi) \neq 0$ and

$\frac{(1-r^2)^2}{(1-\overline{\phi(r\xi)}r\xi)^2} \longrightarrow \frac{0}{(1-\overline{\phi(\xi)}\xi)^2} = 0$ (the denominator of the last function is not zero because $\phi(\xi) \neq \xi$). Thus $\lim_{r\to 1^-} \langle \Delta^* k_{r\xi}, k_{r\xi}\rangle = \bar{\psi}(\xi) \neq 0$, thus establishing that Δ^* and therefore Δ is not compact.

Let $H^2(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$ which are harmonic extensions of functions in $H^2(\mathbb{T})$. It is not very important to distinguish $H^2(\mathbb{D})$ from $H^2(\mathbb{T})$. For details see [33]. Hence in the sequel we shall always refer L_ϕ as a Toeplitz operator on $H^2(\mathbb{T})$ or $H^2(\mathbb{D})$ with symbol $\phi \in L^\infty(\mathbb{T})$. Suppose $\phi \in H(\mathbb{D})$. Then ϕ induces a linear operator $\mathcal{E}_\phi$ on the Hardy space $H^2(\mathbb{D})$ as follows : $\mathcal{E}_\phi f = fo\phi$. The operator $\mathcal{E}_\phi$ is called the composition operator induced by ϕ on $H^2(\mathbb{D})$. Let $L_z f = zf, z \in \mathbb{D}, f \in H^2(\mathbb{D})$.

Theorem 2.3 If $\phi, \psi \in H(\mathbb{D})$, then $(U_z \hat{\otimes} L_z)^{*^n}(C_\phi \hat{\otimes} \mathcal{E}_\psi) \xrightarrow{s} 0$ in $\mathcal{L}(L^2_a \hat{\otimes} H^2)$ for all $z \in \mathbb{D}$, where L_z is the operator of multiplication by z on $H^2(\mathbb{D})$. Further,

$$(U_z \hat{\otimes} L_z)^{*^n}(C_\phi \hat{\otimes} \mathcal{E}_\psi)(U_z \hat{\otimes} L_z)^n \xrightarrow{s} 0$$

and

$$(U_z \hat{\otimes} L_z)^{*^n}(C^*_\phi \hat{\otimes} \mathcal{E}^*_\psi)(U_z \hat{\otimes} L_z)^n \xrightarrow{s} 0.$$

Proof The operator U_z is an unitary operator for all $z \in \mathbb{D}$ and L_z is an isometry on $H^2(\mathbb{D})$. Hence by [22], $U_z \hat{\otimes} L_z$ in $\mathcal{L}(L^2_a(\mathbb{D}) \hat{\otimes} H^2(\mathbb{D}))$ is an unilateral shift for all $z \in \mathbb{D}$. Hence $(U_z \hat{\otimes} L_z)^{*^n} \xrightarrow{s} 0$ and $\|(U_z \hat{\otimes} L_z)\| = \|U_z\|\|L_z\| = 1$.

Further $(U_z \hat{\otimes} L_z)^{*^n}(C_\phi \hat{\otimes} \mathcal{E}_\psi)(U_z \hat{\otimes} L_z)^n = (U_z^{*n} C_\phi U_z^n) \hat{\otimes} (L_z^{*n} \mathcal{E}_\psi L_z^n)$. It follows from [27] that $L_z^{*n} \mathcal{E}_\psi L_z^n \xrightarrow{s} 0$ and $L_z^{*n} \mathcal{E}^*_\psi L_z^n \xrightarrow{s} 0$. The result follows since $\|U_z^{*n} C_\phi U_z^n\| \leq \|C_\phi\|$ for all n. □

Theorem 2.4 Let q be an inner function in $H^\infty(\mathbb{T})$ and L_q be the Toeplitz operator on $H^2(\mathbb{D})$ with symbol q and $\mathcal{E}_\psi$ be the composition operator with symbol ψ on $H^2(\mathbb{D})$. If $\phi, \psi \in H(\mathbb{D})$, then $(U_z \hat{\otimes} L_q)^{*^n}(C_\phi \hat{\otimes} \mathcal{E}_\psi) \longrightarrow 0$ in SOT in $\mathcal{L}\left(L^2_a(\mathbb{D}) \hat{\otimes} H^2(\mathbb{D})\right)$ for all $z \in \mathbb{D}$. Further,

$$(U_z \hat{\otimes} L_q)^{*^n}(C_\phi \hat{\otimes} \mathcal{E}_\psi)(U_z \hat{\otimes} L_q)^n \xrightarrow{s} 0$$

and

$$(U_z \hat{\otimes} L_q)^{*^n}(C^*_\phi \hat{\otimes} \mathcal{E}^*_\psi)(U_z \hat{\otimes} L_q)^n \xrightarrow{s} 0 \text{ for all } z \in \mathbb{D}.$$

Proof The proof is similar to theorem 2.3, as L_q is an isometry on $H^2(\mathbb{D})$ and it follows from [26] that $L_q^{*n} \mathcal{E}_\psi L_q^n \xrightarrow{s} 0$ and $L_q^{*n} \mathcal{E}^*_\psi L_q^n \xrightarrow{s} 0$ in $\mathcal{L}(H^2(\mathbb{D}))$ for all inner functions $q \in H^\infty(\mathbb{T})$. □

Recall that $\{e_n(z)\}_{n=0}^\infty = \{\sqrt{n+1} z^n\}_{n=0}^\infty$ is an orthonormal basis for $L^2_a(\mathbb{D})$. Let $\{u_n\}_{n=0}^\infty$ is an orthonormal basis for $(L^2_a(\mathbb{D}))^\perp$. Reindexing the sequence $\{u_n\}_{n=0}^\infty$ by a subset of $\mathbb{Z} \times \mathbb{Z}$, we obtain the collection $f_{ij} (i \geq 1, j \in \mathbb{Z})$ which is an orthonormal basis for $(L_a^2(\mathbb{D}))^{\perp}$. Thus $L^2(\mathbb{D}, dA)$ is the Hilbert space with an orthonormal basis formed by the vectors $\{e_i\}_{i\geq 0}$ and $\{f_{ij}\}_{i\geq 1, j\in\mathbb{Z}}$.

Define the function $r : \mathbb{N} \to \mathbb{N}$ by $r(k) = s$ whenever $2^{s-1} \leq k < 2^s (k \geq 1, s \geq 1)$. Define $T \in \mathcal{L}(L^2(\mathbb{D}, dA))$ by

$$\begin{aligned}
Tf_{i,j} &= f_{i,j-1} \ (i \geq 1, j \neq 0) \\
Tf_{i,0} &= 4^{-i} f_{i,-1} \ (i \geq 1) \\
Te_j &= e_{j+1} \ \left(j \notin \{3^k : k = 1, 2, ...\}\right) \\
Te_{3^k} &= e_{3^k+1} + f_{r(k),3^k} \ (k = 1, 2....).
\end{aligned}$$

Let $H_0 = L_a^2(\mathbb{D}, dA)$ and $H_i = \bigvee\{f_{ij}, j \in \mathbb{Z}\}, i = 1, 2, ...$. Then $L^2(\mathbb{D}, dA) = \bigoplus_{i=0}^{\infty} H_i$. In this decomposition T can be written in the matrix form as

$$T = \begin{pmatrix} S_0 & 0 & 0 & \cdots \\ Q_1 & S_1 & 0 & \cdots \\ Q_2 & 0 & S_2 & \cdots \\ \cdots & \cdots\cdots & \cdots & \cdots \\ \cdots & \cdots\cdots & \cdots & \cdots \end{pmatrix}$$

where S_0 is the unilateral shift (forward shift) and $S_i (i \geq 1)$ is a bilateral weighted shift. All weights of $S_i (i \geq 1)$ but one are equal to 1 and these are "backward" shifts.

It can be verified similarly as in [25] that the operator T is a power bounded operator $\left(\limsup_{n\to\infty} \|T^n f\| < \infty\right)$ and $T^n \xrightarrow{w} 0$ in $L^2(\mathbb{D}, dA)$. Since any two separable infinite dimensional Hilbert spaces are isometrically isomorphic, there is an unitary map U from $L^2(\mathbb{D}, dA)$ onto $L^2(\mathbb{D}, d\eta)$. Thus $M = UTU^* \in \mathcal{L}(L^2(\mathbb{D}, d\eta))$ and $M = UTU^*$ is power bounded and $M^n = UT^nU^* \xrightarrow{w} 0$ in $\mathcal{L}(L^2(\mathbb{D}, d\eta))$.

Thoerem 2.5 If $\phi, \psi \in H(\mathbb{D})$, then as an operator on $L^2(\mathbb{D}, d\eta), \{M^n C_\phi B^n\}$ converges to 0 in the strong operator topology and $(B^n \hat{\otimes} M^n)(C_\phi \hat{\otimes} C_\psi) \to 0$ in the strong operator topology in $\mathcal{L}\left(L^2(\mathbb{D}, d\eta) \hat{\otimes} L^2(\mathbb{D}, d\eta)\right)$.

Proof: Since $L^1(\mathbb{D}, d\eta) \subset L^1(\mathbb{D}, dA)$, the Berezin transform B is defined on the former space and

$$\begin{aligned}
|\widetilde{f}(w)| &= \left| \int_{\mathbb{D}} f(z) \frac{(1 - |w|^2)^2}{|1 - \overline{w}z|^4} dA(z) \right| \\
&\leq B(|f|)(w).
\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathbb{D}} |\widetilde{f}(w)| \frac{dA(w)}{(1-|w|^2)^2} &\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f(z)| \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4} dA(z) \right) \frac{dA(w)}{(1-|w|^2)^2} \\ &= \int_{\mathbb{D}} |f(z)| \int_{\mathbb{D}} \frac{dA(w)}{|1-\overline{w}z|^4} dA(z) \\ &= \int_{\mathbb{D}} |f(z)| \frac{dA(z)}{(1-|z|^2)^2} \ ,\end{aligned}$$

the change of order of integration being justified by the positivity of the integrand. It follows that B is a contraction on $L^1(\mathbb{D}, d\eta)$. The same is true for $L^\infty(\mathbb{D})$ and from the Marcinkiewicz interpolation theorem it follows that B is a contraction on $L^p(\mathbb{D}, d\eta)$, $1 \leq p \leq \infty$. Further on $L^2(\mathbb{D}, d\eta)$, $Bf = F(\Delta_h)f$ where Δ_h is the self adjoint operator given by $\Delta_h = (1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \overline{z}}$ on $\mathbb{D}$ and

$$\begin{aligned}F(x) &= \prod_{n=1}^{\infty} \left(1 - \frac{x}{n(n+1)}\right)^{-1} \\ &= \frac{\pi x}{\sin \pi(\sqrt{x+\frac{1}{4}} - \frac{1}{2})}.\end{aligned}$$

Thus the spectrum of B must be contained in [0,1]. Let $E(\lambda)$ be the resolution of identity for the self adjoint operator B. Then

$$\|B^n f\|^2 = \int_0^1 |\lambda^n|^2 d\langle E(\lambda)f, f\rangle.$$

Hence by Lebesgue monotone convergence theorem,

$$\|B^n f\|^2 \to \|(I - E(1^-))f\|^2 = \|P_{\ker(B-I)}f\|^2.$$

Since there is no nonzero harmonic function in $L^2(\mathbb{D}, d\eta)$ and $(B-I)g = 0$ implies g is harmonic in $L^2(\mathbb{D}, d\eta)$, hence $\ker(B-I) = \{0\}$ and therefore $\|B^n f\| \to 0$. That is, $B^n \to 0$ strongly in $L^2(\mathbb{D}, d\eta)$. Since M^n is power bounded, there exists a constant k such that $\|M^n\| \leq k$ for all n. Now for $f \in L^2(\mathbb{D}, d\eta)$,

$$\begin{aligned}\|M^n C_\phi B^n f\| &\leq \|M^n\| \|C_\phi\| \|B^n f\| \\ &\leq k \|C_\phi\| \|B^n f\| \longrightarrow 0 \ as \ n \to \infty.\end{aligned}$$

Since $B^n \longrightarrow 0$ strongly and $\|M^n\| \leq k$, hence by [22], the sequence $\{B^n \hat{\otimes} M^n\} \longrightarrow 0$ in strong operator topology in $\mathcal{L}\left(L^2(\mathbb{D}, d\eta) \hat{\otimes} L^2(\mathbb{D}, d\eta)\right)$. Thus

$$(B^n \hat{\otimes} M^n)(C_\phi \hat{\otimes} C_\psi) = (B^n C_\phi \hat{\otimes} M^n C_\psi) \longrightarrow 0 \ in \ SOT \ as \ n \to \infty. \square$$

Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. We say T is power bounded if $\limsup_{n\to\infty} \|T^n f\| < \infty$ for all $f \in L_a^2(\mathbb{D})$. If $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ is similar to a contraction, that is, if there exists an invertible operator $U \in \mathcal{L}(L_a^2(\mathbb{D}))$ with $\|UTU^{-1}\| \leq 1$, then T is power bounded, for in this case

$$\begin{aligned}\|T^n\| &= \|U^{-1}UT^nU^{-1}U\| \\ &\leq \|U^{-1}\|\|UTU^{-1}\|^n\|U\| \\ &\leq \|U^{-1}\|\|U\|\end{aligned}$$

for each $n \in \mathbb{N}$. The converse is not true; a power bounded operator need not be similar to a contraction [17]. The operator $C_\phi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ is an isometry [5] if and only if ϕ is a rotation. Below we find conditions such that C_ϕ is similar to an isometry.

Proposition 2.6 (i) Let C_ϕ be similar to a contraction on $L_a^2(\mathbb{D})$ and the set $\{f \in L_a^2(\mathbb{D}) : \lim_{r\to 1^+} \|(r-1)^{\frac{1}{2}}(r\xi - C_\phi)^{-1}f\| = 0$ for every $\xi \in \mathbb{T}\}$ is dense in $L_a^2(\mathbb{D})$ then $(C_aC_\phi C_a)^n \to 0$ and $(U_aC_\phi U_a)^n \to 0$ in the strong operator topology in $\mathcal{L}(L_a^2(\mathbb{D}))$.

(ii) Let $C_\phi \in \mathcal{L}(L_a^2(\mathbb{D}))$ be such that the sequence $\{\|C_\phi^n f\|\}_{n\geq 1}$ converges for every $f \in L_a^2(\mathbb{D})$, and let B_ϕ be the operator given by

$$\langle B_\phi f, g\rangle = \lim_{n\to\infty} \langle C_\phi^{*n} C_\phi^n f, g\rangle, f, g \in L_a^2(\mathbb{D}).$$

Then C_ϕ is similar to an isometry if and only if B_ϕ is invertible.

Proof:(i) The result follows from [31] as $(C_aC_\phi C_a)^n = C_aC_\phi^nC_a$ and $(U_aC_\phi U_a)^n = U_aC_\phi^nU_a$ for all $n \in N$ and $a \in \mathbb{D}$ and C_ϕ is power bounded.

(ii) Suppose that C_ϕ is similar to an isometry. Then there exists a positive invertible operator $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ with $\|S\| = 1$ such that $C_\phi^* S C_\phi = S$. Thus

$$\langle C_\phi^{*n}(I-S)C_\phi^n f, f\rangle \to \langle (B_\phi - S)f, f\rangle$$

as $n \to \infty$ for every $f \in L_a^2(\mathbb{D})$. Since $I - S \geq 0$ we obtain $B_\phi - S \geq 0$. Thus B_ϕ is invertible because S is positive and invertible in $\mathcal{L}(L_a^2(\mathbb{D}))$. Conversely, if B_ϕ is invertible then since $C_\phi^* B_\phi C_\phi = B_\phi$, we obtain C_ϕ is similar to an isometry. □

The operator $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ is asymptotically regular if $\lim_{n\to\infty} \|T^n f - T^{n+1} f\| = 0$ for all $f \in L_a^2(\mathbb{D})$. If the convergence of this limit is uniform, that is, $\|T^n - T^{n+1}\| \to 0$ as $n \to \infty$, we then say that T is uniformly asymptotically regular. The operator $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ is strongly stable if $\liminf_{n\to\infty} \|T^n f\| = 0$ for all $f \in L_a^2(\mathbb{D})$. For $a \in \mathbb{D}$, let m_a be the geodesic midpoint between 0 and a, that is, $m_a = \frac{1-\sqrt{1-|a|^2}}{|a|^2} a$.

Lemma 2.7 For any $a \in \mathbb{D}$, the operators C_a, U_a are not asymptotically regular and not uniformly asymptotically regular. Further, these operators are not strongly stable.

Proof: The points to note are the following :
(i) $C_a f = f$ if and only if $f = go\phi_{m_a}$ where g is an even function.
(ii) $U_a f = f$ if and only if $f = (go\phi_{m_a})k_{m_a}, g$ is an even function.
For proof of these see [32]. Since $C_a^2 = I$ and $U_a^2 = I$ for all $a \in \mathbb{D}$, hence there exists $f, g \in L_a^2(\mathbb{D})$ such that $\lim_{n\to\infty} \|C_a^n f - C_a^{n+1} f\| \neq 0$ and $\lim_{n\to\infty} \|U_a^n g - U_a^{n+1} g\| \neq 0$. It may be noted here that $\|C_a^n - C_a^{n+1}\| = \|I - C_a\|$ and $\|U_a^n - U_a^{n+1}\| = \|I - U_a\|$ for all $n \in \mathbb{N}$ which does not tend to zero as $n \to \infty$. Further, $\liminf_{n\to\infty} \|C_a^n f\| = \|f\| \neq 0$, if $f = go\phi_{m_a}, g$ even, $g \not\equiv 0$ and $\liminf_{n\to\infty} \|U_a^n f\| = \|f\| \neq 0$ if $f = (go\phi_{m_a})k_{m_a}$, g even, $g \not\equiv 0$. $\square$

The Berezin transform B on $L^2(\mathbb{D}, d\eta)$ has no nonzero fixed points [11] and B is strongly stable as by theorem 2.5, $B^n \to 0$ in the strong operator topology on $L^2(\mathbb{D}, d\eta)$. But it is not known to authors if B is uniformly asymptotic regular. Further if $\phi \in L^\infty(\mathbb{D})$ is such that H_ϕ is a Hilbert schmidt operator then

$$\sum_{n=0}^{\infty} \|B^n\phi - B^{n+1}\phi\|^2_{L^2(\mathbb{D},d\eta)} \leq \|H_\phi\|^2_{HS} < \infty.$$

For details see [2].

A power bounded operator T on a Hilbert space H is called almost weakly stable if 0 is a weak accumulation point of every orbit $\{T^n x : n \in \mathbb{N}\}, x \in H$.

Theorem 2.8 Let $\phi, \psi \in H(\mathbb{D})$. Then there exists a sequence of almost weakly stable unitary operators T_m on $L_a^2(\mathbb{D})$ and a sequence $\{z_m\}_{m=1}^\infty$ in $\mathbb{D}$ with $|z_m| \to 1$ such that $(U_{z_m}\hat{\otimes}T_m)(C_\phi\hat{\otimes}C_\psi) \xrightarrow{w} 0$ in $\mathcal{L}(L_a^2(\mathbb{D})\hat{\otimes}L_a^2(\mathbb{D}))$. If $\lim_{|z|\to 1^-} \dfrac{1-|z|^2}{1-|\phi(z)|^2} = 0$ then $(U_{z_m}\hat{\otimes}T_m)(C_\phi\hat{\otimes}C_\psi) \to 0$ in strong operator topology in $\mathcal{L}\left(L_a^2(\mathbb{D})\hat{\otimes}L_a^2(\mathbb{D})\right)$.
Proof: Since span$\{k_z : z \in \mathbb{D}\}$ is dense in $L_a^2(\mathbb{D})$, it suffices to show that for all $z, w \in \mathbb{D}$, we have $\lim_{m\to\infty} \langle U_{z_m}k_z, k_w\rangle = 0$. Fix $z, w \in \mathbb{D}$. For $m \geq 1$,

$$\begin{aligned}
\langle U_{z_m}k_z, k_w\rangle &= (1-|w|^2)(U_{z_m}k_z)(w) \\
&= (1-|w|^2)k_z(\phi_{z_m}(w))k_{z_m}(w) \\
&= \frac{[(1-|w|^2)(1-|z|^2)(1-|z_m|^2)]}{[(1-\langle\phi_{z_m}(w), z\rangle)(1-\langle w, z_m\rangle)]^2}.
\end{aligned}$$

Since $|\langle\phi_{z_m}(w), z\rangle| \leq |z|$ and $|\langle w, z_m\rangle| \leq |w|$, we obtain

$$|\langle U_{z_m}k_z, k_w\rangle| \leq \frac{(1-|w|^2)(1-|z|^2)(1-|z_m|^2)}{(1-|z|)^2(1-|w|)^2}.$$

It then follows that $\lim_{m\to\infty} \langle U_{z_m}k_z, k_w\rangle = 0$. From [14], it also follows that there exists a sequence of almost weakly stable unitary operators T_m such that $T_m \to I$ in norm in $\mathcal{L}(L_a^2(\mathbb{D}))$. Thus from [22], it follows that $(U_{z_m}\hat{\otimes}T_m) \xrightarrow{w} 0$. Now if

K is compact then $KU_{z_m} \xrightarrow{s} 0$ as $m \to \infty$ in $\mathcal{L}(L_a^2(\mathbb{D}))$. If $\phi : \mathbb{D} \to \mathbb{D}$ is analytic and $\lim_{|z| \to 1^-} \frac{1 - |z|^2}{1 - |\phi(z)|^2} = 0$, then it is shown in [33], that C_ϕ is compact and hence $(U_{z_m} \hat{\otimes} T_m)(C_\phi \hat{\otimes} C_\psi) = U_{z_m} C_\phi \hat{\otimes} T_m C_\psi \to 0$ in SOT. This follows from [22]. $\square$

An operator A on a Hilbert space H is said to be quasinormal if A commutes with A^*A. Moreover A is said to be subnormal if there exists a normal operator Q on a Hilbert space K such that H is a subspace of K, H is invariant under the operator Q, and the restriction of Q to H is A. The operator A is said to be hyponormal if and only if $A^*A - AA^* \geq 0$. The operator A is pure in case, the only subspace of H reducing A on which A is a normal operator is the zero subspace. It is well known that normal operators $\subset$ quasinormal operators $\subset$ subnormal operators $\subset$ hyponormal operators. Excellent references on subnormal and hyponormal operators are [6] and [7].

If $\Upsilon(z) = z + \lambda \bar{z}, |\lambda| < 1$, then the Toeplitz operator T_Υ defined on $L_a^2(\mathbb{D})$ is a pure hyponormal operator. To verify this let G be the smallest reducing subspace of T_Υ which contains the image of the self-commutator of $T_\Upsilon, T_{\overline{\Upsilon}} T_\Upsilon - T_\Upsilon T_{\overline{\Upsilon}}$. By direct computation $T_{\overline{\Upsilon}} T_\Upsilon - T_\Upsilon T_{\overline{\Upsilon}} = (1 - |\lambda|^2)(T_{\bar{z}} T_z - T_z T_{\bar{z}})$. But, for all $i \geq 0$, $(T_{\bar{z}} T_z - T_z T_{\bar{z}})(z^i) = \left(\frac{i+1}{i+2} - \frac{i}{i+1} \right) z^i$. Since the polynomials are dense in $L_a^2(\mathbb{D})$, it follows that $G = L_a^2(\mathbb{D})$, and this will imply that T_Υ is pure. Also, it is clear that T_Υ is hyponormal.

Under a smoothness assumption, Kriete and Cowen [9] have given conditions on ϕ that are necessary and sufficient that C_ϕ^* be subnormal on $H^2(\mathbb{T})$. For ϕ an analytic map of the unit disk into itself, Cowen in [8] has shown that the subnormality of C_ϕ^* on the Hardy space implies its subnormality on the Bergman space $L_a^2(\mathbb{D})$.

Theorem 2.9 Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic. If C_ϕ^* is a hyponormal contraction on $L_a^2(\mathbb{D})$ which is also pure then $C_\phi^{*n} \to 0$ in SOT in $\mathcal{L}(L_a^2(\mathbb{D}))$. Further, if for some $\Upsilon \in L^\infty(\mathbb{D})$, the Toeplitz operator T_Υ is a pure hyponormal contraction, then $T_\Upsilon^n C_\phi T_\Upsilon^{*n} \to 0$ in SOT.

Proof If C_ϕ^* is a pure hyponormal contraction, then from [30] it follows that $C_\phi^{*n} \to 0$ in SOT and if T_Υ is a pure hyponormal contraction then for $f \in L_a^2(\mathbb{D}), \|T_\Upsilon^n C_\phi T_\Upsilon^{*n} f\| \leq \|T_\Upsilon\|^n \|C_\phi\| \|T_\Upsilon^{*n} f\| \leq \|C_\phi\| \|T_\Upsilon^{*n} f\| \to 0$ as $n \to \infty$. $\square$

3. Intertwining Property of Composition Operators

In this section we discuss about some intertwining properties of composition operators and from it derive certain ergodicity properties of composition operators.

Theorem 3.1 Let $T, S \in \mathcal{L}(L_a^2(\mathbb{D}))$ are power bounded and $C_a T = S C_a$ for some $a \in \mathbb{D}$. Then $T^n \to 0$ in WOT if and only if $S^n \to 0$ in WOT. Further, if for each nonincreasing subsequence of positive integers (n_j) and every $f \in L^2_a(\mathbb{D})$ the limit $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} T^{n_j} f$ exists in norm topology then $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} S^{n_j} f$ exists for all $f \in L^2_a(\mathbb{D})$.

Proof Suppose $T^n \to 0$ in WOT. Then $\langle T^n h, h'\rangle \to 0$ for all $h, h' \in L^2_a(\mathbb{D})$. Thus for all $h, k \in L^2_a(\mathbb{D})$ we have, $\langle S^n C_a h, k\rangle = \langle C_a T^n h, k\rangle = \langle T^n h, C_a^* k\rangle \to 0$ as $n \to \infty$. Hence $S^n f \to 0$ weakly for all $f \in L^2_a(\mathbb{D})$ as Range $C_a = L^2_a(\mathbb{D})$. Since S is power bounded, we have $S^n \to 0$ in WOT.

Conversely, suppose that $S^n \to 0$ in WOT. Then $S^{*n} \to 0$ in WOT and $C_a^* S^* = T^* C_a^*$. Hence $T^{*n} \to 0$ in WOT and so $T^n \to 0$ in WOT.

Suppose $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} T^{n_j} f$ exists in the norm topology for each $f \in L^2_a(\mathbb{D})$. Then $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} S^{n_j} C_a f = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} C_a T^{n_j} f$ exists for each $f \in L^2_a(\mathbb{D})$. Since Range $C_a = L^2_a(\mathbb{D})$ and the sequence $\frac{1}{N}\sum_{j=1}^{N} T^{n_j}$ is bounded, the limit $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} S^{n_j} f$ exists for all $f \in L^2_a(\mathbb{D})$. $\square$

This result shows that the intertwining relations with the composition operators $C_a, a \in \mathbb{D}$ is important in describing certain asymptotic properties of composition operators on Bergman space.

Definition 3.2 A function $g(x, \bar{y})$ on $\mathbb{D} \times \mathbb{D}$ is called of positive type (or positive definite),written $g \gg 0$, if

$$\sum_{j,k=1}^{n} c_j \bar{c}_k g(x_j, \bar{x}_k) \geq 0 ....(1)$$

for any n-tuple of complex numbers $c_1, c_2, ...c_n$ and points $x_1, x_2, ...x_n \in \mathbb{D}$. We write $g \gg h$ if $g - h \gg 0$. We shall say $\gamma \in \mathcal{A}$ if $\gamma \in L^\infty(\mathbb{D})$ and is such that

$$\gamma(z) = \Theta(z, \bar{z}) ....(2)$$

where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$ meromorphic in x and conjugate meromorphic in y. It is a fact that (see [19],[21]) Θ as in (2), if it exists, is uniquely determined by γ. The function Θ satisfies the condition $(\star)$ if there exists a constant $C > 0$ such that

$$CK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg -CK(x, \bar{y}).$$

The function γ is said to satisfy the condition $(\star\star)$, if $\gamma \in \mathcal{A}$ and if $\gamma(z) = \Theta(z, \bar{z})$ as in (2), then there exists a constant $C > 0$ such that

$$\Theta_1(x, \bar{y}) = \Theta(x, \bar{y}) + \overline{\Theta(y.\bar{x})}$$

and

$$\Theta_2(x,\bar{y}) = i[\Theta(x,\bar{y}) - \overline{\Theta(y,\bar{x})}]$$

satisfy the condition $(\star)$.

The following theorem gives a necessary and sufficient condition for the existence of a bounded linear operator T on $L_a^2(\mathbb{D})$ with prescribed Berezin transform and is such that $TC_a = C_aT$ for all $a \in \mathbb{D}$.

Theorem 3.3 Let $f \in L^\infty(\mathbb{D})$. Then $f(w) = \int_{\mathbb{D}} f(\phi_z(w))dA(z)$ for all $w \in \mathbb{D}$ and f satisfies $(\star\star)$, if and only if there exists $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\widetilde{T}(z) = f(z)$ and $C_aT = TC_a$ for all $a \in \mathbb{D}$.

Proof: Let $f \in L^\infty(\mathbb{D})$ and f satisfies $(\star\star)$. Then by [11], there exists $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\widetilde{T}(w) = \langle Tk_w, k_w\rangle = f(w)$ for all $w \in \mathbb{D}$. Now $f(w) = \int_{\mathbb{D}} f\left(\phi_z(w)\right) dA(z)$ implies

$$\begin{aligned}
\langle Tk_w, k_w\rangle &= \int_{\mathbb{D}} \left\langle Tk_{\phi_z(w)}, k_{\phi_z(w)}\right\rangle dA(z)\\
&= \int_{\mathbb{D}} \langle U_zTU_zk_w, k_w\rangle\, dA(z)\\
&= \left\langle \left(\int_{\mathbb{D}} U_zTU_z dA(z)\right) k_w, k_w \right\rangle\\
&= \langle \widehat{T}k_w, k_w\rangle
\end{aligned}$$

for all $w \in \mathbb{D}$ where $\widehat{T} = \displaystyle\int_{\mathbb{D}} U_zTU_z dA(z)$. From [11], it follows that $T = \widehat{T}$. It is shown in [32] that if $T = \widehat{T}$ then $TC_a = C_aT$ for all $a \in \mathbb{D}$. Conversely, if there exists $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\widetilde{T}(z) = f(z)$ and $TC_a = C_aT$ for all $a \in \mathbb{D}$, then $\widetilde{T} \in L^\infty(\mathbb{D})$ and $\widetilde{T}$ satisfies the condition $(\star\star)$. For details see [11]. Now since $TC_a = C_aT$ for all $a \in \mathbb{D}$ we have $\widehat{T} = T$. For more details see [32]. Hence

$$\begin{aligned}
f(w) &= \widetilde{T}(w)\\
&= \langle Tk_w, k_w\rangle\\
&= \left\langle \left(\int_{\mathbb{D}} U_zTU_z dA(z)\right) k_w, k_w \right\rangle\\
&= \int_{\mathbb{D}} \langle U_zTU_zk_w, k_w\rangle dA(z)\\
&= \int_{\mathbb{D}} \langle Tk_{\phi_z(w)}, k_{\phi_z(w)}\rangle dA(z)\\
&= \int_{\mathbb{D}} \widetilde{T}(\phi_z(w)) dA(z)\\
&= \int_{\mathbb{D}} f(\phi_z(w)) dA(z).
\end{aligned}$$

The result follows. □

In the following theorem, we find conditions on C_ϕ such that $C_aC_\phi = C_\phi C_a^*$.

Theorem 3.4 Let $a \in \mathbb{D}$ and C_ϕ be a positive composition operator on $L_a^2(\mathbb{D})$. If $C_\phi \leq Re(C_aC_\phi)$, then $C_aC_\phi C_a^* = C_\phi$ and hence $C_a^n C_\phi C_a^{*n} \to C_\phi$ strongly. Further in this case $\phi = go\phi_{m_a}$ where g is an even function.

Proof: For $f \in L_a^2(\mathbb{D})$, by Heinz inequality [20], we obtain

$$\begin{aligned}\langle C_\phi f, f\rangle &\leq \langle Re(C_aC_\phi)f, f\rangle \\ &= Re\langle C_aC_\phi f, f\rangle \\ &\leq |\langle C_aC_\phi f, f\rangle| \\ &= \langle C_\phi f, C_a^* f\rangle| \\ &\leq \langle C_\phi f, f\rangle^{\frac{1}{2}}\langle C_\phi C_a^* f, C_a^* f\rangle^{\frac{1}{2}} \text{................}(3).\end{aligned}$$

Hence $\langle C_\phi f, f\rangle \leq \langle C_aC_\phi C_a^* f, f\rangle$ for all $f \in L_a^2(\mathbb{D})$ and thus

$$C_\phi \leq C_aC_\phi C_a^* \text{..............}(4).$$

In addition to $C_\phi \leq Re(C_aC_\phi)$; if $C_\phi = C_aC_\phi C_a^*$ is assumed, then (3) yields

$$\begin{aligned}\langle C_\phi f, f\rangle &= Re\langle C_aC_\phi f, f\rangle \\ &= |\langle C_aC_\phi f, f\rangle| \\ &= \langle C_aC_\phi f, f\rangle\end{aligned}$$

for all $f \in L_a^2(\mathbb{D})$, and hence $C_\phi = C_aC_\phi$. From (4), since $C_aC_\phi C_a^* - C_\phi \geq 0$, it follows that $C_a(C_aC_\phi C_a^* - C_\phi)C_a^* \geq 0$, that is $C_a^2C_\phi(C_a^*)^2 \geq C_aC_\phi C_a^*$. Repeating this process n times, we have $C_a^{n+1}C_\phi(C_a^*)^{n+1} \geq C_a^nC_\phi(C_a^*)^n$. Thus, $\{C_a^nC_\phi(C_a^*)^n : n = 1, 2......\}$ is an increasing sequence of positive operators. This sequence is bounded, since $C_a^2 = I$. Therefore, it converges to a positive operator on $L_a^2(\mathbb{D})$, say S, in the strong operator topology. Notice that

$$\begin{aligned}C_aSC_a^* &= C_a\left(\lim_{n\to\infty} C_a^nC_\phi(C_a^*)^n\right)C_a^* \\ &= \lim_{n\to\infty} C_a^{n+1}C_\phi(C_a^*)^{n+1} \\ &= S \\ &= \lim_{n\to\infty} C_a^{2n}C_\phi(C_a^*)^{2n} \\ &= C_\phi.\end{aligned}$$

Thus $C_aC_\phi C_a^* = C_\phi$ and therefore $C_a^nC_\phi(C_a^*)^n \to C_\phi$ strongly as $n \to \infty$. From $C_\phi \le \frac{(C_aC_\phi + C_\phi C_a^*)}{2}$, we have

$$C_a^nC_\phi(C_a^*)^n \le \frac{[C_a^n(C_aC_\phi + C_\phi C_a^*)(C_a^*)^n]}{2}$$
$$= \frac{[C_a(C_a^nC_\phi(C_a^*)^n) + (C_a^nC_\phi(C_a^*)^n)C_a^*]}{2}.$$

By letting n tends to ∞, we have $S \le \frac{C_aS+SC_a^*}{2} = Re(C_aS)$. Thus $S = C_aS$. That is, $C_\phi = C_aC_\phi$. Hence for all $f \in L_a^2(\mathbb{D}), C_\phi f$ is a fixed point of C_a. In particular, taking $f(z) = z$, we obtain that $C_\phi z = C_aC_\phi z$. That is $\phi = \phi o\phi_a$. From [32], it follows that $\phi = go\phi_{m_a}$ where g is an even function. $\square$

Let T be a contraction (that is, a linear operator of norm ≤ 1) on the Hilbert space $L_a^2(\mathbb{D})$. For $T \in \mathcal{L}(L_a^2(\mathbb{D}))$, by $\mathcal{M}_T$ we shall denote the uniformly closed subalgebra of $\mathcal{L}(L_a^2(\mathbb{D}))$ generated by T and I. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\sigma(T)$ denote the spectrum of T. Let $\sigma_u(T) = \sigma(T) \cap \mathbb{T}$, the unitary spectrum of T. By lemma 2.7, the operators C_a, U_a are neither strongly stable nor asymptotically regular. But the following holds:

Theorem 3.5 Let C_ϕ be a contraction on $L_a^2(\mathbb{D})$. Then if the Gelfand transform of $\Gamma \in \mathcal{M}_{C_\phi}$ vanishes on $\sigma_u(C_\phi)$, then $\|U_aC_\phi^n\Gamma C_af\| \to 0$ as $n \to \infty$ for all $a \in \mathbb{D}$ and for all $f \in L_a^2(\mathbb{D})$.

Proof: Since C_ϕ is a contraction on $L_a^2(\mathbb{D})$, hence for every $f \in L_a^2(\mathbb{D})$, the limit $\lim\limits_{n\to\infty} \|C_\phi^n f\|$ exists and is equal to $\inf\limits_{n\in\mathbb{N}} \|C_\phi^n f\|$. Define an inner product on $L_a^2(\mathbb{D})$ by the formula $[f, g] = \langle B_\phi f, g\rangle$ where $\langle B_\phi f, g\rangle = \lim\limits_{n\to\infty} \langle C_\phi^{*n} C_\phi^n f, g\rangle$. It is easy to see that the limit on the right hand side exists. This induced a semi-norm on $L_a^2(\mathbb{D})$ defined by $p(f) = \langle B_\phi f, f\rangle^{\frac{1}{2}}$. Let $E = \ker p$. It is clear that E is a closed, invariant subspace of C_ϕ. If $E = L_a^2(\mathbb{D})$, then there is nothing to prove. Hence, we may assume that $E \ne L_a^2(\mathbb{D})$. Let $J : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})/E$ be the quotient mapping. Then the semi-norm p induces a norm $\widetilde{\widetilde{p}}$ on $K_0 = L_a^2(\mathbb{D})/E$ by $\widetilde{\widetilde{p}}(Jf) = p(f)$ and we have $\|Jf\| = (\lim\limits_{n\to\infty} \|C_\phi^n f\|^2)^{\frac{1}{2}}$. Let K be the completion of K_0 with respect to the norm $\widetilde{\widetilde{p}}$. Define $V_0 : K_0 \to K_0$ by $V_0J = JC_\phi$. Since $\|V_0Jf\| \le \|C_\phi\|\|Jf\|$, V_0 extends to a bounded operator V on K. Then we have $VJ = JC_\phi$, where $J : L_a^2(\mathbb{D}) \to K$ has dense range. Also since $[Vp(f), Vp(g)]$=$[p(f), p(g)]$,$f, g \in L_a^2(\mathbb{D})$, V is an isometry on K. Let $\lambda \notin \sigma(C_\phi)$. Since $p((\lambda - C_\phi)^{-1}f) \le \|(\lambda - C_\phi)^{-1}\|p(f), f \in L_a^2(\mathbb{D})$, it follows that $\lambda \notin \sigma(V)$. Thus $\sigma(V) \subset \sigma(C_\phi)$. Now assume V is an isometry but not unitary. Then $\sigma(V) = \bar{\mathbb{D}}$ and consequently, $\sigma(C_\phi) = \bar{\mathbb{D}}$. Hence, $\sigma_u(C_\phi) = \mathbb{T}$. Let $\Gamma \in \mathcal{M}_{C_\phi}$ be such that $\hat{\Gamma}(\xi) \equiv 0$ on $\sigma_u(C_\phi)$ where $\hat{\Gamma}$ denote the Gelfand transform of Γ. Since $\Gamma \in \mathcal{M}_{C_\phi}$, there exists a sequence $\{p_n(z)\}_{n=1}^\infty$ of polynomials such that

$\|p_n(C_\phi) - \Gamma\| \to 0$ and $p_n(z) \to 0$ uniformly on $\mathbb{T}$. By the Von-Newmann inequality, $\|p_n(C_\phi)\| \leq \sup_{\xi \in \mathbb{T}} |p_n(\xi)| \to 0$, and so $\Gamma = 0$. Hence we may assume that V is an unitary operator. As above, there exists a sequence of polynomials $\{p_n(z)\}_{n=1}^{\infty}$ such that $\|p_n(C_\phi) - \Gamma\| \to 0$. It follows that $p_n(z) \to 0$ uniformly on $\sigma_u(C_\phi)$. Since $\sigma(V) \subset \sigma_u(C_\phi)$, we have $\|p_n(V)\| \to 0$. Now from the identity $p_n(V)J = Jp_n(C_\phi)$, we obtain that $J\Gamma = 0$. Hence we have that $\lim_{n\to\infty} \|C_\phi^n \Gamma f\| = 0$ for all $f \in L_a^2(\mathbb{D})$. Since $C_a^2 = I$ for all $a \in \mathbb{D}$ and U_a is an unitary operator and $U_a^2 = I$ for all $a \in \mathbb{D}$, we obtain $\lim_{n\to\infty} \|U_a C_\phi^n \Gamma C_a f\| = 0$ for all $a \in \mathbb{D}$ and for all $f \in L_a^2(\mathbb{D})$.□

In the following theorem, we prove that if there is a self-adjoint composition operator C_ϕ lying between the self-adjoint composition operator C_ψ and $U_z C_\psi U_z$ for some $z \in \mathbb{D}$ and if either $\sigma(C_\phi)$ or $\sigma(C_\psi)$ is a null set with respect to Lebesgue measure on $\mathbb{R}$ then $C_\psi = C_\phi = U_z C_\psi U_z$.

Theorem 3.6 Let C_ϕ and C_ψ be self-adjoint operators on $L_a^2(\mathbb{D})$. If either $\sigma(C_\phi)$ or $\sigma(C_\psi)$ is a null set with respect to Lebesgue measure on $\mathbb{R}$ and $C_\psi \leq C_\phi \leq U_z C_\psi U_z$ for some $z \in \mathbb{D}$ then $\mathcal{M}_z = \{(go\phi_{m_z})k_{m_z} : g \text{ even}\}$ is a reducing subspace of $U_z C_\psi U_z = C_\psi = C_\phi$ and if $C_\psi \leq C_\phi \leq U_z C_\psi U_z$ for all $z \in \mathbb{D}$ then $C_\psi = C_\phi = \alpha I$. In both the cases $U_z^n C_\psi U_z^n \to C_\psi$ strongly.

Proof : Without loss of generality, assume $\sigma(C_\psi)$ is a null set with respect to Lebesgue measure on $\mathbb{R}$. Choose $\lambda > 0$ such that $C_\psi + \lambda I$ is positive and invertible. Let $T = (C_\psi + \lambda I)^{\frac{1}{2}} U_z$. Then by our assumption, $TT^* \leq C_\phi + \lambda I \leq T^*T$ and $|T| = U_z(C_\psi + \lambda I)^{\frac{1}{2}} U_z = U_z T$. Thus T is a hyponormal operator on $L_a^2(\mathbb{D})$. Notice that $\sigma(|T|)$ is a null set with respect to Lebesgue measure on $\mathbb{R}$ because $\sigma((C_\psi + \lambda I)^{\frac{1}{2}})$ is also a null set with respect to Lebesgue measure on $\mathbb{R}$. Now $T = U_z|T|$ as $U_z^2 = I$. Since $U_z^* = U_z$; thus U_z is a self-adjoint unitary operator and $U_z \neq \pm I$. This implies that the spectrum $\sigma(U_z) = \{-1, 1\}$. By [29], T is normal. Thus $C_\psi = C_\phi = U_z C_\psi U_z$. By the spectral decomposition, there exists a unique orthogonal projection P_z on $L_a^2(\mathbb{D})$ such that $U_z = P_z - P_z^\perp$. Let M_z be the range of P_z. Thus $U_z f = f$ if and only if $P_z f = f$ for any $f \in L_a^2(\mathbb{D})$. By [32], $M_z = \{k_{m_z}(go\phi_{m_z}) : g \text{ even}\}$. Thus $C_\psi U_z = U_z C_\psi$ for some $z \in \mathbb{D}$ if and only if $C_\psi P_z = P_z C_\psi$. This is true if and only if M_z is a reducing subspace of C_ψ. If $C_\psi = C_\phi = U_z C_\psi U_z$ for all $z \in \mathbb{D}$ then $C_\psi U_z = U_z C_\psi$ for all $z \in \mathbb{D}$, and therefore $C_\psi = \alpha I$. For details see [11]. □

For $\epsilon \in \mathbb{T}$ and f a function on $\mathbb{D}$, let $(R_\epsilon f)(z) = f(\epsilon z)$. Then $R_\epsilon^{-1} = R_\epsilon^* = R_{\overline{\epsilon}}$ and R_ϵ is an unitary operator on $L_a^2(\mathbb{D})$ and $L^\infty(\mathbb{D})$. Thus the transformation $\Xi(T) = R_\epsilon^* T R_\epsilon$ is an isometry in $\mathcal{L}(L_a^2(\mathbb{D}))$ since R_ϵ is unitary. The map Ξ also maps compact operators onto the set of compact operators and it has the following interesting properties.

Theorem 3.7 (i) If $\phi(z) = \alpha z, \alpha \in \mathbb{C}$ and $\psi(z) = \beta z, \beta \in \mathbb{C}$ then the following hold: $R_\epsilon^{*n} C_\phi R_\epsilon^n = C_\phi, R_\epsilon^{*n} C_\phi^* R_\epsilon^n = C_\phi^*, R_\epsilon^{*n} C_\phi C_\psi R_\epsilon^n = C_\phi C_\psi$ for all n, $\epsilon \in \mathbb{T}$ and hence $\{R_\epsilon^{*n} C_\phi R_\epsilon^n\}_{n=0}^\infty, \{R_\epsilon^{*n} C_\phi^* R_\epsilon^n\}_{n=0}^\infty, \{R_\epsilon^{*n} C_\phi C_\psi R_\epsilon^n\}_{n=0}^\infty$ converges in norm to $C_\phi, C_\phi^*, C_\phi C_\psi$ respectively as $n \to \infty$.

(ii) If $\phi \in L^\infty(\mathbb{D})$, then $R_\epsilon^{*n} C_{R_\epsilon^n \phi} \to C_\phi$ in norm as $n \to \infty$.

(iii) If T_ϕ, H_ϕ, S_ϕ are the Toeplitz, Hankel, little Hankel operators on the Bergman space $L_a^2(\mathbb{D})$ respectively with symbol $\phi \in L^\infty(\mathbb{D})$, then the following hold:

$$R_\epsilon^{*n} T_\phi R_\epsilon^n = T_{R_\epsilon^{*n}\phi}, R_\epsilon^{*n} H_\phi R_\epsilon^n = H_{R_\epsilon^{*n}\phi}, R_\epsilon^{*n} S_\phi R_\epsilon^n = S_{R_\epsilon^{*n}\phi}$$

for all n and therefore

$$\{R_\epsilon^{*n} T_{R_\epsilon^n \phi} R_\epsilon^n\}_{n=0}^\infty, \{R_\epsilon^n H_{R_\epsilon^n \phi} R_\epsilon^n\}_{n=0}^\infty, \{R_\epsilon^{*n} S_{R_\epsilon^n \phi} R_\epsilon^n\}_{n=0}^\infty$$

converges in norm to T_ϕ, H_ϕ, S_ϕ as $n \to \infty$ respectively.

Proof:(i) Since for all $z \in \mathbb{D}, f \in L_a^2(\mathbb{D})$

$$\begin{aligned}
(R_\epsilon^{*n} C_\phi R_\epsilon^n f)(z) &= (C_\phi R_\epsilon^n f)(\bar{\epsilon}^n z) \\
&= (R_\epsilon^n f o \phi)(\bar{\epsilon}^n z) \\
&= (R_\epsilon^n f)(\phi(\bar{\epsilon}^n z)) \\
&= (R_\epsilon^n f)(\alpha \bar{\epsilon}^n z) \\
&= f(\epsilon^n \alpha \bar{\epsilon}^n z) \\
&= f(\alpha z) \\
&= (f o \phi)(z) \\
&= (C_\phi f)(z)
\end{aligned}$$

hence $R_\epsilon^{*n} C_\phi R_\epsilon^n = C_\phi$ for all $n \in \mathbb{N}$ and $R_\epsilon^{*n} C_\phi^* R_\epsilon^n = C_\phi^*$ for all $n \in \mathbb{N}$. Thus $R_\epsilon^{*n} C_\phi R_\epsilon^n \to C_\phi$ in norm as $n \to \infty$ and $R_\epsilon^{*n} C_\phi^* R_\epsilon^n \to C_\phi^*$ in norm as $n \to \infty$. Further

$$\begin{aligned}
R_\epsilon^{*n} C_\phi C_\psi R_\epsilon^n &= R_\epsilon^{*n} C_\phi R_\epsilon^n R_\epsilon^{*n} C_\psi R_\epsilon^n \\
&= C_\phi C_\psi
\end{aligned}$$

for all n. Hence $\{R_\epsilon^{*n} C_\phi C_\psi R_\epsilon^n\}$ converges in norm to $C_\phi C_\psi$ on $L_a^2(\mathbb{D})$ as $n \to \infty$.
(ii)Since

$$\begin{aligned}(R_\epsilon^{*n} C_{R_\epsilon^n \phi} f)(z) &= C_{R_\epsilon^n \phi} f(\bar{\epsilon}^n z) \\ &= (f o R_\epsilon^n \phi)(\bar{\epsilon}^n z) \\ &= f(R_\epsilon^n \phi(\bar{\epsilon}^n z)) \\ &= f(\phi(\epsilon^n \bar{\epsilon}^n z)) \\ &= f(\phi(z)) \\ &= (C_\phi f)(z)\end{aligned}$$

for all $z \in \mathbb{D}$ and $f \in L_a^2(\mathbb{D})$, hence $R_\epsilon^{*n} C_{R_\epsilon^n \phi} = C_\phi$ for all n and $R_\epsilon^{*n} C_{R_\epsilon^n \phi} \to C_\phi$ in norm as $n \to \infty$.
(iii) For $z \in \mathbb{D}, f \in L_a^2(\mathbb{D}), (R_\epsilon^{*n} T_\phi R_\epsilon^n f)(z) = (T_\phi R_\epsilon^n f)(\bar{\epsilon}^n z)$ and

$$\begin{aligned}(T_\phi R_\epsilon^n f)(\bar{\epsilon}^n z) &= P(\phi(\bar{\epsilon}^n z) f(z)) \\ &= P((R_{\bar{\epsilon}}^n \phi) f)(z) \\ &= (T_{R_{\bar{\epsilon}}^n \phi} f)(z).\end{aligned}$$

Thus $(R_\epsilon^{*n} T_\phi R_\epsilon^n f)(z) = (T_{R_{\bar{\epsilon}\phi}^n} f)(z)$ for all $z \in \mathbb{D}, f \in L_a^2(\mathbb{D})$ and therefore $R_\epsilon^{*n} T_\phi R_\epsilon^n = T_{R_{\bar{\epsilon}}^n \phi}$ on $L_a^2(\mathbb{D})$ and

$$R_\epsilon^{*n} T_{R_\epsilon^n \phi} R_\epsilon^n = T_{R_{\bar{\epsilon}}^n R_\epsilon^n \phi} = T_\phi.$$

Also

$$\begin{aligned}(R_\epsilon^{*n} H_\phi R_\epsilon^n f)(z) &= (H_\phi R_\epsilon^n f)(\bar{\epsilon}^n z) \\ &= (I - P)(\phi(\bar{\epsilon}^n z) f(z)) \\ &= (H_{R_\epsilon^{*n} \phi} f)(z)\end{aligned}$$

and the result follows. If S_ϕ is the little Hankel operator on $L_a^2(\mathbb{D})$, we proceed similarly. For $z \in \mathbb{D}, f \in L_a^2(\mathbb{D})$,

$$\begin{aligned}(R_\epsilon^{*n} S_\phi R_\epsilon^n f)(z) &= S_\phi R_\epsilon^n f(\bar{\epsilon}^n z) \\ &= PJ((\phi R_\epsilon^n f)(\bar{\epsilon}^n z)) \\ &= PJ(\phi(\bar{\epsilon}^n z) R_\epsilon^n f(\bar{\epsilon}^n z)) \\ &= PJ(\phi(\bar{\epsilon}^n z) f(z)) \\ &= (S_{R_\epsilon^{*n} \phi} f)(z).\end{aligned}$$

Hence $R_\epsilon^{*n} S_\phi R_\epsilon^n = S_{R_{\bar{\epsilon}}^n \phi}$ and therefore $R_\epsilon^{*n} S_{R_\epsilon^n \phi} R_\epsilon^n \to S_\phi$ in norm as $n \to \infty$. $\square$

If n is a nonnegative integer and $z \in \mathbb{D}$, the function $K_z^{(n)}(w) = \frac{1}{(1-\bar{z}w)^{2+n}}, w \in \mathbb{D}$ is the reproducing kernel of z in the weighted Bergman space $L_a^2(dA_n)$, where

$$dA_n(w) = (n+1)(1-|w|^2)^n dA(w).$$

The n-Berezin transform of an operator $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ is defined as
$$(B_nS)(z) = (n+1)(1-|z|^2)^{2+n} \sum_{j=0}^{n} \binom{n}{j}(-1)^j \left\langle S(w^j K_z^{(n)}), w^j K_z^{(n)} \right\rangle.$$
It is clear that $B_nS \in L^\infty(\mathbb{D})$ for every $S \in \mathcal{L}(L_a^2(\mathbb{D}))$. Using the fact that

$$\sum_{j=0}^{n} \binom{n}{j}(-1)^j |w|^{2j} = (1-|w|^2)^n,$$

we see that if $S = T_\phi$ with $\phi \in L^\infty(\mathbb{D})$, then
$$\begin{aligned}
&(B_n\phi)(z) = (B_nT_\phi)(z)\\
&= (n+1)(1-|z|^2)^{2+n} \sum_{j=0}^{n} \binom{n}{j}(-1)^j \int_{\mathbb{D}} \frac{\phi(w)|w|^{2j}}{|1-\bar{z}w|^{2(2+n)}} dA(w)\\
&= \int_{\mathbb{D}} \phi(w) \frac{(1-|z|^2)^{2+n}}{|1-\bar{z}w|^{2(2+n)}} (n+1)(1-|w|^2)^n dA(w)\\
&= \int_{\mathbb{D}} \phi(\phi_z(\rho))(n+1)(1-|\rho|^2)^n dA(\rho),
\end{aligned}$$
where the last equality comes from the change of variables $w = \phi_z(\rho)$. Notice that $\|B_n(\phi)\|_\infty \le \|\phi\|_\infty$ for all $\phi \in L^\infty(\mathbb{D})$. The 0-Berezin transform of an operator is the usual Berezin transform, which has been extensively used in this work. The n-Berezin transforms of functions (not necessarily bounded) were introduced by Berezin in [4]. It is not difficult to verify that for $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $n \ge 0$;

$$(n+2)(1-|z|^2)B_n\left(S - T_{\bar{w}}ST_w\right)(z) = (n+1)B_{n+1}\left(T_{1-\bar{w}z}ST_{1-w\bar{z}}\right)(z)$$

for every $z \in \mathbb{D}$ and $\|B_nS\|_\infty \le (n+2)2^n\|S\|$.
In the following theorem we use the concept of n-Berezin transform to describe an intertwining property of composition operators. This theorem can be compared with theorem 3.3. For $\phi \in L^\infty(D, dA)$, we define a function $\widehat{\phi}$ on $\mathbb{D}$ as follows:

$$\widehat{\phi}(z) = \int_{\mathbb{D}} \phi(\phi_w(z)) dA(w).$$

This should not be confused with the Berezin transform defined previously. As one can verify, the function $\widehat{\phi}$ differs spectacularly from $B\phi$ for harmonic functions ϕ.

Theorem 3.8 Let C_ϕ be a composition operator on $L_a^2(\mathbb{D})$ and suppose there is $p > 3$ such that

$$\sup_{z\in\mathbb{D}} \|T_{(B_nC_\phi)o\phi_z}1\|_p < C \text{ and } \sup_{z\in\mathbb{D}} \|T^*_{(B_nC_\phi)o\phi_z}1\|_p < C............(5)$$

where $C > 0$ is independent of n. Then $C_a C_\phi C_a = C_\phi$ for all $a \in \mathbb{D}$ if and only if $\widetilde{C}_\phi(w) = \widetilde{\psi}(w)$ where $\psi = \lim_{n\to\infty} \widehat{B_n C_\phi}$ for all $w \in \mathbb{D}$.

Proof : Let $C_\phi \in \mathcal{L}(L^2_a(\mathbb{D}))$ and satisfies the condition (5). It follows from [28] that, $T_{B_n C_\phi} \to C_\phi$ in $\mathcal{L}(L^2_a(\mathbb{D}))$ norm. It is shown in [32] that $C_a C_\phi C_a = C_\phi$ if and only if $\widetilde{C}_\phi(w) = \int_{\mathbb{D}} \langle U_z C_\phi U_z k_w, k_w\rangle dA(z)$. Thus

$$\begin{aligned}
\widetilde{C}_\phi(w) &= \left\langle \left(\int_{\mathbb{D}} U_z C_\phi U_z dA(z) \right) k_w, k_w \right\rangle \\
&= \left\langle \left(\int_{\mathbb{D}} U_z (\lim_{n\to\infty} T_{B_n C_\phi}) U_z dA(z) \right) k_w, k_w \right\rangle \\
&= \lim_{n\to\infty} \left\langle \left(\int_{\mathbb{D}} (U_z T_{B_n C_\phi} U_z) dA(z) \right) k_w, k_w \right\rangle \\
&= \lim_{n\to\infty} \left\langle T_{\widehat{B_n C_\phi}} k_w, k_w \right\rangle \\
&= \left\langle T_{\lim\limits_{n\to\infty} \widehat{B_n C_\phi}} k_w, k_w \right\rangle \\
&= B \left(\lim_{n\to\infty} \widehat{B_n C_\phi} \right)(w) \\
&= B(\psi)(w) = \widetilde{\psi}(w)
\end{aligned}$$

where $\psi = \lim_{n\to\infty} \widehat{B_n C_\phi}$. $\square$

Corollary 3.9 If $C_\phi \in \mathcal{L}(L^2_a(\mathbb{D}))$ and there exists a sequence $\{\psi_n\}$ such that $T_{\psi_n} \to C_\phi$ strongly then there exists a sequence $\{\phi_n\}$ such that $T_{\phi_n} \to C_\phi$ strongly and $T^*_{\phi_n} \to C^*_\phi$ strongly in $\mathcal{L}(L^2_a(\mathbb{D}))$.

Proof: It is well known [18] that if $\{A_n\}$ is a sequence of operators on a Hilbert space H and $A_n \to A$ in strong operator topology, then there exists $\{B_n\}, B_n = \sum_{k=1}^{m_n} \gamma_k A_k$ such that $\gamma_k \geq 0, \sum \gamma_k = 1, B_n \to A$ strongly and $B^*_n \to A^*_n$ strongly. The corollary is a direct consequence of this. $\square$

4. Applications

The applications of these asymptotic results and the intertwining property of composition operators in this article are manifold (see [3],[16],[27]). We shall present here only one application of these results relating to distances between the unitary operator U_a and $C_\phi, a \in \mathbb{D}$. Notice that if $\phi \in H(\mathbb{D})$ (the space of all holomorphic functions from $\mathbb{D}$ into itself) $a \in \mathbb{D}$, then $\|U_a - C_\phi\| < 1$ implies the operator C_ϕ is invertible. This is so for the following reasons : since U_a is unitary and $U_a^2 = I$ hence $\|I - U_a C_\phi\| = \|U_a^2 - U_a C_\phi\| = \|U_a - C_\phi\| < 1$. Thus $U_a C_\phi$ is invertible and therefore $C_\phi = U_a(U_a C_\phi)$ is invertible. This implies ϕ is conformal [24]. The following is also true.

Theorem 4.1 (i) Let $\phi \in H(\mathbb{D})$ be conformal and (x_n) be a sequence of real

numbers such that $x_n = o(1)$. Suppose $\|U_a f - rT_{B_nC_\phi} f\| < 1 + x_n$ for some real number $r > 1$ and for all $f \in L^2_a(\mathbb{D})$ and $n \in \mathbb{N}$. Then $\|U_a - C_\phi\| < 1$.
(ii) Let $\phi \in H(\mathbb{D})$ be a rotation, that is, $\phi(z) = \alpha z, |\alpha| = 1$. Suppose (x_n) is a sequence of real numbers such that $x_n = o(1)$. If $\|U_a f - T_{B_nC_\phi} f\| = 2 + x_n$ for all $f \in L^2_a(\mathbb{D})$ and for all $n \in \mathbb{N}$ then $U_a + C_\phi$ is not invertible.

Proof:(i) Suppose $\phi \in H(\mathbb{D})$ is conformal and $\|U_a f - rT_{B_nC_\phi} f\| = 1 + x_n$ for all $n \in \mathbb{N}$ and for all $f \in L^2_a(\mathbb{D})$. Then C_ϕ is invertible [24] and since $T_{B_nC_\phi} \to C_\phi$ in strong operator topology, we have for all $f \in L^2_a(\mathbb{D})$,

$$\begin{aligned}\|U_a f - rC_\phi f\| &= \lim_{n\to\infty} \|U_a f - rT_{B_nC_\phi} f\| \\ &= \lim_{n\to\infty} (1 + x_n) = 1.\end{aligned}$$

Thus $\|U_a - rC_\phi\| = 1$. By [23], 0 is not in the approximate point spectrum of the operator $rC_\phi = \|U_a - rC_\phi\|U_a + (rC_\phi - U_a)$ if and only if $\|(r-1)U_a - (rC_\phi - U_a)\| < (r-1) + \|rC_\phi - U_a\| = r$. Hence, $r\|U_a - C_\phi\| = \|rU_a - rC_\phi\| = \|(r-1)U_a - (rC_\phi - U_a)\| < r$ and so $\|U_a - C_\phi\| < 1$. This proves (i). For proof of (ii), let $\phi \in H(\mathbb{D})$ be a rotation. By [5], this implies C_ϕ is an isometry on $L^2_a(\mathbb{D})$ and so $\|C_\phi\| = 1$. Let $r > 1 = \|C_\phi\|$ and $W = \frac{U_a + C_\phi}{r}$. Then $C_\phi = rW - U_a$ and $\|U_a - rW\| = \|C_\phi\| = 1$. Proceeding as in (i), one can show that $W = \frac{U_a + C_\phi}{r}$ is not invertible if and only if $\|U_a - \frac{U_a + C_\phi}{r}\| \geq 1$. That is, if and only if $\|(r-1)U_a - C_\phi\| = r$. Equivalently, if and only if $\|U_a - C_\phi\| = 1 + \|C_\phi\| = 2$. Thus we have shown that $\|U_a - C_\phi\| = 2$ if and only if $U_a + C_\phi$ is not invertible. In (ii), it is given that $\|U_a f - T_{B_nC_\phi} f\| = 2 + x_n$ for all $f \in L^2_a(\mathbb{D})$ and for all $n \in \mathbb{N}$. Since $T_{B_nC_\phi} \to C_\phi$ in strong operator topology, hence for all $f \in L^2_a(\mathbb{D}), \|U_a f - C_\phi f\| = \lim_{n\to\infty} \|U_a f - T_{B_nC_\phi} f\| = \lim_{n\to\infty} (2 + x_n) = 2$. That is, $\|U_a - C_\phi\| = 2$ and hence $U_a + C_\phi$ is not invertible. $\square$

References

[1] S.W. Brown, *Contractions with spectral boundary*, Integral equations and operator theory, **11** (1988), 49-63.

[2] W. Bauer, and K. Furutani, *Hilbert Schmidt Hankel operators and Berezin iteration*, Tokyo. J. Math., **31** (2) (2008), 293-319.

[3] J. Barria and P.R. Halmos, *Asymptotic Toeplitz operators*, Trans. Amer. Math. Soc., **273** (1982), 621-630.

[4] F.A. Berezin, *Covariant and Contravariant symbols of operators*, Math. USSRIzv, **6** (1972), 1117-1151.

[5] B.J. Carswell and C. Hammond, *Composition operators with maximal norm on weighted Bergman spaces*, Proc. Amer. Math. Soc., **134** (9) (2006), 2599-2605.

[6] K. Clancey, *Seminormal Operators*, Lect. Notes, Vol. 742, Springer-Verlag, New York, 1979.

[7] J.B. Conway, *Subnormal Operators*, Pitman, Boston, 1981.
[8] C.C. Cowen, *Transferring subnormality of adjoint composition operators*, Integral equations and operator theory, **15** (1992), 167-171.
[9] C.C. Cowen and T.L. Kriete, *Subnormality and composition operators on* H^2, J. Funct. Anal., **81** (1988), 298-329.
[10] P.L. Duren, *Theory of* H^p*-spaces*, Academic press, New York, 1970.
[11] N. Das, *The Berezin transform of bounded linear operators*, The Journal of Indian Mathematical society 2010 (to appear).
[12] M. Englis, *Functions invariant under Berezin transform*, J. Funct. Anal. **121** (1994), 233-254.
[13] M. Englis, *Some density theorems for Toeplitz operators on Bergman spaces*, Czech. Math. Journal **40** (115) (1990), 419-502.
[14] T. Eisner and A. Sereny, *Category theorems for stable operators on Hilbert spaces*, arxiv : 0805.1016v1 [math.FA] 7 may 2008, 1-9.
[15] A. Feintuch, *On Hankel operators associated with a class of Non-Toeplitz operators*, J. Funct. Anal., **94** (1990), 1-13.
[16] A. Feintuch, *On asymptotic Toeplitz and Hankel operators*, Operator theory, Adv. Appl. **41** (1989), 241-254.
[17] S.R. Foguel, *A counter example to a problem of Sz-Nagy*, Proc. Amer. Math. Soc. **15** (1964), 788-790.
[18] P.G. Ghatage, *Lifting Hankel operators from the Hardy space to the Bergman space*, Rocky Mountain Journal of Mathematics **20** (2) (1990), 433-438.
[19] G.M. Goluzin, *Geometric theory of functions of complex variables*, Nauka, 1966, (Russian).
[20] E. Heinz, *On an inequality for linear operators in Hilbert space*, Report on Operator theory and group representations, Publ. No. 387, Nat. Acad. Sci-Nat. Res. Council, Washington, D.C., (1995), 27-29.
[21] S.G. Krantz, *Function Theory of Several Complex Variables*, John Wiley, New York, 1982.
[22] C.S. Kubrusly and P.C.M. Vieira, *Convergence and decomposition for tensor products of Hilbert space operators*, Operators and Matrices **2** (3) (2008), 407-416.
[23] C.S. Lin, *The unilateral shift and a norm equality for bounded linear operators*, Proc. Amer. Math. Soc. **127** (6) (1999), 1693-1696.
[24] C.K. Mohapatra and N. Das, *Composition operators on weighted Bergman spaces*, Journal of Orissa Mathematical society, **11, 12** (1991-92), 81-93.
[25] V. Muller and Y. Tomilov, *Quasisimilarity of power bounded operator and Blum-Hanson property*, J. Funct. Anal. **246** (2) (2007), 385-399.

[26] V. Matache, *S-Toeplitz composition operators*, Complex and harmonic analysis, DEStech Publ., Inc., Lancaster, PA, 2007, 189-204.

[27] F. Nazarov and J.H. Shapiro, *On the Toeplitzness of composition operators*, Complex Variables and Elliptic equations **52** (2,3) (2007), 193-210.

[28] K. Nam, D. Zheng and C. Zhong, *m-Berezin transform and compact operators*, Rev. Math. Iberoamericana **22** (3) (2006), 867-892.

[29] T. Okayasu and Y. Ueta, *A condition under which $B = A = U^*BU$ follows from $B \leq A \leq U^*BU$*, Proc. Amer. Math. Soc. **135** (5), (2007), 1399-1403.

[30] C.R. Putnam, *Hyponormal contractions and strong power convergence*, Pacific J. Math. **57** (2) (1975), 531-538.

[31] Y. Tomilov, *A resolvent approach to stability of operator semigroups*, J. Operator theory, **46** (2001), 63-98.

[32] K. Zhu, *On certain unitary operators and composition operators*, Proc. Symp. Pure. Math. **51**, 371-385, Providence, 1990.

[33] K. Zhu, *Operator Theory in Fuction Spaces*, Monographs and textbooks in pure and applied Mathematics, 139, Marcel Dekker, New York, 1990.

Namita Das
P.G.Department of Mathematics,
Utkal University, Vani
Vihar,Bhubaneswar- 751004
Orissa, India,
namitadas440@yahoo.co.in

Pabitra Kumar Jena
P.G.Department of Mathematics,
Utkal University, Vani
Vihar, Bhubaneswar-751004
Orissa, India
pabitramath@gmail.com