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EVEN-HOLE-FREE GRAPHS: A SURVEY

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The class of even-hole-free graphs is structurally quite similar to the class of perfect graphs, which was the key initial motivation for their study. The techniques developed in the study of even-hole-free graphs were generalized to other complex hereditary graph classes, and in the case of perfect graphs led to the famous resolution of the Strong Perfect Graph Conjecture and their polynomial time recognition. The class of even-hole-free graphs is also of independent interest due to its relationship to β-perfect graphs. In this survey we describe all the different structural characterizations of even-hole-free graphs, focusing on their algorithmic consequences.

1. INTRODUCTION

All graphs in this paper are finite, simple and undirected. We say that a graph G *contains* a graph F if F is isomorphic to an induced subgraph of G. A graph G is F*-free* if it does not contain F. Let $\mathcal{F}$ be a (possibly infinite) family of graphs. A graph G is $\mathcal{F}$*-free* if it is F-free for every $F \in \mathcal{F}$.

A *hole* is a chordless cycle of length at least four. A hole is *even* (resp. *odd*) if it contains an even (resp. odd) number of nodes. A hole of length n is also called an n*-hole*. In this survey we focus on the class of *even-hole-free* graphs, i.e. graphs that are $\mathcal{F}$-free where $\mathcal{F}$ denotes the family of all even holes.

Many interesting classes of graphs can be characterized as being $\mathcal{F}$-free for some family $\mathcal{F}$. The most famous such example is the class of perfect graphs. A graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$, where $\chi(H)$ denotes the *chromatic number* of H, i.e. the minimum number of colors needed to color the vertices of H so that no two adjacent vertices receive the same color,

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and $\omega(H)$ denotes the size of a largest clique, where a *clique* is a graph in which every pair of vertices are adjacent. The famous Strong Perfect Graph Theorem (conjectured by Berge [**2**], and proved by Chudnovsky, Robertson, Seymour and Thomas [**10**]) states that a graph is perfect if and only if it does not contain an odd hole nor an odd antihole (where an *antihole* is a complement of a hole). The graphs that do not contain an odd hole nor an odd antihole are known as Berge graphs.

The structure of even-hole-free graphs is in many ways quite similar to the structure of Berge graphs. Note that by excluding the 4-hole, one also excludes all antiholes of length at least 6, so the similarity between even-hole-free graphs and Berge graphs is higher than simply with the class odd-hole-free graphs.

Another motivation for the study of even-hole-free graphs is their connection to β-perfect graphs introduced by Markossian, Gasparian and Reed [**34**]. For a graph G, let $\delta(G)$ be the minimum degree of a vertex in G. Consider the following total order on $V(G)$: order the vertices by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Coloring greedily on this order gives the upper bound $\chi(G) \leq \beta(G)$, where $\beta(G) = \max\{\delta(G') + 1 : G'$ is an induced subgraph of $G\}$. A graph is *β-perfect* if for each induced subgraph H of G, $\chi(H) = \beta(H)$. It is easy to see that β-perfect graphs belong to the class of even-hole-free graphs, and that this containment is proper.

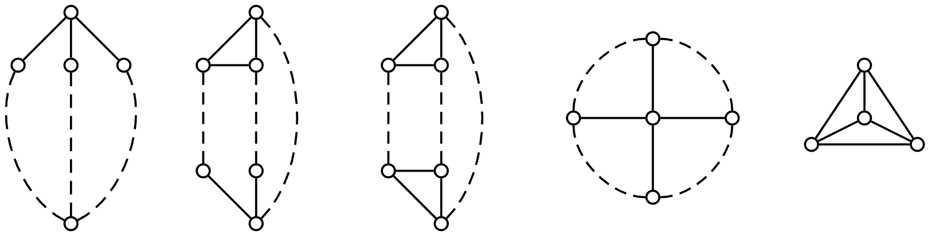
The first major structural study of even-hole-free graphs was done by Conforti, Cornuéjols, Kapoor and Vušković in [**15**] and [**16**]. They were focused on showing that even-hole-free graphs can be recognized in polynomial time (a problem that at that time was not even known to be in NP), and their primary motivation was to develop techniques which can then be used in the study of perfect graphs. In [**15**] a decomposition theorem is obtained for even-hole-free graphs, based on which the first known polynomial time recognition algorithm for even-hole-free graphs is constructed in [**16**]. This research kick-started a number of other studies of even-hole-free graphs which we survey in this paper.

The essence of even-hole-free graphs is actually captured by their generalization to signed graphs, called the odd-signable graphs, which we introduce in Section 2. Decomposition theorems for even-hole-free graphs are described in Section 3, recognition algorithms in Section 4, and combinatorial optimization algorithms in Section 5.

2. ODD-SIGNABLE GRAPHS

In 1982 Truemper proved a theorem that characterizes graphs whose edges can be labeled so that all chordless cycles have prescribed parities. Truemper's interest in this theorem at the time was to obtain a co-NP characterization of balanceable matrices, that are a generalization of regular matrices. This theorem also provided the first insight into the structure of even-hole-free and odd-hole-free graphs.

Theorem 2.1. (TRUEMPER [**47**]) *Let β be a $\{0,1\}$ vector whose entries are in one-to-one correspondence with the chordless cycles of a graph G. Then there exists a subset F of the edge set of G such that $|F \cap C| \equiv \beta_C \bmod 2$ for all chordless cycles C of G, if and only if for every induced subgraph G' of G that is a* TRUEMPER *configuration* (*Figure* 1), *there exists a subset F' of the edge set of G' such that $|F' \cap C| \equiv \beta_C \bmod 2$, for all chordless cycles C of G'.*



$3PC(\cdot,\cdot)$ $3PC(\Delta,\cdot)$ $3PC(\Delta,\Delta)$ wheel K_4

Figure 1. TRUEMPER configurations

TRUEMPER configurations are depicted in Figure 1, where a solid line denotes an edge and a dashed line denotes a chordless path containing one or more edges. We now define these configurations.

The first three configurations in Figure 1 are referred to as *3-path configurations* ($3PC$'s). They are structures induced by three paths P_1, P_2 and P_3, in such a way that the nodes of $P_i \cup P_j$, $i \neq j$, induce a hole. More specifically, a $3PC(x,y)$ is a structure induced by three paths that connect two nonadjacent nodes x and y; a $3PC(x_1x_2x_3, y)$, where $x_1x_2x_3$ is a triangle, is a structure induced by three paths having endnodes x_1, x_2 and x_3 respectively and a common endnode y; a $3PC(x_1x_2x_3, y_1y_2y_3)$, where $x_1x_2x_3$ and $y_1y_2y_3$ are two node-disjoint triangles, is a structure induced by three paths P_1, P_2 and P_3 such that, for $i = 1,2,3$, path P_i has endnodes x_i and y_i. We say that a graph G contains a $3PC(\cdot,\cdot)$ if it contains a $3PC(x,y)$ for some $x, y \in V(G)$, a $3PC(\Delta,\cdot)$ if it contains a $3PC(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$, and it contains a $3PC(\Delta,\Delta)$ if it contains a $3PC(x_1x_2x_3, y_1y_2y_3)$ for some $x_1, x_2, x_3, y_1, y_2, y_3 \in V(G)$. Note that the condition that nodes of $P_i \cup P_j$, $i \neq j$, must induce a hole, implies that all paths of a $3PC(\cdot,\cdot)$ have length greater than one, and at most one path of a $3PC(\Delta,\cdot)$ has length one. $3PC(\cdot,\cdot)$'s are also known as *thetas* (as in [**8**]), $3PC(\Delta,\Delta)$'s are also known as *prisms* (as in [**8**]), and $3PC(\Delta,\cdot)$'s are also known as *pyramids* (as in [**7**]).

A *wheel* consists of a hole and a node called the *center* that has at least three neighbors on the hole. Finally, a K_4 is a clique on four vertices. We note that in [**47**] K_4's are also referred to as wheels, but in this paper we choose to separate these two structures.

We *sign* a graph by assigning $0,1$ weights to its edges. A graph is *odd-signable* if there exists a signing that makes every triangle odd weight and every hole odd weight. This class is clearly a generalization of even-hole-free graphs. In analogous way odd-hole-free graphs can be generalized. A graph is *even-signable* if there exists a signing that makes every triangle odd weight and every hole even weight. Odd-signable and even-signable graphs were first introduced in [**14**], where the following two characterizations of these classes are obtained. These characterizations in fact follow easily from Theorem 2.1.

We say that a wheel (H, x) is *even* if x has an even number of neighbors on H, and it is *odd* if the center x and the hole H induce an odd number of triangles. Note that a wheel may be both even and odd.

Theorem 2.2. [**14**] *A graph is odd-signable if and only if it does not contain an even wheel, a* $3PC(\cdot,\cdot)$ *nor a* $3PC(\Delta,\Delta)$.

Theorem 2.3. [**14**] *A graph is even-signable if and only if it does not contain an odd wheel nor a* $3PC(\Delta,\cdot)$.

All decomposition theorems for even-hole-free graphs described in the next section are in fact proved for 4-hole-free odd-signable graphs, and the above characterization of odd-signable graphs is repeatedly used in the proofs. Similarly when one works with odd-hole-free graphs one relies on odd wheels and $3PC(\Delta,\cdot)$'s as excluded structures.

3. DECOMPOSITION THEOREMS

In a connected graph G, a subset S of nodes and edges is a *cutset* if its removal disconnects G. If S consists only of nodes then it is referred to as a *node cutset*, and if it consists only of edges then it is referred to as an *edge cutset*. A *decomposition theorem* for a class of graphs $\mathcal{C}$ states that every graph in $\mathcal{C}$ either has a particular type of a cutset or belongs to a basic (i.e. undecomposable) subclass of $\mathcal{C}$. The following cutsets are used in decomposition of even-hole-free graphs.

For $A \subseteq V(G)$, $G[A]$ denotes the subgraph of G induced by A, $N(A)$ denotes the *neighborhood* of A (i.e. the nodes of $V(G) \setminus A$ that have a neighbor in A), and $N[A] = N(A) \cup A$. A node cutset $S \subseteq V(G)$ is a *k-star cutset* of G if S is comprised of a clique C of size k and nodes with at least one neighbor in C, i.e. $C \subseteq S \subseteq N[C]$. We refer to C as the *center* of S. A 1-star is also referred to as a *star*, a 2-star as a *double star*, and 3-star as a *triple star*. If $S = N[C]$, then S is called a *full k-star*.

The following edge cutset was first introduced by Cornuéjols and Cunningham [**21**]. A graph G has a *2-join* $V_1|V_2$, with special sets (A_1, A_2, B_1, B_2), if the nodes of G can be partitioned into sets V_1 and V_2 so that the following hold.

(i) For $i = 1, 2$, $A_i \cup B_i \subseteq V_i$, and A_i and B_i are nonempty and disjoint.

(ii) Every node of A_1 is adjacent to every node of A_2, every node of B_1 is adjacent to every node of B_2, and these are the only adjacencies between V_1 and V_2.

(iii) For $i = 1, 2$, the graph induced by V_i, $G[V_i]$, contains a path with one endnode in A_i and the other in B_i. Furthermore, $G[V_i]$ is not a chordless path.

We note that slightly different definitions of 2-joins are used in different papers.

Let Σ be a $3PC(\Delta, \cdot)$. If one of the paths of Σ is of length 1, then Σ is also a wheel that is called a *bug*. If all of the paths of Σ are of length greater than 1, then Σ is called a *long* $3PC(\Delta, \cdot)$. Note that long $3PC(\Delta, \cdot)$'s are even-hole-free but have no k-star cutset nor a 2-join. So in a decomposition theorem for even-hole-free graphs that uses k-star cutsets and 2-joins, long $3PC(\Delta, \cdot)$'s form a basic class. We now introduce another basic class.

Let L be the line graph of a tree. Note that every edge of L belongs to exactly one maximal clique, and every node of L belongs to at most two maximal cliques. The nodes of L that belong to exactly one maximal clique are called *leaf nodes*. A clique of L is *big* if it is of size at least 3. In the graph obtained from L by removing all edges in big cliques, the connected components are chordless paths (possibly of length 0). Such a path P is an *internal segment* if it has its endnodes in distinct big cliques (when P is of length 0, it is called an internal segment when the node of P belongs to two big cliques). The other paths P are called *leaf segments*. Note that one of the endnodes of a leaf segment is a leaf node.

A *nontrivial basic graph* R is defined as follows: R contains two adjacent nodes x and y, called the *special* nodes. The graph L induced by $R \setminus \{x, y\}$ is the line graph of a tree and contains at least two big cliques. In R, each leaf node of L is adjacent to exactly one of the two special nodes, and no other node of L is adjacent to special nodes. The last condition for R is that no two leaf segments of L with leaf nodes adjacent to the same special node have their other endnode in the same big clique. The *internal segments* of R are the internal segments of L, and the *leaf segments* of R are the leaf segments of L together with the node in $\{x, y\}$ to which the leaf segment is adjacent to.

Let G be a graph that contains a nontrivial basic graph R with special nodes x and y. R^* is an *extended nontrivial basic graph* of G if R^* consists of R and all nodes $u \in V(G) \setminus V(R)$ such that for some big clique K of R and for some $z \in \{x, y\}$, $N(u) \cap V(R) = V(K) \cup \{z\}$. We also say that R^* is an *extension* of R. See Figure 2.

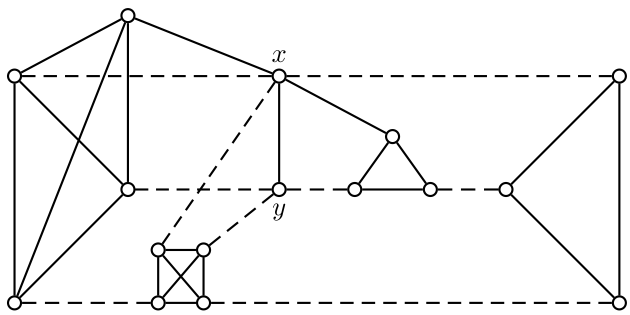


Figure 2. An extended nontrivial basic graph.

The following was the first decomposition theorem obtained for even-hole-free graphs.

Theorem 3.1. (CONFORTI, CORNUÉJOLS, KAPOOR, VUŠKOVIĆ [**15**]) *A connected 4-hole-free odd-signable graph is either a clique, a hole, a long $3PC(\Delta, \cdot)$ or a nontrivial basic graph, or it has a 2-join or k-star cutset, for $k \leq 3$.*

This theorem was strong enough to be used in construction of the first known polynomial time recognition algorithm for even-hole-free graphs [**16**], as we shall see in Section 3, but even at that time it was suspected that a stronger decomposition theorem was possible. After that work was completed the efforts were concentrated on trying to apply the same techniques to the study of perfect graphs. This approach turned out to be fruitful. CONFORTI, CORNUÉJOLS and VUŠKOVIĆ in [**17**] proved the Strong Perfect Graph Conjecture for 4-hole-free graphs, by decomposing 4-hole-free BERGE graphs using star cutsets and 2-joins into bipartite graphs and line graphs of bipartite gaphs. Finally, using the same approach, the famous Strong Perfect Graph Conjecture was proved by CHUDNOVSKY, ROBERTSON, SEYMOUR and THOMAS in [**10**], by decomposing BERGE graphs using skew cutsets, 2-joins and their complements. A node cutset S is a *skew cutset* (or *skew partition*) if there exists a partition (S_1, S_2) of S such that every node of S_1 is adjacent to every node of S_2. Star cutsets and skew cutsets were first introduced by CHVÁTAL [**12**]. Note that skew cutsets are a generalization of star cutsets, and a special case of double star cutsets.

In a graph that does not contain a 4-hole, a skew cutset reduces to a star cutset, and a 2-join in the complement implies a star cutset. The decomposition of BERGE graphs with skew cutsets, 2-joins and their complements provided a motivation to believe that it is also possible to decompose even-hole-free graphs with just the star cutsets and 2-joins. This strengthening of the decomposition theorem was obtained by DA SILVA and VUŠKOVIĆ in [**42**].

Let us say that an even-hole-free graph is *basic* if it is one of the following graphs:

- a clique,
- a hole,
- a long $3PC(\Delta, \cdot)$, or
- an extended nontrivial basic graph.

Theorem 3.2. (DA SILVA, VUŠKOVIĆ [**42**]) *A connected 4-hole-free odd-signable graph is either basic or it has a 2-join or a star cutset.*

Here is a simple restatement of Theorem 3.2, that is sufficient for algorithms described in the following sections. A graph is a *clique tree* if each of its maximal 2-connected components is a clique. A graph is an *extended clique tree* if it can be obtained from a clique tree by adding at most two vertices.

Corollary 3.3. *A connected even-hole-free graph is either an extended clique tree, or it has a* 2*-join or a star cutset.*

The key difference in the proof of Theorem 3.1 and the proof of Theorem 3.2 is that in the proof of Theorem 3.1 bugs are decomposed with double star cutsets. Since only star cutsets are used in Theorem 3.2, it is not possible to decompose all bugs in the graph, and hence the class of basic (undecomposable) graphs needed to be enlarged to include the extended nontrivial basic graphs.

The following intermediate result, that is used as one of the steps in the proof of Theorem 3.2, is of an independent interest, as we shall see in Section 5.3. It is used to prove that (diamond, even hole)-free graphs are β-perfect (where a *diamond* is the graph obtained from a clique on 4 nodes by removing an edge). A *bisimplicial cutset* is a node cutset that either induces a clique or two cliques with exactly one common node. Note that a bisimplicial cutset is a very special type of a star cutset.

Theorem 3.4. (Kloks, Müller, Vušković [**30**]) *A connected* (*diamond,* 4*-hole*)*-free odd-signable graph is either basic, or it has a bisimplicial cutset or a* 2*-join.*

4. RECOGNITION ALGORITHMS

In this section we describe polynomial-time recognition algorithms for even-hole-free graphs [**16, 9, 42**]. Perfect graphs can also be recognized in polynomial time [**7**], but it is still not known whether odd-hole-free graphs can.

A recognition algorithm for even-hole-free graphs can be used to find an even hole in a graph G, if one exists, in the following way. Let $v_1, \dots, v_n$ denote the nodes of G and let $H = G$. In Iteration i, test whether $H \setminus \{v_i\}$ contains an even hole. If the answer is yes, set $H = H \setminus \{v_i\}$ and otherwise keep H unchanged. Perform n iterations. At termination, the graph H is the desired even hole.

With two calls to the recognition algorithm we can also check whether for a given graph G and a node v of G, all the even holes of G contain v. On the other hand, it is NP-complete to decide for a given graph G and a node v of G, whether there exists an even (resp. odd) hole that contains v [**3**].

We now describe the ideas behind a decomposition based recognition algorithm. To use a decomposition theorem to recognize a class of graphs $\mathcal{C}$, basic graphs need to be simple in the sense that they can easily be recognized, and the cutsets need to have the following property. The removal of a cutset S from a graph G disconnects G into two or more connected components. From these components *blocks of decomposition* are constructed by adding some more nodes and edges. A decomposition is $\mathcal{C}$-preserving if it satisfies the following: G belongs to $\mathcal{C}$ if and only if all the blocks of decomposition belong to $\mathcal{C}$. A recognition algorithm takes a graph G as input and decomposes it using $\mathcal{C}$-preserving decompositions into a polynomial number of basic blocks, which are then checked, in polynomial time, whether they belong to $\mathcal{C}$.

This is an ideal scenario, which works for example for obtaining a recognition algorithm for regular matroids [**40**]. On the other hand, it does not work for obtaining a recognition algorithm for even-hole-free graphs. The problem is that for star cutsets one does not know how to construct the blocks of decomposition that would, at the same time, be class-preserving as well as guarantee polynomiality of the decomposition tree. For 2-joins this is possible, by constructing blocks of decomposition G_1 and G_2 of a graph G with respect to a 2-join $X_1|X_2$ with special sets (A_1, A_2, B_1, B_2) in the following way: G_1 is obtained from G by replacing X_2 by a *marker path* P_2 that is a chordless path from a vertex a_2 complete to A_1 to a vertex b_2 complete to B_1, and whose interior vertices are all of degree two in G_1. Block G_2 is obtained similarly by replacing X_1 by a marker path P_1. For $i = 1, 2$ let Q_i be any chordless path from A_i to B_i whose intermediate vertices are in $X_i \setminus (A_i \cup B_i)$, and let the length of marker path P_i be of the same parity as Q_i. It can then be shown that G is even-hole-free if and only if G_1 and G_2 are even-hole-free. Furthermore, it can be shown that a graph can be completely decomposed with 2-joins (into blocks that do not have any 2-joins) using linearly many decompositions.

Let S be a node cutset of a graph G, and let $C_1, \dots, C_k$ be the connected components of $G \setminus S$. A standard way to construct blocks of decomposition w.r.t. a node cutset would be to define blocks to be graphs $G_1, \dots, G_k$, where $G_i = G[C_i \cup S]$ for $i = 1, \dots, k$. Such definition of blocks is not preserving for the class of even-hole-free graphs. For example when the graph G on the left in Figure 3 is decomposed with star cutset $S = N[x]$, then both of the blocks of decomposition, G_1 and G_2, are like the graph on the right. Observe that G does contain an even hole, but blocks G_1 and G_2 do not.

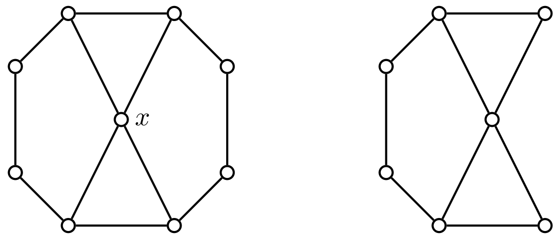


Figure 3. A graph and its block of decomposition w.r.t. star cutset $S = N[x]$.

This problem was first encountered when trying to construct a polynomial time recognition algorithm for balanced matrices (that correspond to bipartite graphs in which all holes are of length 0 mod 4). At that time a technique called *cleaning* was developed by Conforti and Rao [**19**] that enabled them to recognize, in polynomial time, linear balanced matrices. This technique was further developed and used in obtaining decomposition based polynomial time recognition algorithms for balanced matrices [**18**], balanced $0, \pm 1$ matrices [**13**], and a new level of cleaning needed to be invented for recognition of even-hole-free graphs [**16**], which was also used in the cleaning for recognition of perfect graphs [**7**].

We describe the cleaning procedure in the context of its use for recognizing even-hole-free graphs. Given an input graph G, the cleaning procedure produces, in polynomial time, a clean graph G', such that G is even-hole-free if and only if G' is even-hole-free, and if G contains an even hole then G' contains a clean even hole (i.e. an even hole for which there are no nodes outside the hole that have problematic neighbors on the hole, which can be used as clique centers of star cutsets to break the hole). Once we have a clean graph, decomposition can be applied safely, since it will now be class-preserving.

As expected, cleaning was also the key to obtaining a polynomial time recognition algorithm for Berge graphs [**7**]. What was surprising, as Chudnovsky and Seymour observed, was that once the cleaning is performed, one does not need the decomposition based recognition algorithm, one can simply look for the "bad structure" (in this case an odd hole) directly. So in [**7**] two recognition algorithms for Berge graphs are given: an $\mathcal{O}(n^9)$ Chudnovsky/Seymour style algorithm that uses the direct method, and an $\mathcal{O}(n^{18})$ decomposition based recognition algorithm. Then Zambelli [**49**] showed that by using the cleaning with the direct method, the complexity of the recognition algorithm for balanced $0, \pm 1$ matrices dramatically drops, in comparison with their original recognition [**13**] based on the decomposition method.

The original recognition algorithm for even-hole-free graphs from [**16**] uses Theorem 3.1 and is of complexity of about $\mathcal{O}(n^{40})$. In [**9**] Chudnovsky, Kawarabayashi and Seymour obtain an $\mathcal{O}(n^{31})$ recognition algorithm for even-hole-free graphs, using cleaning with the direct method. In the same paper they sketch another more complicated algorithm that, they claim, runs in time $\mathcal{O}(n^{15})$. This algorithm first needs to test for $3PC(\cdot,\cdot)$'s (thetas) and $3PC(\Delta,\Delta)$'s (prisms) in that time. It turns out that testing for thetas can be done in time $\mathcal{O}(n^{11})$ [**11**]. Detecting a prism is NP-complete in general [**31**]. In [**9**] it is claimed (without a proof) that under the assumption that the graph does not contain a theta one can use cleaning to test for prisms in time $\mathcal{O}(n^{15})$. This turns out to be false so far. Detecting a theta or a prism using the method outlined in [**9**] ends up being of complexity $\mathcal{O}(n^{35})$ [**8**].

It was then shown in [**42**] that using Theorem 3.2 one can construct a decomposition based recognition algorithms for even-hole-free graphs that runs in time $\mathcal{O}(n^{19})$. Interestingly, this is the first example in which a decomposition based method performs faster than the direct method (when both methods yield algorithms). We note that there are examples of recognition problems for which algorithms exist using only one of the methods, for example recognizing $3PC(\Delta,\cdot)$-free graphs is only known by the direct method [**7**] and recognizing graphs with no cycle with a unique chord is only known by the decomposition method [**45**].

The algorithm in [**42**] actually uses Corollary 3.3, so testing whether a basic block is even-hole-free reduces to testing whether an extended clique tree is even-hole-free, which can be done efficiently as follows. Clearly, clique trees contain no holes. Moreover, in a clique tree there is at most one induced path between any pair of vertices. So, if $G \setminus \{x\}$ is a clique tree, to determine if G is even-hole-free we need only test, for every pair y and z of neighbors of x, whether there is an induced path from y to z in $G \setminus \{x\}$ which contains no other neighbor of x and is of even length. Since there is at most one path to test, this can be done efficiently. A similar algorithm allows us to test if an extended clique tree contains an even hole.

5. COMBINATORIAL OPTIMIZATION

Finding a maximum clique, a maximum independent set and an optimal coloring are problems that are NP-complete in general, but are all known to be polynomial for perfect graphs [**27, 28**]. This result of GRÖTSCHEL, LOVÁSZ and SCHRIJVER uses the ellipsoid method and consequently is impractical. The question remains whether these optimization problems can be solved for perfect graphs by polynomial time purely combinatorial algorithms, avoiding the numerical instability of the ellipsoid method. Once again studying these problems on the class of even-hole-free graphs might shed some light on how one would go about solving them for perfect graphs.

The complexities of finding a maximum independent set and an optimal coloring are not known for even-hole-free graphs nor for the odd-hole-free graphs. Finding a maximum clique for odd-hole-free graphs is NP-complete (follows from 2-subdivision [**37**]), but can be efficiently solved for even-hole-free graphs, as we shall see in Section 5.1. In Section 5.2 we discuss the boundedness of the chromatic number for even-hole-free graphs. β-perfect graphs are a subclass of even-hole-free graphs for which there exists an efficient coloring algorithm. In Section 5.3 we show how this approach can be used to efficiently color the class of (even-hole, diamond)-free graphs. In Section 5.4 we show how one can obtain combinatorial optimization algorithms for even-hole-free graphs that do not contain star cutsets.

5.1. The clique number

One can find a maximum clique of an even-hole-free graph in polynomial time, since as observed by FARBER [**23**] 4-hole-free graphs have $\mathcal{O}(n^2)$ maximal cliques and hence one can list them all in polynomial time. For a graph G let k denote the number of maximal cliques in G, n the number of nodes in G and m the number of edges of G. In [**48**] an $\mathcal{O}(nmk)$ algorithm for generating all maximal cliques of a graph is given, and in [**5**] this complexity is improved to $\mathcal{O}(m^{1.5}k)$. The complexity is further improved for dense graphs by the $\mathcal{O}(M(n)k)$ algorithm in [**33**], where $M(n)$ denotes the time needed to multiply two $n \times n$ matrices. Matrix multiplication can be done in $\mathcal{O}(n^{2.376})$ time [**20**]. So one can generate all the maximal cliques of a 4-hole-free graph in time $\mathcal{O}(m^{1.5}n^2)$ or $\mathcal{O}(n^{4.376})$.

The following structural characterization of odd-signable graphs that do not contain a 4-hole leads to a faster algorithm for computing a maximum clique in an even-hole-free graph. For $x \in V(G)$, the graph $G[N(x)]$ is called the *neighborhood* of x. A graph is *triangulated* if it does not contain a hole.

Theorem 5.1. (DA SILVA, VUŠKOVIĆ [**41**]) *Every* 4*-hole-free odd-signable graph has a node whose neighborhood is triangulated.*

Exactly the same characterization of 4-hole-free BERGE graphs (i.e. graphs that do not contain a 4-hole nor an odd hole) is obtained by PARFENOFF, ROUSSEL and RUSU in [**36**]. Note that 4-hole-free graphs in general need not have this property, see Figure 7.

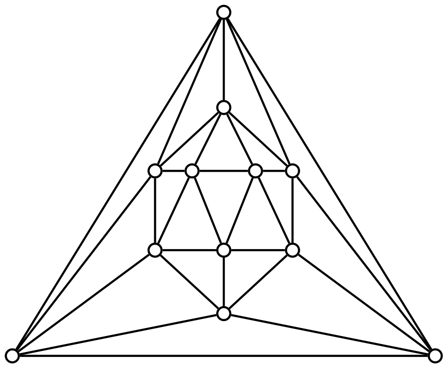


Figure 4. A 4-hole-free graph that has no vertex whose neighborhood is triangulated.

We now show that Theorem 5.1 implies that there are at most $n+2m$ maximal cliques in a 4-hole-free odd-signable graph, and it yields an algorithm that generates all the maximal cliques of a 4-hole-free odd-signable graph in time $\mathcal{O}(n^2m)$. In particular, in a weighted graph, a maximum weight clique can be found in time $\mathcal{O}(n^2m)$.

Let $\mathcal{C}$ be any class of graphs closed under taking induced subgraphs, such that for every G in $\mathcal{C}$, G has a node whose neighborhood is triangulated. Consider the following algorithm for generating all maximal cliques of graphs in $\mathcal{C}$.

Find a node x_1 of G whose neighborhood is triangulated (if no such node exists, G is not in $\mathcal{C}$, or in particular, G is not 4-hole-free odd-signable graph by Theorem 5.1). Let $G_1 = G[N[x_1]]$ and $G^1 = G \setminus \{x_1\}$. Every maximal clique of G belongs to G_1 or G^1. Recursively construct triangulated graphs $G_1, \dots, G_n$ as follows. For $i \geq 2$, find a node x_i of G^{i-1} whose neighborhood is triangulated and let $G_i = G[N_{G^{i-1}}[x_i]]$ and $G^i = G^{i-1} \setminus \{x_i\} = G \setminus \{x_1, \dots, x_i\}$.

Clearly every maximal clique of G belongs to exactly one of the graphs $G_1, \dots, G_n$. A triangulated graph on n vertices has at most n maximal cliques [**24**]. So for $i = 1, \dots, n$, graph G_i has at most $1 + d(x_i)$ maximal cliques (where $d(x)$ denotes the degree of vertex x). It follows that the number of maximal cliques of G is at most $\sum_{i=1}^{n} (1 + d(x_i)) = n + 2m$.

Checking whether a graph is triangulated can be done in time $\mathcal{O}(n + m)$ (using lexicographic breadth-first search [**39**]). So finding a vertex with triangulated neighborhood can be done in time $\mathcal{O}(\sum_{x \in V(G)} (d(x) + m)) = \mathcal{O}(nm)$. Hence constructing the graphs $G_1, \dots, G_n$ takes time $\mathcal{O}(n^2m)$.

Generating all maximal cliques in a triangulated graph can be done in time $\mathcal{O}(n + m)$ (see, for example, [**26**]). Hence the overall complexity of generating all maximal cliques in a 4-hole-free odd-signable graph is dominated by the construction of the sequence $G_1, \dots, G_n$, i.e. it is $\mathcal{O}(n^2 m)$.

Note that this algorithm is *robust* in Spinrad's sense [**43**]: given any graph G, the algorithm either verifies that G is not in $\mathcal{C}$ (or in particular that G is not a 4-hole-free odd-signable graph) or it generates all the maximal cliques of G. Note that, when G is not in $\mathcal{C}$, the algorithm might still generate all the maximal cliques of G.

The proof of Theorem 5.1 is obtained by using the following general technique developed in [**32**] (and used there to obtain a combinatorial optimization algorithm for a subclass of Berge graphs that generalizes both 4-hole-free Berge graphs and claw-free Berge graphs). A class $\mathcal{F}$ of graphs satisfies *property (*) w.r.t. a graph G* if the following holds: for every node x of G such that $G \setminus N[x] \neq \varnothing$, and for every connected component C of $G \setminus N[x]$, if $F \in \mathcal{F}$ is contained in $G[N(x)]$, then there exists a node of F that has no neighbor in C.

Theorem 5.2. (Maffray, Trotignon, Vušković [**32**]) *Let $\mathcal{F}$ be a class of graphs such that for every $F \in \mathcal{F}$, no node of F is adjacent to all the other nodes of F. If $\mathcal{F}$ satisfies property (*) w.r.t. a graph G, then G has a node whose neighborhood is $\mathcal{F}$-free.*

In [**41**] it is shown that property (*) holds for 4-hole-free odd-signable graphs when $\mathcal{F}$ is the set of all holes, and then the proof of Theorem 5.1 follows from Theorem 5.2. We observe that the fact that property (*) holds also implies the following decomposition result, which is used as one of the steps in the proof of Theorem 3.2. A wheel (H, x) is a *universal wheel* if x is adjacent to all nodes of H.

Theorem 5.3. (da Silva, Vušković [**41**]) *Let G be a 4-hole-free odd-signable graph. If G contains a universal wheel, then G has a star cutset.*

In a graph G, for any node x, let $C_1, \dots, C_k$ be the connected components of $G \setminus N[x]$, with $|C_1| \geq \dots \geq |C_k|$, and let the numerical vector $(|C_1|, \dots, |C_k|)$ be associated with x. The nodes of G can thus be ordered according to the lexicographic ordering of the numerical vectors associated with them. Say that a node x is *lex-maximal* if the associated numerical vector is lexicographically maximal over all nodes of G. Theorem 5.2 actually shows that for a lex-maximal node x, $N(x)$ is $\mathcal{F}$-free. This implies the following.

Theorem 5.4. (da Silva, Vušković [**41**]) *Let G be a 4-hole-free odd-signable graph, and let x be a lex-maximal node of G. Then the neighborhood of x is triangulated.*

Possibly a more efficient algorithm for listing all maximal cliques can be constructed by searching for a lex-maximal node.

5.2. The boundedness of the chromatic number

The clique number of a graph $\omega(G)$ is a lower bound for the chromatic number $\chi(G)$. This bound can be tight, as in the case of perfect graphs, but it can also be arbitrarily bad. GYÁRFÁS introduced a family of χ-bounded graphs [**29**] as a natural extension of the family of perfect graphs: a family of graphs $\mathcal{G}$ is *χ-bounded* with *χ-binding function* f if, for every induced subgraph G' of $G \in \mathcal{G}$, $\chi(G') \leq f(\omega(G'))$. Note that perfect graphs are a χ-bounded family of graphs with the χ-binding function $f(x) = x$.

A natural question to ask is: what choices of forbidden induced subgraphs guarantee that a family of graphs is χ-bounded? Much research has been done in this area, for a survey see [**38**]. We note that most of that research has been done on classes of graphs obtained by forbidding a finite number of graphs. Since there are graphs with arbitrarily large chromatic number and girth [**22**], in order for a family of graphs defined by forbidding a finite number of graphs (as induced subgraphs) to be χ-bounded, at least one of these forbidden graphs needs to be acyclic. In [**1**] it is shown that even-hole-free graphs (a class defined by forbidding a family of graphs none of which is acyclic) are χ-bounded. This result follows easily from the following characterization of even-hole-free graphs. A *bisimplicial vertex* is a vertex whose set of neighbors induces a graph that is a union of two cliques.

Theorem 5.5. (ADDARIO-BERRY, CHUDNOVSKY, HAVET, REED, SEYMOUR [**1**]) *Every even-hole-free graph has a bisimplicial vertex.*

Corollary 5.6. [**1**] *If G is even-hole-free then $\chi(G) \leq 2\omega(G) - 1$.*

Proof. By theorem 5.5 let v be a bisimplicial vertex of G. Inductively color $G \setminus \{v\}$ with $2\omega(G) - 1$ colors. Since v is bisimplicial, its degree is at most $2\omega(G) - 2$, and hence G can be colored with $2\omega(G) - 1$ colors. □

It is interesting to observe that Theorem 5.5 is also obtained using decomposition, although in [**1**] not all even-hole-free graphs are decomposed, but enough structures are decomposed using special double star cutsets to obtain the desired result.

The following star cutset decomposition is proved in [**1**]. A *twin wheel* is a wheel with exactly two short sectors and one long sector. A *proper wheel* is a wheel that is neither a bug nor a twin wheel.

Theorem 5.7. (ADDARIO-BERRY, CHUDNOVSKY, HAVET, REED, SEYMOUR [**1**]) *Let G be a 4-hole-free odd-signable graph. If G contains a proper wheel that is not a universal wheel, then G has a star cutset.*

We note that in [**1**] the statement of the above theorem is for even-hole-free graphs, but since in the proof of that theorem only the exclusion of 4-holes, even wheels, $3PC(\cdot,\cdot)$'s and $3PC(\Delta,\Delta)$'s is used, the above statement is actually proved.

Theorem 5.7 and Theorem 5.3 imply the following, which is used as one of the steps in the proof of Theorem 3.2.

Theorem 5.8. *Let G be a 4-hole-free odd-signable graph. If G contains a proper wheel, then G has a star cutset.*

5.3. Coloring (even-hole, diamond)-free graphs

Recall that the complexity of finding an optimal coloring of an even-hole-free graph is not known. β-perfect graphs, introduced in [**34**], are a subclass of even-hole-free graphs that can be efficiently colored, by coloring greedily on a particular easily constructable ordering of vertices, see Section 1. Unfortunately it is not known whether β-perfect graphs can be recognized in polynomial time. We now present some results on subclasses of β-perfect graphs that can be recognized in polynomial time.

Recall that a *diamond* is a cycle of length 4 that has exactly one chord. A *cap* is a cycle of length greater than four that has exactly one chord, and this chord forms a triangle with two edges of the cycle. In [**34**] Markossian, Gasparian and Reed shown that (even-hole, diamond, cap)-free graphs are β-perfect, and in [**25**] de Figueiredo and Vušković show that (even-hole, diamond, cap-on-6-vertices)-free graphs are β-perfect. These results were extended by Kloks, Müller and Vušković who show in [**30**] that (even-hole, diamond)-free graphs are β-perfect. This result follows from the following characterization of (even-hole, diamond)-free graphs. A vertex is *simplicial* if its neighborhood set induces a clique, and it is a *simplicial extreme* if it is either simplicial or of degree 2.

Theorem 5.9. (Kloks, Müller, Vušković [**30**]) *Every* (*even-hole, diamond*)*-free graph has a simplicial extreme.*

Theorem 5.9 and the following property of minimal β-imperfect graphs, imply that (even-hole, diamond)-free graphs are β-perfect.

Lemma 5.10. (Markossian, Gasparian, Reed [**34**]) *A minimal β-imperfect graph that is not an even hole, contains no simplicial extreme.*

Corollary 5.11. [**30**] *Every* (*even-hole, diamond*)*-free graph is β-perfect.*

Note that the fact that (even-hole, diamond)-free graphs have simplicial extremes implies that for such a graph G, $\chi(G) \leq \omega(G) + 1$ (observe that if v is a simplicial extreme of G, then its degree is at most $\omega(G)$, and hence G can be colored with at most $\omega(G) + 1$ colors).

The proof of Theorem 5.9 is obtained as a consequence of Theorem 3.4. Theorem 5.9 was actually conjectured to be true by de Figueiredo and Vušković [**25**]. In [**25**] they prove that every (even-hole, diamond, cap-on-6-vertices)-free graph is β-perfect by showing the following property of this class of graphs.

Theorem 5.12. (de Figueiredo, Vušković [**25**]) *If G is an (even-hole, diamond, cap-on-*6*-vertices)-free graph, then one of the following holds.*

(1) *G is triangulated.*

(2) *For every edge xy, G has a simplicial extreme in $G \setminus N[\{x, y\}]$.*

A similar property was used in [**1**] to prove that every even-hole-free graph has a bisimplicial vertex.

Theorem 5.13. (Addario-Berry, Chudnovsky, Havet, Reed, Seymour [**1**]) *If G is even-hole-free then the following hold.*

(1) *If K is a clique of G of size at most 2 such that $N[K] \neq V(G)$, then G has a bisimplicial vertex in $G \setminus N[K]$.*

(2) *If H is a hole of G such that $N[H] \neq V(G)$, then G has a bisimplicial vertex in $G \setminus N[H]$.*

Such characterizations allowed for certain types of double star cutsets to be used in the inductive proofs of Theorem 5.12 and Theorem 5.13. For assume that Theorem 5.12 (resp. Theorem 5.13) holds for all graphs with fewer vertices than G, and suppose that for an edge xy, $N[\{x, y\}]$ is a double star cutset of G. Then we can conclude that for every connected component C of $G \setminus N[\{x, y\}]$, there exists a simplicial extreme (resp. bisimplicial vertex) of G in C.

For the class of (even-hole, diamond)-free graphs it is not even the case that for every vertex there is a simplicial extreme outside the neighborhood of that vertex. The graph in Figure 5 is (even-hole, diamond)-free, and its only simplicial extremes are in the neighborhood of vertex x. Note that this graph contains a cap on 6 vertices. Also, all the vertices of this graph, except x, are bisimplicial vertices, so for any edge there is a bisimplicial vertex outside of the neighborhood of that edge.

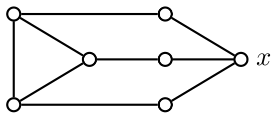


Figure 5. An (even-hole, diamond)-free graph whose only simplicial extremes are in the neighborhood of x.

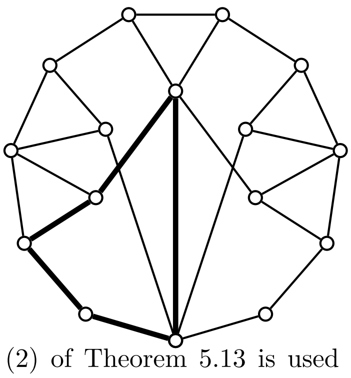


Figure 6. An (even-hole, diamond)-free graph G, bold edges denote a hole H such that no vertex of $G - N[H]$ is a simplicial extreme of G.

(2) of Theorem 5.13 is used to help prove (1). Figure 6 shows that analogous property does not hold for (even-hole, diamond)-free graphs and simplicial extremes: bold edges denote a hole H such that no vertex of $G \setminus N[H]$ is a simplicial extreme of G.

Theorem 5.9 is proved by showing the following property of (even-hole, diamond)-free graphs.

Theorem 5.14. (Kloks, Müller, Vušković [**30**]) *If G is an (even-hole, diamond)-free graph, then one of the following holds.*

(1) *G is a clique.*

(2) *G contains two nonadjacent simplicial extremes.*

This property does not allow for the use of double star cutset decompositions in the proof, not even star cutset decompositions. But bisimplicial cutsets from Theorem 3.4 do suffice.

We close this section by observing that there are (even-hole, cap)-free graphs that are not β-perfect, see Figure 7. Total characterization of β-perfect graphs remains open.

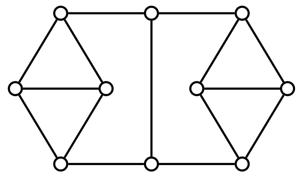


Figure 7. An (even-hole, cap)-free graph that is not β-perfect.

5.4. Combinatorial optimization with 2-joins

Decomposition can also be used to construct optimization algorithms. The general paradigm would be as follows: given a decomposition tree T for a graph $G \in \mathcal{C}$ obtained by using decompositions from some decomposition theorem for class $\mathcal{C}$, with the property that for every leaf L of T one can solve an optimization problem (such as coloring or finding the size of the largest clique or a stable set), can we construct an algorithm to solve the corresponding problem on G? This general paradigm sometimes works nicely (as in the case of triangulated graphs and clique cutsets), but most of the time it is difficult to apply to classes whose decomposition theorems require "strong cutsets", such as star cutsets.

All of the decomposition theorems mentioned in this survey use star cutsets (or their generalizations) and 2-joins. The problem with the star cutset is that it can be very big (as big as all of the vertex set except two vertices), and in the cutset itself the edges are unconstrained, so there is not much structure one can work with. 2-Joins on the other hand have quite a bit of structure within the cutset. There is an interesting relationship between star cutset decomposition and 2-join decomposition, in classes of even-hole-free and Berge graphs, that was observed when constructing decomposition based recognition algorithms in [**16**] and [**7**]. One can build a decomposition tree by first doing star cutset decompositions and then 2-join decompositions, with leaves being undecomposable blocks. Analogous separation of skew cutsets and 2-joins exists in Berge graphs [**44**].

In [**46**] Trotignon and Vušković (taking the bottom-up approach) consider the class of even-hole-free graphs that do not contain star cutsets. By Theorem 3.2, this class is decomposable into basic graphs just by 2-joins. In the same paper another class decomposable by 2-joins is considered, namely Berge graphs with no skew cutset nor homogeneous pair (which follows from [**10**], [**6**] and [**44**]). This allowed them to focus on developing techniques for combinatorial optimization with

2-joins. In [**46**] combinatorial polynomial time algorithms are given for finding the size of a largest independent set in even-hole-free graphs with no star cutset; as well as finding the size of a largest independent set, the size of a largest clique and an optimal coloring for Berge graphs with no skew cutset nor homogeneous pair. In [**4**] it is shown that the required 2-joins can be found in time $\mathcal{O}(n^2m)$ time, and hence the above mentioned coloring algorithm can be implemented to run in $\mathcal{O}(n^7)$ time and all the other ones in $\mathcal{O}(n^6)$ time. Coloring of Berge graphs actually follows from being able to compute the size of a largest independent set and largest clique ([**27, 28**]), so these two problems are the focus of the work in [**46**].

Using 2-joins in combinatorial optimization algorithms requires building blocks of decomposition and asking at least two questions for at least one block, while for recognition algorithms one question suffices. Applying this process recursively can lead to an exponential blow-up even when the decomposition tree is linear in size of the input graph. In [**46**] this problem is bypassed by using *extreme 2-joins*, i.e. 2-joins whose one block of decomposition is basic. Graphs in general do not have extreme 2-joins, this is a special property of 2-joins in graphs with no star cutset. The graph G in Figure 8 has exactly two 2-joins, one is represented with bold lines, and the other is equivalent to it. Both of the blocks of decomposition are isomorphic to graph H (where dotted lines represent paths of arbitrary length, possibly of length 0), and H has a 2-join whose edges are represented with bold lines. So G does not have an extreme 2-join.

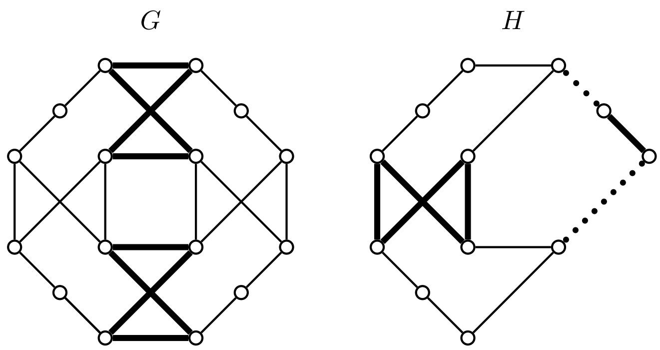


Figure 8. A graph G with no extreme 2-join

We first give a method from [**46**] that can be used to solve the maximum weighted clique problem for any class of graphs that can be decomposed with extreme 2-joins into basic graphs for which the problem can be solved efficiently. (To be able to apply arguments inductively one actually needs to switch to the weighted version of the problem). Let G be a weighted graph with a weight function $w : \mathcal{N}^+ \longrightarrow V(G)$. When H is an induced subgraph of G or a subset of $V(G)$, $w(H)$ denotes the sum of the weights of vertices in H. Here, $\omega(G)$ denotes the weight of a maximum weighted clique of G.

Let $X_1|X_2$ be a 2-join with special sets (A_1, A_2, B_1, B_2), and let G_1 and G_2

be the blocks of decomposition w.r.t. this 2-join constructed as in Section 3. Let us also assume that the lengths of marker paths are at least 3 (this is important in [**46**] because there it is not just important that the parity of holes is preserved in the blocks, but also the property of not having a star cutset). Let $P_1 = a_1, x_1, \dots, x_k, b_1$ be the marker path of G_2, where a_1 is adjacent to all of A_2 and b_1 is adjacent to all of B_2. The weights of vertices of G_2 are modified as follows:

- for every $u \in X_2$, $w_{G_2}(u) = w_G(u)$;
- $w_{G_2}(a_1) = \omega(G[A_1])$;
- $w_{G_2}(b_1) = \omega(G[B_1])$;
- $w_{G_2}(x_1) = \omega(G[X_1]) - \omega(G[A_1])$;
- $w_{G_2}(x_i) = 0$, for $i = 2, \dots, k$.

With such modification of weights it can be shown that $\omega(G) = \omega(G_2)$ [**46**]. Now if $X_1|X_2$ is an extreme 2-join, we may assume that block G_1 is undecomposable and hence basic in the sense that the maximum weighted clique problem can be solved on that block efficiently. In particular, all of the weights needed to be computed for modifying the weights of G_2 as above can be computed efficiently. We note that this method of computing a maximum stable set in the case of even-hole-free graphs (with no star cutset) is not so interesting since the algorithm described in Section 5.1 is more efficient.

Using 2-joins to compute a maximum stable set is more difficult since stable sets can completely overlap both sides of the 2-join. In [**46**] a simple class of graphs $\mathcal{C}$ decomposable along extreme 2-joins into bipartite graphs and line graphs of cycles with one chord is given for which computing a maximum stable set is NP-hard. Here is how $\mathcal{C}$ is constructed. A *gem-wheel* is a graph made of an induced cycle of length at least 5 together with a vertex adjacent to exactly four consecutive vertices of the cycle. Note that a gem-wheel is a line-graph of a cycle with one chord. A *flat path* of a graph G is a path of length at least 2, whose interior vertices all have degree 2 in G, and whose ends have no common neighbors outside the path. *Extending a flat path* $P = p_1, \dots, p_k$ of a graph means deleting the interior vertices of P and adding three vertices x, y, z and the following edges: p_1x, xy, yp_k, zp_1, zx, zy, zp_k. *Extending a graph* G means extending all paths of $\mathcal{M}$, where $\mathcal{M}$ is a set of flat paths of length at least 3 of G. Class $\mathcal{C}$ is the class of all graphs obtained by extending 2-connected bipartite graphs. From the definition, it is clear that all graphs of $\mathcal{C}$ are decomposable along extreme 2-joins. One leaf of the decomposition tree is the underlying bipartite graph, and all the others leaves are gem-wheels. The following is shown by Naves [**35**], and the proof of it can be found in [**46**].

Theorem 5.15. (Naves [**35**, **46**]) *The problem whose instance is a graph G from $\mathcal{C}$ and an integer k, and whose question is "Does G contain a stable set of size at least k" is NP-complete.*

Let $\mathcal{C}^{PARITY}$ be the class of graphs in which all holes have the same parity. In [**46**] it is shown how to use 2-joins to compute a maximum stable set in $\mathcal{C}^{PARITY}$.

Let G be a graph with a weight function w on the vertices and $X_1|X_2$ a 2-join of G with special sets (A_1, A_2, B_1, B_2). For $i = 1, 2$, $C_i = X_i \setminus (A_i \cup B_i)$. For any graph H, $\alpha(H)$ denotes the weight of a maximum weighted stable set of H. Let $a = \alpha(G[A_1 \cup C_1])$, $b = \alpha(G[B_1 \cup C_1])$, $c = \alpha(G[C_1])$ and $d = \alpha(G[X_1])$.

Blocks of decomposition w.r.t. a 2-join that would be useful for computing a largest stable set can be done as follows.

A *flat claw* of a weighted graph G is any set $\{q_1, q_2, q_3, q_4\}$ of vertices such that:

- the only edges between the q_i's are q_1q_2, q_2q_3 and q_4q_2;
- q_1 and q_3 have no common neighbor in $V(G) \setminus \{q_2\}$;
- q_4 has degree 1 in G and q_2 has degree 3 in G.

Define the *even block* G_2 with respect to a 2-join $X_1|X_2$ in the following way. Keep X_2 and replace X_1 by a flat claw on $q_1, \dots, q_4$ where q_1 is complete to A_2 and q_3 is complete to B_2. Give the following weights: $w(q_1) = d-b$, $w(q_2) = c$, $w(q_3) = d-a$, $w(q_4) = a + b - d$. It can be shown that all weights are in fact non-negative.

A *flat vault* of graph G is any set $\{r_1, r_2, r_3, r_4, r_5, r_6\}$ of vertices such that:

- the only edges between the r_i's are such that r_3, r_4, r_5, r_6, r_3 is a 4-hole;
- $N(r_1) = N(r_5) \setminus \{r_4, r_6\}$;
- $N(r_2) = N(r_6) \setminus \{r_3, r_5\}$;
- r_1 and r_2 have no common neighbors;
- r_3 and r_4 have degree 2 in G.

Define the *odd block* G_2 with respect to a 2-join $X_1|X_2$ in the following way. Replace X_1 by a flat vault on $r_1, \dots, r_6$. Moreover r_1, r_5 are complete to A_2 and r_2, r_6 are complete to B_2. Give the following weights: $w(r_1) = d - b$, $w(r_2) = d - a$, $w(r_3) = w(r_4) = c$, $w(r_5) = w(r_6) = a + b - c - d$. It can be shown that all weights are non-negative, if $c + d \leq a + b$ holds.

By adequately choosing when to use even and when odd blocks, it can be shown that for a 2-join in an even-hole-free graph G (or more generally a graph in $\mathcal{C}^{PARITY}$), $\alpha(G_2) = \alpha(G)$.

We observe that such construction of blocks is not class-preserving, so it would not allow for inductive use of the decomposition theorems. This problem is avoided in [**46**] by building the decomposition tree in two stages. First using blocks of decomposition constructed as in Section 4 (that are class-preserving). In the second stage the decomposition tree is reprocessed to replace marker paths by gadgets designed for even and odd blocks. This results in the leaves of the decomposition tree that are not basic as in the decomposition theorems used, but some extensions of these basic classes, for which it is shown that the weighted stable set problem can be computed efficiently.

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