Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **x** (xxxx), 129–142. doi:10.2298/AADM130211004B

LYAPUNOV-TYPE INEQUALITIES FOR PLANAR LINEAR DYNAMIC HAMILTONIAN SYSTEMS

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We give new Lyapunov-type inequalities for linear Hamiltonian systems on arbitrary time scales, which improve recently published results and hence all the related ones in the literature. As an application, we obtain new diconjugacy criteria for linear Hamiltonian systems.

1. INTRODUCTION

In this paper, we establish Lyapunov-type inequalities for the planar Hamiltonian system

$$x^{\Delta} = \alpha(t)x^{\sigma} + \beta(t)u, \quad u^{\Delta} = -\gamma(t)x^{\sigma} - \alpha(t)u, \tag{1.1}$$

where α, β, γ are real-valued rd-continuous functions defined on a given arbitrary time scale $\mathbb{T}$.

Lyapunov-type inequalities have proved to be very useful in studying the qualitative behavior of solutions such as oscillation, disconjugacy, and eigenvalue problems for differential and difference equations. Although Lyapunov-type inequalities are well developed for the continuous case after the appearance of Lyapunov's well-known inequality, discrete Lyapunov-type inequalities and their time scale versions are in early stages and therefore need to be improved.

Recently, He et al. [**8**] have obtained several Lyapunov-type inequalities for the Hamiltonian system (1.1), which improved the earlier results given by Jiang and Zhou [**9**], and hence the related ones in [**1**, **2**, **5**–**7**]. The following theorem seems to be the best result for (1.1) thus far.

2010 Mathematics Subject Classification. 34N05, 34A40, 34B24.
Keywords and Phrases. Hamiltonian systems, time scales, dynamic equations, time scales exponential function, Lyapunov inequality.

Theorem 1.1 (See [**8**, Theorem 3.1]). *Suppose that*

$$1 - \mu(t)\alpha(t) > 0 \quad \textit{for all} \quad t \in \mathbb{T} \tag{1.2}$$

and

$$\beta(t) \geq 0 \quad \textit{for all} \quad t \in \mathbb{T}. \tag{1.3}$$

Let $a, b \in \mathbb{T}^\kappa$ with $\sigma(a) \leq b$. Assume (1.1) *has a real solution (x, y) such that x is nontrivial and has generalized zeros at a and b, i.e., either $x(a) = 0$ or $x(a)x^\sigma(a) < 0$; either $x(b) = 0$ or $x(b)x^\sigma(b) < 0$. Then one has the inequality*

$$\int_a^b |\alpha(t)|\Delta t + \left[\int_a^{\sigma(b)} \beta(t)\Delta t \int_a^b \gamma^+(t)\Delta t\right]^{1/2} \geq 2, \tag{1.4}$$

where we put as usual $\lambda^+ = \max\{\lambda, 0\}$ for any $\lambda \in \mathbb{R}$.

In all Lyapunov-type inequalities given for (1.1) in the literature, the condition (1.2) is a must. We show in this paper that this condition can be completely dropped. To do this, we will introduce a new definition for a generalized zero, motivated by the one given in [**11**] for the discrete case.

Note that inequality (1.4) is trivial if

$$\int_a^b |\alpha(t)|\Delta t \geq 2.$$

Let

$$\int_a^b |\alpha(t)|\Delta t < 2,$$

then inequality (1.4) is equivalent to

$$\int_a^{\sigma(b)} \beta(t)\Delta t \int_a^b \gamma^+(t)\Delta t \geq \left[2 - \int_a^b |\alpha(t)|\Delta t\right]^2. \tag{1.5}$$

As an improvement as well as an alternative to inequality (1.5), we will also show that if

$$1 - \mu(t)\alpha(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}, \tag{1.6}$$

then a Lyapunov-type inequality of the form

$$\int_a^{\sigma(b)} \beta(t)\Delta t \int_a^b \gamma^+(t)\Delta t \geq 4\exp\left(-\int_a^b |\psi_{\mu(t)}(-\alpha(t))|\Delta t\right) \tag{1.7}$$

holds, where

$$\psi_h(z) = \begin{cases} \dfrac{\log|1 + hz|}{h}, & h \neq 0,\ 1 + hz \neq 0 \\ z, & h = 0. \end{cases}$$

In fact, inequality (1.5) follows from (1.7) under an additional condition implying (1.2), see Remark 3.17 below.

Definition 1.2. *A real nontrivial solution* (x, u) *of* (1.1) *is said to have a relative generalized zero* (*with respect to* x) *at* $t_0 \in \mathbb{T}$ *if either* $x(t_0) = 0$ *or* $x^*(t_0) < 0$, *where*

$$x^*(t) := [1 - \mu(t)\alpha(t)]x(t)x(\sigma(t)). \tag{1.8}$$

Definition 1.3. *The Hamiltonian system* (1.1) *is said to be relatively disconjugate* (*with respect to* x) *on* $[a, b]_{\mathbb{T}}$ *if there is no real solution* (x, u) *with* x *having more than one generalized zero in* $[a, b]_{\mathbb{T}}$.

The paper is organized as follows. In the next section, we give some properties of the time scale exponential function and introduce some estimates for a time scale exponential bound function (see Definition 2.8). Lyapunov-type inequalities will be given in Section 3. The last section is devoted to a simple application, namely new disconjugacy criteria are given for linear Hamiltonian systems.

2. TIME SCALES EXPONENTIAL FUNCTION

In this section, we let $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous and regressive, i.e.,

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T},$$

and we let $s, t, r \in \mathbb{T}$.

Definition 2.4. *The time scales exponential function is defined by*

$$e_p(t, s) := \exp\left(\int_s^t \xi_{\mu(t)}(p(\tau))\Delta t\right),$$

where

$$\xi_h(z) := \begin{cases} \dfrac{\operatorname{Log}(1 + hz)}{h}, & h \neq 0,\ 1 + hz \neq 0 \\ z, & h = 0 \end{cases}$$

is called the cylinder transformation.

Some of the properties enjoyed by the time scales exponential function are given next.

Theorem 2.5 (See [**4**, Theorem 2.36])**.** *We have*

$$e_{\ominus p}(t, s) = e_p(s, t) = \frac{1}{e_p(t, s)}, \quad \textit{where} \quad \ominus p := \frac{1}{1 + \mu p}, \tag{2.9}$$

$$e_p(t, s)e_p(s, r) = e_p(t, r), \quad e_p(t, t) = 1, \tag{2.10}$$

$$e_p^\sigma(\cdot, s) = (1 + \mu p)e_p(\cdot, s), \quad e_p^\sigma(s, \cdot) = \frac{e_p(s, \cdot)}{1 + \mu p}, \tag{2.11}$$

and

$$e_p^\Delta(\cdot, s) = pe_p(\cdot, s), \quad e_p^\Delta(s, \cdot) = -pe_p^\sigma(s, \cdot). \tag{2.12}$$

The following variation of parameter formula holds.

Theorem 2.6 (See [**4**, Theorem 2.74]). *Suppose $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Then x solves*

$$x^{\Delta} = -p(t)x^{\sigma} + f(t)$$

if and only if

$$x(t) = e_p(s,t)x(s) + \int_s^t e_p(\tau,t)f(\tau)\Delta\tau.$$

Theorem 2.7 (See [**3**, Proof of Theorem 3.4]). *We have*

$$|e_p(t,s)| = \exp\left(\int_s^t \psi_{\mu(t)}(p(\tau))\Delta t\right),$$

where

$$\psi_h(z) := \begin{cases} \dfrac{\log|1+hz|}{h}, & h \neq 0,\ 1+hz \neq 0 \\ z, & h = 0. \end{cases}$$

We now introduce a function that will serve as a bound for the absolute value of the exponential function on time scales.

Definition 2.8. *The time scales exponential bound function is defined by*

$$E_p(t,s) := \exp\left(\int_s^t \left|\psi_{\mu(t)}(p(\tau))\right| \Delta t\right).$$

For later use in this paper and also for future reference, some of the properties satisfied by the time scales exponential bound function are gathered next.

Theorem 2.9. *We have*

$$1 \leq E_p(t,s) \leq E_p(\tilde{t},\tilde{s}) \quad \text{if} \quad \tilde{s} \leq s \leq t \leq \tilde{t}, \tag{2.13}$$

$$E_p(s,t) \leq |e_p(t,s)| \leq E_p(t,s) \quad \text{for} \quad t \geq s, \tag{2.14}$$

$$E_p(t,s) \leq |e_p(t,s)| \leq E_p(s,t) \quad \text{for} \quad t \leq s, \tag{2.15}$$

$$E_p\left(\min\{s,t\}, \max\{s,t\}\right) \leq |e_p(t,s)| \leq E_p\left(\max\{s,t\}, \min\{s,t\}\right), \tag{2.16}$$

$$E_{\ominus p}(t,s) = E_p(t,s) = \frac{1}{E_p(s,t)}, \tag{2.17}$$

$$E_p(t,s)E_p(s,r) = E_p(t,r), \quad E_p(t,t) = 1, \tag{2.18}$$

and

$$E_p^{\sigma}(\cdot,s) = \max\left\{|1+\mu p|, \frac{1}{|1+\mu p|}\right\} E_p(\cdot,s). \tag{2.19}$$

Proof. Clearly, (2.13) and (2.18) follow from the definition of E. The second equality of (2.17) follows from (2.18). Now note that

$$\begin{aligned}\psi_{\mu(t)}((\ominus p)(t)) &= \begin{cases} \dfrac{\log|1+\mu(t)(\ominus p)(t)|}{\mu(t)}, & \mu(t)\neq 0\\ (\ominus p)(t), & \mu(t)=0\end{cases}\\ &= \begin{cases} \dfrac{\log\left|\frac{1}{1+\mu(t)p(t)}\right|}{\mu(t)}, & \mu(t)\neq 0\\ -p(t), & \mu(t)=0\end{cases}\\ &= -\psi_{\mu(t)}(p(t))\end{aligned}$$

implies

$$\left|\psi_{\mu(t)}((\ominus p)(t))\right| = \left|\psi_{\mu(t)}(p(t))\right|.$$

This shows the first equality of (2.17). Now let $t\geq s$. Then we have

$$|e_p(t,s)| = \exp\left(\int_s^t \psi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \leq \exp\left(\int_s^t \left|\psi_{\mu(\tau)}(p(\tau))\right|\Delta\tau\right) = E_p(t,s).$$

This shows the second inequality of (2.14). Moreover, by using (2.9), (2.17), and the second inequality of (2.14), we obtain

$$|e_p(t,s)| = \frac{1}{|e_{\ominus p}(t,s)|} \geq \frac{1}{E_{\ominus p}(t,s)} = \frac{1}{E_p(t,s)} = E_p(s,t).$$

This shows the first inequality of (2.14). Next let $t\leq s$. Then we can use (2.9), the second inequality of (2.14), and (2.17) to obtain

$$|e_p(t,s)| = \frac{1}{|e_p(s,t)|} \geq \frac{1}{E_p(s,t)} = E_p(t,s),$$

which shows the second inequality of (2.15). Moreover, by using (2.9), the second inequality of (2.15), and (2.17), we obtain

$$|e_p(t,s)| = \frac{1}{|e_{\ominus p}(t,s)|} \geq \frac{1}{E_{\ominus p}(s,t)} = \frac{1}{E_p(s,t)} = E_p(t,s).$$

This shows the first inequality of (2.15). Finally, (2.16) follows by combining (2.14) and (2.15).

3. LYAPUNOV-TYPE INEQUALITIES

Theorem 3.10. *Let $a,b\in\mathbb{T}^\kappa$ with $\sigma(a)\leq b$. Assume* (1.6) *and*

$$\beta(t)\geq 0,\quad \beta(t)\not\equiv 0,\quad t\in[a,b]_{\mathbb{T}}. \tag{3.20}$$

If (1.1) *has a real solution* (x,u) *such that* $x(a)=0$ *and* $x(b)=0$, *and if* $x(t)\neq 0$ *for all* $t\in[a,b]_{\mathbb{T}}$, *then*

$$\int_a^b \beta(t)\Delta t \int_a^b \gamma^+(t)\Delta t \geq 4\exp\left(-\int_a^b |\psi_{\mu(t)}(-\alpha(t))|\Delta t\right). \tag{3.21}$$

Proof. By the variation of parameters formula (Theorem 2.6), we write

$$x(t) = e_{-\alpha}(s,t)x(s) + \int_s^t e_{-\alpha}(\tau,t)\beta(\tau)u(\tau)\Delta\tau. \tag{3.22}$$

Put $s=a$ and use $x(a)=0$ in (3.22). Then

$$|x(t)| \leq \int_a^t |e_{-\alpha}(\tau,t)|\beta(\tau)|u(\tau)|\Delta\tau. \tag{3.23}$$

For $a\leq\tau<t\leq b$, we use (2.15) and (2.13) to obtain

$$|e_{-\alpha}(\tau,t)| \leq E_{-\alpha}(t,\tau) \leq E_{-\alpha}(t,a),$$

which together with (3.23) shows

$$|x(t)| \leq E_{-\alpha}(t,a)\int_a^t \beta(\tau)|u(\tau)|\Delta\tau. \tag{3.24}$$

Next, putting $s=b$ and using $x(b)=0$ in (3.22) leads to

$$|x(t)| \leq \int_t^b |e_{-\alpha}(\tau,t)|\beta(\tau)|u(\tau)|\Delta\tau. \tag{3.25}$$

For $a\leq t\leq\tau<b$, we use (2.14) and (2.13) to obtain

$$|e_{-\alpha}(\tau,t)| \leq E_{-\alpha}(\tau,t) \leq E_{-\alpha}(b,t),$$

which together with (3.25) shows

$$|x(t)| \leq E_{-\alpha}(b,t)\int_t^b \beta(\tau)|u(\tau)|\Delta\tau. \tag{3.26}$$

Now let

$$Q_1 = \frac{|x(t)|}{E_{-\alpha}(t,a)}, \quad Q_2 = \frac{|x(t)|}{E_{-\alpha}(b,t)}.$$

Then (2.18), the arithmetic-geometric inequality, (3.24), (3.26), and (2.13) yield

$$\begin{aligned}\frac{|x(t)|}{\sqrt{E_{-\alpha}(b,a)}} &= \frac{|x(t)|}{\sqrt{E_{-\alpha}(b,t)E_{-\alpha}(t,a)}}\\ &= \sqrt{Q_1Q_2} \le \frac{Q_1+Q_2}{2} = \frac{|x(t)|}{2E_{-\alpha}(t,a)} + \frac{|x(t)|}{2E_{-\alpha}(b,t)}\\ &\le \frac{E_{-\alpha}(t,a)\int_a^t \beta(\tau)|u(\tau)|\Delta\tau}{2E_{-\alpha}(t,a)} + \frac{E_{-\alpha}(b,t)\int_t^b \beta(\tau)|u(\tau)|\Delta\tau}{2E_{-\alpha}(b,t)}\\ &= \frac{1}{2}\int_a^b \beta(s)|u(s)|\Delta s\end{aligned}$$

and thus, by the Cauchy–Schwarz inequality (see [**4**, Theorem 6.15]),

$$\frac{4x^2(t)}{E_{-\alpha}(b,a)} \le \left[\int_a^b \beta(s)|u(s)|\Delta s\right]^2 \le \int_a^b \beta(s)\Delta s \int_a^b \beta(s)u^2(s)\Delta s. \tag{3.27}$$

Next, we use the time scales product rule (see [**4**, Theorem 1.20]) and (1.1) to calculate

$$(xu)^\Delta = x^\Delta u + x^\sigma u^\Delta = (\alpha x^\sigma + \beta u)u - (\gamma x^\sigma + \alpha u)x^\sigma = \beta u^2 - \gamma(x^\sigma)^2. \tag{3.28}$$

Hence

$$0 = \int_a^b \left\{\beta(\tau)u^2(\tau) - \gamma(\tau)(x^\sigma(\tau))^2\right\}\Delta\tau$$

and thus

$$\int_a^b \beta(\tau)u^2(\tau)\Delta\tau = \int_a^b \gamma(\tau)(x^\sigma(\tau))^2\Delta\tau \le \int_a^b \gamma^+(\tau)(x^\sigma(\tau))^2\Delta\tau. \tag{3.29}$$

Using (3.29) in (3.27), we find

$$\frac{4x^2(t)}{E_{-\alpha}(b,a)} \le \int_a^b \beta(s)\Delta s \int_a^b \gamma^+(s)x^2(\sigma(s))\Delta s. \tag{3.30}$$

Pick now $t^* \in [a,\sigma(b)]$ such that

$$|x(t_*)| = \max_{a\le t\le\sigma(b)} |x(t)| > 0.$$

As in [**8**], by treating b left-scattered and left-dense seperately, (3.30) yields

$$\frac{4x^2(t_*)}{E_{-\alpha}(b,a)} \le x^2(t_*)\int_a^b b(s)\Delta s \int_a^b \gamma^+(s)\Delta s,$$

which clearly results in (3.21).

Theorem 3.11. *Let $a, b \in \mathbb{T}^\kappa$ with $\sigma(a) \le b$. Assume* (1.6) *and* (3.20)*. If* (1.1) *has a real solution (x, u) such that $x(a) = 0$ and $x^*(b) < 0$, then*

$$\int_a^{\sigma(b)} \beta(t)\Delta t \int_a^b \gamma^+(t)\Delta t \ge 4\exp\left(-\int_a^b \left|\psi_{\mu(t)}(-\alpha(t))\right| \Delta t\right). \tag{3.31}$$

Proof. We proceed as in the proof of Theorem 3.10 and arrive at (3.24). Replacing s by b in (3.22), we obtain

$$x(t) = e_{-\alpha}(b, t)x(b) - \int_t^b e_{-\alpha}(\tau, t)\beta(\tau)u(\tau)\Delta\tau. \tag{3.32}$$

Multiply the first equation in (1.1) by $\mu(t)$ and use $x^\sigma = x + \mu x^\Delta$ (see [**4**, Theorem 1.16]) to obtain

$$(1 - \mu(t)\alpha(t))x^\sigma(t) = x(t) + \beta(t)\mu(t)u(t). \tag{3.33}$$

Let

$$k_b := -\frac{x^*(b)}{x^2(b)} > 0.$$

Then (3.33) yields

$$x(b) = -\frac{1}{k_b + 1}\beta(b)\mu(b)u(b), \tag{3.34}$$

and hence (3.32) leads to

$$x(t) = -\frac{1}{k_b + 1}\beta(b)\mu(b)u(b)e_{-\alpha}(b, t) - \int_t^b e_{-\alpha}(\tau, t)\beta(\tau)u(\tau)\Delta\tau$$

and thus, by (2.14) and (2.13),

$$\begin{aligned} |x(t)| &\le E_{-\alpha}(b, t)\left[\frac{1}{k_b + 1}\beta(b)\mu(b)|u(b)| + \int_t^b \beta(\tau)|u(\tau)|\Delta\tau\right] \\ &= E_{-\alpha}(b, t)\int_t^{\sigma(b)} \beta_b(\tau)|u(\tau)|\Delta\tau \end{aligned} \tag{3.35}$$

(use [**4**, Theorem 1.75]), where

$$\beta_b(t) = \begin{cases} \beta(t), & t \ne b \\ \dfrac{1}{k_b + 1}\beta(b), & t = b. \end{cases}$$

Note that since $1/(k_b + 1) < 1$, we have

$$\beta_b(t) \le \beta(t) \quad \text{for all} \quad t \in \mathbb{T}. \tag{3.36}$$

As in the proof of Theorem 3.10, applying the arithmetic-geometric inequality with

$$Q_1 = \frac{|x(t)|}{E_{-\alpha}(t,a)}, \quad Q_2 = \frac{|x(t)|}{E_{-\alpha}(b,t)}$$

and using (2.18), (3.24), (3.35), (2.13), and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}(3.37)\quad \frac{4x^2(t)}{E_{-\alpha}(b,a)} &\le \left[\int_a^{\sigma(b)} \beta_b(\tau)|u(\tau)|\Delta\tau\right]^2 \le \int_a^{\sigma(b)} \beta_b(\tau)\Delta\tau \int_a^{\sigma(b)} \beta_b(\tau)u^2(\tau)\Delta\tau \\ &\le \int_a^{\sigma(b)} \beta(\tau)\Delta\tau \int_a^{\sigma(b)} \beta_b(\tau)u^2(\tau)\Delta\tau,\end{aligned}$$

where we also have used (3.36). On the other hand, integrating (3.28) from a to b and using (3.34) yields

$$\int_a^b \beta(\tau)u^2(\tau)\Delta\tau + \frac{1}{k_b+1}\beta(b)\mu(b)u^2(b) = \int_a^b \gamma(\tau)(x^\sigma(\tau))^2\Delta\tau,$$

and hence

$$(3.38)\qquad \int_a^{\sigma(b)} \beta_b(\tau)u^2(\tau)\Delta\tau = \int_a^b \gamma(\tau)(x^\sigma(\tau))^2\Delta\tau \le \int_a^b \gamma^+(\tau)(x^\sigma(\tau))^2\Delta\tau.$$

Combining (3.37) and (3.38), we arrive at (3.31).

Theorem 3.12. *Let $a, b \in \mathbb{T}^\kappa$ with $\sigma(a) \le b$. Assume* (1.6) *and* (3.20)*. If* (1.1) *has a real solution (x, u) such that $x^*(a) < 0$ and $x(b) = 0$, then* (3.21) *is satisfied.*

Proof. As in the proof of Theorem 3.10, we see that (3.26) is satisfied. Replacing s by a in (3.22), we obtain

$$(3.39)\qquad x(t) = e_{-\alpha}(a,t)x(a) + \int_a^t e_{-\alpha}(\tau,t)\beta(\tau)u(\tau)\Delta\tau.$$

Let

$$k_a := -\frac{x^*(a)}{x^2(a)} > 0.$$

From (3.33), we have

$$(3.40)\qquad x(a) = -\frac{1}{k_a+1}\beta(a)\mu(a)u(a).$$

Using (3.40) in (3.39) gives

$$\begin{aligned}(3.41)\quad x(t) &= -\frac{1}{k_a+1}\beta(a)\mu(a)u(a)e_{-\alpha}(a,t) + \int_a^t e_{-\alpha}(\tau,t)\beta(\tau)u(\tau)\Delta\tau \\ &= \left(1 - \frac{1}{k_a+1}\right)\beta(a)\mu(a)u(a)e_{-\alpha}(a,t) + \int_{\sigma(a)}^t e_{-\alpha}(\tau,t)\beta(\tau)u(\tau)\Delta\tau \\ &= \int_a^t e_{-\alpha}(\tau,t)\beta_a(\tau)u(\tau)\Delta\tau,\end{aligned}$$

where

$$\beta_a(t) = \begin{cases} \beta(t), & t \neq a \\ \dfrac{k_a}{k_a+1}\beta(a), & t = a. \end{cases}$$

Note that $k_a/(k_a+1) < 1$ implies

$$\beta_a(t) \leq \beta(t) \quad \text{for all} \quad t \in \mathbb{T}. \tag{3.42}$$

From (3.41), using (2.15) and (2.13), we get

$$|x(t)| \leq E_{-\alpha}(t,a) \int_a^t \beta_a(\tau)|u(\tau)|\Delta\tau. \tag{3.43}$$

As before by employing the arithmetic-geometric inequality with

$$Q_1 = \frac{|x(t)|}{E_{-\alpha}(t,a)}, \quad Q_2 = \frac{|x(t)|}{E_{-\alpha}(b,t)}$$

and then using the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \frac{4x^2(t)}{E_{-\alpha}(b,a)} &\leq \left[\int_a^b \beta_b(\tau)|u(\tau)|\Delta\tau\right]^2 \leq \int_a^b \beta_a(\tau)\Delta\tau \int_a^b \beta_a(\tau)u^2(\tau)\Delta\tau \\ &\leq \int_a^b \beta(\tau)\Delta\tau \int_a^b \beta_a(\tau)u^2(\tau)\Delta\tau, \end{aligned} \tag{3.44}$$

where the last inequality follows from (3.42). Now from (3.28), we see that

$$\begin{aligned} &\int_a^b \gamma(\tau)(x^\sigma(\tau))^2\Delta\tau = \int_a^b \beta(\tau)u^2(\tau)\Delta\tau - \frac{1}{k_a+1}\beta(a)\mu(a)u^2(a) \\ &= \int_{\sigma(a)}^b \beta(\tau)u^2(\tau)\Delta\tau + \left(1 - \frac{1}{k_a+1}\right)\beta(a)\mu(a)u^2(a) = \int_a^b \beta_a(\tau)u^2(\tau)\Delta\tau, \end{aligned}$$

and hence

$$\int_a^b \beta_a(\tau)u^2(\tau)\Delta\tau \leq \int_a^b \gamma^+(\tau)(x^\sigma(\tau))^2\Delta\tau. \tag{3.45}$$

Combining (3.44) and (3.45), we see that (3.21) holds.

Theorem 3.13. *Let $a, b \in \mathbb{T}^\kappa$ with $\sigma(a) \leq b$. Assume* (1.6) *and* (3.20). *If* (1.1) *has a real solution (x,u) such that $x^*(a) < 0$ and $x^*(b) < 0$, then* (3.31) *is satisfied.*

Proof. The proof can be easily accomplished by combining the arguments in the last two theorems. □

From Theorems 3.10–3.13, we easily deduce the following theorem.

Theorem 3.14. *Let $a, b \in \mathbb{T}^\kappa$ with $\sigma(a) \le b$. Assume* (1.6) *and* (3.20). *If* (1.1) *has a real solution (x, u) with generalized zeros at a and b, and if $x(t) \ne 0$ for all $t \in [a, b]_\mathbb{T}$, then* (3.31) *is satisfied.*

By using similar arguments, we will next show that inequality (1.4) is valid without the condition (1.2). The result follows from the following counterpart of Theorem 3.14. Since the condition (1.6) is dropped, we deduce that (1.2) in Theorem 1.1 is superfluous. The proof is relatively less complicated because no exponential bound function is involved. The main difference is the use of

$$x(t) = x(s) + \int_s^t \alpha(\tau)x(\sigma(\tau))\Delta\tau + \int_s^t \beta(\tau)u(\tau)\Delta\tau \tag{3.46}$$

instead of the variation of parameters formula (3.22). The equality (3.46) simply follows from integrating the first equation in (1.1).

Theorem 3.15. *Let $a, b \in \mathbb{T}^\kappa$ with $\sigma(a) \le b$. Assume* (1.3). *If* (1.1) *has a real solution (x, u) with generalized zeros at a and b, and if $x(t) \ne 0$ for all $t \in [a, b]_\mathbb{T}$, then*

$$\int_a^b \alpha(t)\Delta t + \left[\int_a^{\sigma(b)} \beta(t)\Delta t\right]^{1/2} \left[\int_a^b \gamma^+(t)\Delta t\right]^{1/2} \ge 2. \tag{3.47}$$

Pproof. We will only give the proof when $x(a) = 0$ and $x^*(b) < 0$, i.e., the case contained in Theorem 3.11. From (3.46), we write that

$$x(t) = \int_a^t \alpha(\tau)x(\sigma(\tau))\Delta\tau + \int_a^t \beta(\tau)u(\tau)\Delta\tau \tag{3.48}$$

and

$$x(t) = x(b) - \int_t^b \alpha(\tau)x(\sigma(\tau))\Delta\tau - \int_t^b \beta(\tau)u(\tau)\Delta\tau. \tag{3.49}$$

From (3.48), we have

$$|x(t)| \le \int_a^t |\alpha(\tau)||x(\sigma(\tau))|\Delta\tau + \int_a^t \beta(\tau)|u(\tau)|\Delta\tau. \tag{3.50}$$

As in the proof of Theorem 3.11, with k_b defined as there, we obtain (3.34). Using (3.34) in (3.49) leads to

$$x(t) = -\frac{1}{k_b + 1}\beta(b)\mu(b)u(b) - \int_t^b \alpha(\tau)x(\sigma(\tau))\Delta\tau - \int_t^b \beta(\tau)u(\tau)\Delta\tau$$

and hence

$$\begin{aligned} |x(t)| &\le \frac{1}{k_b + 1}\beta(b)\mu(b)|u(b)| + \int_t^b |\alpha(\tau)||x(\sigma(\tau))|\Delta\tau + \int_t^b \beta(\tau)|u(\tau)|\Delta\tau \\ &\le \int_t^b |\alpha(\tau)||x(\sigma(\tau))|\Delta\tau + \int_t^{\sigma(b)} \beta_b(\tau)|u(\tau)|\Delta\tau, \end{aligned} \tag{3.51}$$

where β_b is defined as in the proof of Theorem 3.11. Note that $1/(k_b+1) < 1$ implies that (3.36) holds. By using the inequalities (3.50) and (3.51), (3.36), and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
2|x(t)| &\le \int_a^b |\alpha(\tau)||x(\sigma(\tau))|\Delta\tau + \int_a^{\sigma(b)} \beta_b(\tau)|u(\tau)|\Delta\tau \\
&\le \int_a^b |\alpha(\tau)||x(\sigma(\tau))|\Delta\tau + \left[\int_a^{\sigma(b)} \beta(\tau)\Delta\tau\right]^{\frac{1}{2}} \left[\int_a^{\sigma(b)} \beta_b(\tau)u^2(\tau)\Delta\tau\right]^{\frac{1}{2}}.
\end{aligned} \tag{3.52}
$$

On the other hand, (3.38) remains valid. In view of (3.52) and (3.38), we arrive at (3.47).

Remark 3.16. If the condition (3.20) is replaced by

$$\beta(t) \ge 0 \quad \text{for all} \quad t \in [a,b]_{\mathbb{T}}$$

with

$$\beta(t) \not\equiv 0 \quad \text{on any subinterval} \quad J \subset [a,b]_{\mathbb{T}},$$

then inequalities (3.31) and (3.47) become strict. In case $\mathbb{T} = \mathbb{R}$, we thus recover [**10**, Theorem 2.4] from Theorem 3.14 and Theorem 3.15.

Remark 3.17. Assume (1.2). If $\mu(t) = 0$, then

$$\psi_{\mu(t)}(-\alpha(t)) = -\alpha(t)$$

and if $\mu(t) > 0$, then

$$
\begin{aligned}
\psi_{\mu(t)}(-\alpha(t)) &= \frac{\log|1-\mu(t)\alpha(t)|}{\mu(t)} = \frac{\log(1-\mu(t)\alpha(t))}{\mu(t)} \\
&= -\alpha(t) + \frac{\log(1-\mu(t)\alpha(t)) + \mu(t)\alpha(t)}{\mu(t)} \le -\alpha(t)
\end{aligned}
$$

as

$$\log(1+x) \le x \quad \text{for all} \quad x \ge -1.$$

Hence we conclude

$$\psi_{\mu(t)}(-\alpha(t)) \le -\alpha(t) \quad \text{for all} \quad t \in \mathbb{T}. \tag{3.53}$$

In case of $\alpha(t) \le 0$ for all $t \in \mathbb{T}$, (1.2) is satisfied and (3.53) implies

$$\left|\psi_{\mu(t)}(-\alpha(t))\right| \le |\alpha(t)|,$$

and so (3.31) implies

$$\int_a^{\sigma(b)} \beta(t)\Delta t \int_a^b \gamma^+(t)\Delta t \ge 4\exp\left(-\int_a^b |\alpha(t)|\Delta t\right). \tag{3.54}$$

In view of $(2-\eta)^2 < 4e^{-\eta}$ for $\eta \in (0,2)$, by taking

$$\eta = \int_a^b |\alpha(t)|\Delta t,$$

we see that the Lyapunov-type inequality (3.47) follows from (3.54). So we may say in this case that the inequality (3.31) is better than (3.47). In the special case $\mathbb{T} = \mathbb{R}$, the inequality (3.31) implies (3.47) in view of $\psi_{\mu(t)}(-\alpha(t)) = -\alpha(t)$.

4. DISCONJUGACY CRITERIA

In this section, we give a simple application. Consider the Hamiltonian system (1.1) on $[a,b]_{\mathbb{T}}$.

Theorem 4.18. *Let $a,b \in \mathbb{T}^\kappa$ with $\sigma(a) \le b$. Assume* (1.6) *and* (3.20)*. If*

$$\int_a^{\sigma(b)} \beta(t)\Delta t \int_a^b \gamma^+(t)\Delta t < 4\exp\left(-\int_a^b \left|\psi_{\mu(t)}(-\alpha(t))\right| \Delta t\right), \tag{4.55}$$

then the system (1.1) *is relatively disconjugate on* $[a,b]_{\mathbb{T}}$.

Proof. Suppose that system (1.1) is not relatively disconjugate on $[a,b]_{\mathbb{T}}$. Then there exists a real solution (x,u) with x nontrivial and such that $x(a)=0$ and that x has a next generalized zero at $c \in (a,b]_{\mathbb{T}}$. We have either $x(c)=0$ or $x^*(c)<0$. Applying Theorem 3.10 and Theorem 3.11, we see that

$$\int_a^{\sigma(c)} \beta(t)\Delta t \int_a^c \gamma^+(t)\Delta t \ge 4\exp\left(-\int_a^c \left|\psi_{\mu(t)}(-\alpha(t))\right| \Delta t\right),$$

and hence

$$\int_a^{\sigma(b)} \beta(t)\Delta t \int_a^b \gamma^+(t)\Delta t \ge 4\exp\left(-\int_a^b \left|\psi_{\mu(t)}(-\alpha(t))\right| \Delta t\right). \tag{4.56}$$

The inequalities (4.55) and (4.56) contradict each other. □

In a similar manner, we can prove the following theorem.

Theorem 4.19. *Let $a,b \in \mathbb{T}^\kappa$ with $\sigma(a) \le b$. Assume* (1.6) *and* (3.20)*. If*

$$\int_a^b \alpha(t)\Delta t + \left[\int_a^{\sigma(b)} \beta(t)\Delta t\right]^{1/2} \left[\int_a^b \gamma^+(t)\Delta t\right]^{1/2} < 2, \tag{4.57}$$

then the system (1.1) *is relatively disconjugate on* $[a,b]_{\mathbb{T}}$.

Remark 4.20. Note that the second-order equation

$$(p(t)x^\Delta)^\Delta + q(t)x^\sigma = 0 \tag{4.58}$$

can be expressed as an equivalent Hamiltonian system of type (1.1) with

$$\alpha(t) \equiv 0, \quad \beta(t) = \frac{1}{p(t)}, \quad \gamma(t) = q(t).$$

Therefore, one can easily rewrite the corresponding theorems for (4.58).

Remark 4.21. In the special case $\mathbb{T} = \mathbb{Z}$, our results coincide with the corresponding ones in [**11**], where additionally the stability criteria are also given in connection with Lyapunov-type inequalities when the system is periodic. The stability problem for (1.1) on an arbitrary time scale has been studied in [**12**].

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(Received June 29, 2012)
(Revised February 11, 2013)

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