

## ON POSITIVE SOLUTIONS TO NONLOCAL FRACTIONAL AND INTEGER-ORDER DIFFERENCE EQUATIONS

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In this paper, we consider a discrete fractional boundary value problem of the form  $-\Delta^\nu y(t) = f(t + \nu - 1, y(t + \nu - 1))$ ,  $y(\nu - 2) = \psi(y)$ ,  $y(\nu + b) = \phi(y)$ , where  $t \in [0, b]_{\mathbb{N}_0}$ ,  $f : [\nu - 1, \dots, \nu + b - 1]_{\mathbb{N}_{\nu-2}} \times \mathbb{R} \rightarrow [0, +\infty)$  is continuous,  $\psi, \phi : \mathcal{C}([\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}) \rightarrow \mathbb{R}$  are given functionals, and  $1 < \nu \leq 2$ . We show that provided that both  $\psi$  and  $\phi$  are linear functionals, then under certain conditions the fractional boundary value problem will have at least one positive solution even if neither  $\psi$  nor  $\phi$  is nonnegative for all  $y \geq 0$ . This provides new results not only for the fractional boundary value problem but also in the case when  $\nu = 2$ . Our results also generalize some recent work on the conjugate fractional boundary value problem. We conclude with two examples to illustrate our results.

### 1. INTRODUCTION

In this paper we consider a discrete fractional boundary value problem (FBVP) of the form

$$(1.1) \quad -\Delta^\nu y(t) = f(t + \nu - 1, y(t + \nu - 1)),$$

$$(1.2) \quad y(\nu - 2) = \psi(y),$$

$$(1.3) \quad y(\nu + b) = \phi(y),$$

where  $t \in [0, b]_{\mathbb{N}_0}$ ,  $f : [\nu - 1, \dots, \nu + b - 1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \rightarrow [0, +\infty)$  is a continuous function,  $\psi, \phi : \mathcal{C}([\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}) \rightarrow \mathbb{R}$  are given functionals, and  $1 < \nu \leq 2$ .

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The boundary conditions (1.2)–(1.3) are generally called nonlocal conditions. The contribution of this article is to show that provided that each of  $\psi$  and  $\phi$  is a linear functional, then (1.1)–(1.3) can have at least one positive solution even if neither  $\psi$  nor  $\phi$  is nonnegative for all  $y$ . Because our results are true in case  $\nu = 2$  (i.e., the integer-order case), we believe that we provide new results even in this case. While assuming that both  $\psi$  and  $\phi$  are nonnegative for *all*  $y \geq 0$  is a typical assumption to make when analyzing the existence of positive solutions to nonlocal boundary value problems, it is useful to see in what way this condition may be weakened. In this paper, we give one possible answer to this question. We carry out this program by using a cone introduced by INFANTE and WEBB in a recent paper [20].

To place problem (1.1)–(1.3) in an appropriate context, we note first that recently there has been much progress made in developing the basic theory of fractional difference equations. In particular, ATICI and ELOE [6] analyzed a fractional conjugate BVP, whereas the present author analyzed a fractional right-focal BVP in [14]. On the other hand, a variety of other results on discrete fractional IVPs and BVPs have been provided by ATICI and ELOE [4] and the present author [13, 15, 16, 17, 18, 19]. Furthermore, a recent paper by ATICI and ŞENGÜL [7] demonstrated that fractional difference equations may provide a useful context in which to model tumor growth. There has also been some initial work completed in developing the discrete fractional calculus with nabla derivative – see [5]. In addition, BASTOS et al. [9] have addressed some initial problems in the discrete fractional calculus of variations – specifically necessary conditions for the existence of a minimizer. Finally, very recent and interesting papers by both BASTOS et al. [8, 10] and ANASTASSIOU [2, 3] have provided some initial extensions of the discrete fractional calculus to more general time scales.

Secondly, over the past several years there have appeared several works on multipoint and nonlocal discrete boundary value problems of order two. For example, CHEUNG et al. [12] considered the delta-nabla equation  $u^{\Delta\nabla}(t) + f(t, u(t)) = 0$  subject to either the boundary conditions  $u(0) - \beta\Delta u(0) = 0$ ,  $u(T+1) = \alpha u(\ell)$  or  $\Delta u(0) = 0$ ,  $u(T+1) = \alpha u(\ell)$ . KAUFMANN [21] earlier had considered a similar delta-nabla BVP subject to the multipoint boundary condition  $u(0) = 0 = \alpha u(\eta) - u(T)$ . On the other hand, a second-order delta BVP of the form  $\Delta^2 u(t-1) + a(t)f(u(t)) = 0$ ,  $u(0) = 0$ ,  $\alpha u(\eta) = u(N)$  was considered by MA and RAFFOUL [22]. Finally TIAN et al. [23] considered a related second-order delta BVP of the form  $\Delta^2 u(k-1) + f(k, u(k), \Delta u(k)) = 0$ ,  $u(0) = au(\ell_1)$ ,  $u(T+1) = bu(\ell_2)$ . We should also note in passing that there has recently appeared some works on *continuous fractional* BVPs with nonlocal conditions – see [11].

None of these works, however, considers a more general multipoint condition. Moreover, none supposes that the coefficients in the multipoint condition could be negative. The main contribution of this work is that we explore some of these extensions. It should be noted that even though we approach this problem in the more general fractional setting, *our results provide completely new results and techniques even in the integer-order setting* – i.e., when  $\nu = 2$ . In particular, we shall complete this program by adapting some recent ideas of INFANTE and WEBB

[20]. A couple of recent works by the present author, see [16, 17], have also used the ideas introduced in [20]. In [16], a first-order  $p$ -Laplacian BVP on a time scale was considered, whereas in [17] a system of discrete fractional BVPs was considered in the context of eigenvalues. However, in [17] somewhat different conditions on the nonlinearities were assumed than the conditions assumed in this paper, and as a consequence the conclusions of that work are different from the ones here. Moreover, a direct comparison of the results in [17] reveals that the results stated here are stronger due to the treatment of only one BVP versus the system of BVPs treated in [17]. In addition, our results here extend and generalize those given in [6, 12, 21, 22, 23] among others, and we shall explicitly demonstrate these improvements by way of two numerical examples in Section 4. Finally, our results also extend and complement those given recently in a paper by the author [15].

## 2. PRELIMINARIES

We first wish to collect some basic lemmas that will be important to us in the sequel. These and other related results and their proofs can be found in any of the recent papers in the literature (e.g., [6]).

**Definition 2.1.** We define  $t^\nu := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ , for any  $t$  and  $\nu$  for which the right-hand side is defined. We also appeal to the convention that if  $t+1-\nu$  is a pole of the Gamma function and  $t+1$  is not a pole, then  $t^\nu = 0$ .

**Definition 2.2.** The  $\nu$ -th fractional sum of a function  $f$ , for  $\nu > 0$ , is  $\Delta^{-\nu} f(t) = \Delta^{-\nu} f(t; a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} f(s)$ , for  $t \in \{a+\nu, a+\nu+1, \dots\} =: \mathbb{N}_{a+\nu}$ .

We also define the  $\nu$ -th fractional difference for  $\nu > 0$  by  $\Delta^\nu f(t) := \Delta^N \Delta^{\nu-N} f(t)$ , where  $t \in \mathbb{N}_{a+\nu}$  and  $N \in \mathbb{N}$  is chosen so that  $0 \leq N-1 < \nu \leq N$ .

**Lemma 2.3.** Let  $t$  and  $\nu$  be any numbers for which  $t^\nu$  and  $t^{\nu-1}$  are defined. Then  $\Delta t^\nu = \nu t^{\nu-1}$ .

**Lemma 2.4.** Let  $0 \leq N-1 < \nu \leq N$ . Then  $\Delta^{-\nu} \Delta^\nu y(t) = y(t) + C_1 t^{\nu-1} + C_2 t^{\nu-2} + \dots + C_N t^{\nu-N}$ , for some  $C_i \in \mathbb{R}$ , with  $1 \leq i \leq N$ .

We now begin proving some new results that are necessary in the sequel.

**Lemma 2.5.** Let  $1 < \nu \leq 2$  and  $h : [\nu-1, \nu+b-1]_{\mathbb{N}_{\nu-1}} \rightarrow \mathbb{R}$  be given. The unique solution of the FBVP  $-\Delta^\nu y(t) = h(t+\nu-1)$ ,  $y(\nu-2) = 0 = y(\nu+b)$  is given by  $y(t) = \sum_{s=0}^b G(t, s) h(s+\nu-1)$ , where  $G : [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0, b]_{\mathbb{N}_0} \rightarrow \mathbb{R}$  is defined by

$$(2.1) \quad G(t, s) := \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t^{\nu-1}(\nu+b-s-1)^{\nu-1}}{(\nu+b)^{\nu-1}} - (t-s-1)^{\nu-1}, & 0 \leq s < t-\nu+1 \leq b \\ \frac{t^{\nu-1}(\nu+b-s-1)^{\nu-1}}{(\nu+b)^{\nu-1}}, & 0 \leq t-\nu+1 \leq s \leq b \end{cases}.$$

**Proof.** In [6, Theorem 3.1], ATICI and ELOE proved this theorem in case the right boundary condition is  $y(\nu + b + 1) = 0$ . So, an obvious modification of their proof yields this result.  $\square$

**Lemma 2.6.** *The Green's function  $G(t, s)$  given in Lemma 2.5 satisfies:*

1.  $G(t, s) \geq 0$  for each  $(t, s) \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}} \times [0, b]_{\mathbb{N}_0}$ ;
2.  $\max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} G(t, s) = G(s + \nu - 1, s)$  for each  $s \in [0, b]_{\mathbb{N}_0}$ ; and
3. there exists a number  $\gamma \in (0, 1)$  such that

$$\min_{\frac{b+\nu}{4} \leq t \leq \frac{3(b+\nu)}{4}} G(t, s) \geq \gamma \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} G(t, s) = \gamma G(s + \nu - 1, s),$$

for  $s \in [0, b]_{\mathbb{N}_0}$ .

**Proof.** An obvious modification of the proof of [6, Theorem 3.2] yields this result. We omit the details here.  $\square$

REMARK 2.7. Observe that  $G(\nu - 2, s) = G(\nu + b, s) = 0$ , for each  $s \in [0, b]_{\mathbb{N}_0}$ .

Our next result provides for a representation of the solution to (1.1)–(1.3), provided that it exists.

**Theorem 2.8.** *Let  $f : [\nu - 1, \dots, \nu + b - 1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \rightarrow [0, +\infty)$  and  $\psi, \phi \in \mathcal{C}([\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}, \mathbb{R})$  be given. A function  $y$  is a solution of the discrete FBVP (1.1)–(1.3) if  $y(t)$ , for  $t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}$ , is a fixed point of the operator*

$$(2.2) \quad Ty(t) := \alpha(t)\psi(y) + \beta(t)\phi(y) + \sum_{s=0}^b G(t, s)f(s + \nu - 1, y(s + \nu - 1)),$$

where

$$(2.3) \quad \alpha(t) := \frac{1}{\Gamma(\nu - 1)} \left[ t^{\nu-2} - \frac{1}{b+2} t^{\nu-1} \right],$$

$$(2.4) \quad \beta(t) := \frac{t^{\nu-1}}{(\nu + b)^{\nu-1}},$$

and  $G(t, s)$  is as given in Lemma 2.5.

**Proof.** We consider first the simpler problem in which (1.1) is replaced by the equation  $-\Delta^\nu y(t) = h(t + \nu - 1)$ , for  $h : [\nu - 1, \nu + b - 1]_{\mathbb{N}_{\nu-1}} \rightarrow \mathbb{R}$  given. Then from Lemma 2.4, we find that a general solution to this problem is

$$(2.5) \quad y(t) = -\Delta^{-\nu} h(t + \nu - 1) + c_1 t^{\nu-1} + c_2 t^{\nu-2}.$$

Applying the boundary condition (1.2) to  $y(t)$  as defined in (2.5) implies at once that

$$(2.6) \quad c_2 = \frac{\psi(y)}{\Gamma(\nu-1)}.$$

On the other hand, applying the boundary condition (1.3) to  $y(t)$  implies, after some routine simplification, that

$$(2.7) \quad c_1 = \frac{\phi(y)}{(\nu+b)^{\nu-1}} - \frac{\psi(y)(\nu+b)^{\nu-2}}{\Gamma(\nu-1)(\nu+b)^{\nu-1}} \\ + \frac{1}{(\nu+b)^{\nu-1}\Gamma(\nu)} \sum_{s=0}^b (\nu+b-s-1)^{\nu-1} h(s+\nu-1).$$

Finally, putting (2.5)–(2.7) together and using both the definition of  $t^\zeta$  and the form of  $G(t, s)$  as given in Lemma 2.5, we find that

$$(2.8) \quad y(t) = \psi(y) \left[ -\frac{t^{\nu-1}}{(b+2)\Gamma(\nu-1)} + \frac{t^{\nu-2}}{\Gamma(\nu-1)} \right] \\ + \phi(y) \left[ \frac{t^{\nu-1}}{(\nu+b)^{\nu-1}} \right] + \sum_{s=0}^b G(t, s) h(s+\nu-1).$$

Consequently, by (2.3) and (2.4) we see that (2.8) implies that whenever  $y$  is a solution of (1.1)–(1.3),  $y$  is a fixed point of (2.2), as desired.  $\square$

**REMARK 2.9.** Note that Theorem 2.8 does not require that either  $\psi$  or  $\phi$  is linear. Hence, we did not make such an assumption in the statement of the theorem. However, as will be seen in Section 3, we shall assume in the sequel that each of  $\psi$  and  $\phi$  are linear and, in particular, multipoint conditions – see condition (G1) in Section 3.

**REMARK 2.10.** We note that in many works on multipoint BVPs, the multipoint functions  $\psi$  and  $\phi$  appearing in (2.2) are rewritten in terms of summations (or, in the time scales case, integrals) involving  $f(t, y)$ . We shall not do that in the sequel, however, for the representation given in (2.2) will suffice for our purposes.

Our next lemma shows that the coefficient functions  $\alpha(t)$  and  $\beta(t)$  given above in (2.3) and (2.4), respectively, satisfy certain conditions, which will be important in Section 3.

**Lemma 2.11.** *The function  $\alpha(t)$  is strictly decreasing in  $t$ , for  $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$ . In addition,  $\min_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} \alpha(t) = 0$  and  $\max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} \alpha(t) = 1$ . On the other hand, the function  $\beta(t)$  is strictly increasing in  $t$ , for  $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$ . In addition,  $\min_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} \beta(t) = 0$  and  $\max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} \beta(t) = 1$ .*

**Proof.** Note that  $\Delta_t \beta(t) = \frac{(\nu-1)t^{\nu-2}}{(\nu+b)^{\nu-1}}$ . Therefore, it is obvious that  $\Delta_t \beta(t) > 0$  for all  $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$ . So, the first claim about  $\beta(t)$  holds. On the other hand,

since  $\beta(\nu - 2) = 0$  and  $\beta(\nu + b) = 1$ , which may be easily verified, it follows that the second claim about  $\beta(t)$  holds.

It may be shown in a similar way that  $\alpha$  satisfies the properties given in the statement of this lemma. We omit the details, however.  $\square$

We conclude this section with an easy but important corollary of Lemma 2.11.

**Corollary 2.12.** *Let  $I := \left[ \frac{b + \nu}{4}, \frac{3(b + \nu)}{4} \right]$ . There are constants  $M_\alpha, M_\beta \in (0, 1)$  such that  $\min_{t \in I} \alpha(t) = M_\alpha \|\alpha\|$  and  $\min_{t \in I} \beta(t) = M_\beta \|\beta\|$ , where  $\|\cdot\|$  is the usual maximum norm.*

**Proof.** Note that since  $\alpha(\nu - 2) = 1, \alpha(\nu + b) = 0$ , and  $\alpha(t)$  is strictly decreasing in  $t$ , it follows that there exists a positive constant, say  $M_\alpha$ , such that  $\min_{t \in I} \alpha(t) = M_\alpha = M_\alpha \|\alpha\|$ . Similarly, since  $\beta(\nu - 2) = 0, \beta(\nu + b) = 1$ , and  $\beta(t)$  is strictly increasing in  $t$ , it follows that there exists a positive constant, say  $M_\beta$ , such that  $\min_{t \in I} \beta(t) = M_\beta = M_\beta \|\beta\|$ . It is obvious that  $M_\alpha, M_\beta \in (0, 1)$ , as desired.  $\square$

### 3. EXISTENCE OF A POSITIVE SOLUTION

We now show that (1.1)–(1.3) has a positive solution under certain conditions on  $\psi(y), \phi(y)$ , and  $f(t, y)$ . Of course, there are numerous reasonable and nontrivial assumptions one may make, and so our approach in the sequel is representative rather than definitive; we do not record here all of the possible extensions and alternative forms of our basic result, which is Theorem 3.3. Indeed, from the existing literature it is clear how our methods can be modified to allow for other assumptions on the nonlinearity  $f(t, y)$ , but we omit the details here.

In the sequel, we let

$\eta := \left[ \sum_{s=0}^b G(s + \nu - 1, s) \right]^{-1}, \lambda := \left[ \sum_{s=\lceil (\nu+b)/4-\nu+1 \rceil}^{\lfloor 3(\nu+b)/4-\nu+1 \rfloor} G\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) \right]^{-1}$ , and  $\tilde{\gamma} := \min\{\gamma, M_\alpha, M_\beta\}$ , where  $\gamma$  is the number given in Lemma 2.6. Observe that  $\tilde{\gamma} \in (0, 1)$ . We now present the conditions on  $\psi, \phi$ , and  $f$  that we presume in the sequel.

**F1:** There exists a number  $r > 0$  such that  $f(t, y) \leq \frac{1}{2}\eta r$  whenever  $0 \leq y \leq r$ .

**F2:** There exists a number  $r > 0$  such that  $f(t, y) \geq \lambda r$  whenever  $\tilde{\gamma} r \leq y \leq r$ .

**G1:** The functionals  $\psi$  and  $\phi$  are linear. In particular, we assume that

$$\psi(y) = \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2} y(i) \text{ and } \phi(y) = \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2} y(k),$$

for  $c_{i-\nu+2}, d_{k-\nu+2} \in \mathbb{R}$ .

**G2:** We have both that

$$\sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2}G(i, s) \geq 0 \text{ and } \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2}G(k, s) \geq 0,$$

$$\text{for each } s \in [0, b]_{\mathbb{N}_0}, \text{ and that } \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2} + \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2} \leq \frac{1}{2}.$$

**G3:** We have that each of  $\psi(\alpha)$ ,  $\phi(\alpha)$ ,  $\psi(\beta)$ , and  $\phi(\beta)$  is nonnegative.

Let  $\mathcal{B}$  represent the Banach space of maps from  $[\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}$  into  $\mathbb{R}$  equipped with the maximum norm,  $\|\cdot\|$ . For our cone, we use

$$(3.1) \quad \mathcal{K} := \left\{ y \in \mathcal{B} : y(t) \geq 0, \min_{t \in [\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}]} y(t) \geq \tilde{\gamma}\|y\|, \psi(y) \geq 0, \phi(y) \geq 0 \right\},$$

which is evidently adapted from [20]. We show now that  $T$  leaves  $\mathcal{K}$  invariant.

**Lemma 3.1.** *Assume that (G1)–(G3) hold, and let  $T$  be the operator defined in (2.2). Then  $T : \mathcal{K} \rightarrow \mathcal{K}$ .*

**Proof.** Let  $T$  be the operator given in (2.2). We show first that  $\psi(Ty) \geq 0$  whenever  $y \in \mathcal{K}$ . To see this, notice that

$$(3.2) \quad \begin{aligned} \psi(Ty) &= \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2}(Ty)(i) \\ &= \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2} \sum_{s=0}^b G(i, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &\quad + \sum_{i=\nu-2}^{\nu+b} \sum_{j=\nu-2}^{\nu+b} c_{i-\nu+2} c_{j-\nu+2} y(j) \alpha(i) + \sum_{i=\nu-2}^{\nu+b} \sum_{k=\nu-2}^{\nu+b} c_{i-\nu+2} d_{k-\nu+2} y(k) \beta(i) \\ &= \psi \left( \sum_{s=0}^b G(t, s) f(s + \nu - 1, y(s + \nu - 1)) \right) + \psi(\alpha)\psi(y) + \psi(\beta)\phi(y). \end{aligned}$$

But by assumptions (G2) and (G3) together with the nonnegativity of  $f(t, y)$  and the fact that  $y \in \mathcal{K}$ , we find from (3.2) that  $\psi(Ty) \geq 0$ . An entirely dual argument, which we omit, shows that  $\phi(Ty) \geq 0$ , too. On the other hand, it follows from both Lemma 2.6 and Corollary 2.12 that

$$(3.3) \quad \begin{aligned} \min_{t \in [\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}]} (Ty)(t) &\geq \min_{t \in [\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}]} \sum_{s=0}^b G(t, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &\quad + \min_{t \in [\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}]} \alpha(t) \psi(y) + \min_{t \in [\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}]} \beta(t) \phi(y) \end{aligned}$$

$$\begin{aligned} &\geq \tilde{\gamma} \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^b G(t, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &\quad + \tilde{\gamma} \|\alpha\| \psi(y) + \tilde{\gamma} \|\beta\| \phi(y) \geq \tilde{\gamma} \|Ty\|, \end{aligned}$$

whence  $\min_{t \in [\frac{b+\nu}{4}, \frac{3(b+\nu)}{4}]} (Ty)(t) \geq \tilde{\gamma} \|Ty\|$ . Finally, it is evident that  $(Ty)(t) \geq 0$  whenever  $y \in \mathcal{K}$ . So, we conclude that  $T : \mathcal{K} \rightarrow \mathcal{K}$ , and the proof is complete.  $\square$

We now prove our main theorem, Theorem 3.3. We first recall the Krasnosel'skiĭ fixed point theorem (see [1]), which we shall use to prove Theorem 3.3.

**Lemma 3.2.** *Let  $\mathcal{B}$  be a Banach space and let  $\mathcal{K} \subseteq \mathcal{B}$  be a cone. Assume that  $\Omega_1$  and  $\Omega_2$  are open sets contained in  $\mathcal{B}$  such that  $0 \in \Omega_1$  and  $\overline{\Omega}_1 \subseteq \Omega_2$ . Assume, further, that  $T : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow \mathcal{K}$  is a completely continuous operator. If either*

1.  $\|Ty\| \leq \|y\|$  for  $y \in \mathcal{K} \cap \partial\Omega_1$  and  $\|Ty\| \geq \|y\|$  for  $y \in \mathcal{K} \cap \partial\Omega_2$ ; or
2.  $\|Ty\| \geq \|y\|$  for  $y \in \mathcal{K} \cap \partial\Omega_1$  and  $\|Ty\| \leq \|y\|$  for  $y \in \mathcal{K} \cap \partial\Omega_2$ ;

then  $T$  has at least one fixed point in  $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ .

**Theorem 3.3.** *Assume that there exist distinct numbers  $r_1, r_2 > 0$  such that condition (F1) holds at  $r_1$  and condition (F2) holds at  $r_2$ . Moreover, assume that (G1)–(G3) hold. Then (1.1)–(1.3) has at least one positive solution.*

**Proof.** That  $T$  is completely continuous is trivially true here. Moreover, we have already shown in Lemma 3.1 that  $T : \mathcal{K} \rightarrow \mathcal{K}$ . Without loss, assume that  $0 < r_1 < r_2$ . Put  $\Omega_1 := \{y \in \mathcal{B} : \|y\| < r_1\}$  and  $\Omega_2 := \{y \in \mathcal{B} : \|y\| < r_2\}$ . Observe first that for  $y \in \partial\Omega_1 \cap \mathcal{K}$ , we find that

$$\begin{aligned} (3.4) \quad \|Ty\| &\leq \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} (\alpha(t)\psi(y)) + \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} (\beta(t)\phi(y)) \\ &\quad + \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} \sum_{s=0}^b G(t, s) f(s + \nu - 1, y(s + \nu - 1)) \\ &\leq \psi(y) + \phi(y) + \frac{1}{2} r_1 \leq r_1 \left[ \sum_{i=\nu-2}^{\nu+b} c_{i-\nu+2} + \sum_{k=\nu-2}^{\nu+b} d_{k-\nu+2} + \frac{1}{2} \right] \leq r_1, \end{aligned}$$

where we have used the fact that  $(Ty)(t) \geq 0$  whenever  $y \in \mathcal{K}$  as well as the special choice of  $\eta$  given above. Therefore, it follows that  $\|Ty\| \leq \|y\|$ .

On the other hand, for  $y \in \partial\Omega_2 \cap \mathcal{K}$ , we find that

$$\begin{aligned} (3.5) \quad (Ty) \left( \left\lfloor \frac{b+1}{2} \right\rfloor + \nu \right) &\geq \sum_{s=0}^b G \left( \left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s \right) f(s + \nu - 1, y(s + \nu - 1)) \\ &\geq \lambda r_2 \sum_{s=\lceil \frac{\nu+b}{4} - \nu + 1 \rceil}^{\lfloor \frac{3(\nu+b)}{4} - \nu + 1 \rfloor} G \left( \left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s \right) = r_2, \end{aligned}$$

so that  $\|Ty\| \geq \|y\|$ . Combining (3.4)–(3.5) and applying Lemma 3.2, we find that (1.1)–(1.3) has a positive solution, as claimed.  $\square$

REMARK 3.4. We emphasize that Theorem 3.3 does *not* presume that either of the functionals  $\psi$  and  $\phi$  is nonnegative for all  $y \geq 0$ . In particular, the result of Theorem 3.3 allows for both  $\psi(y)$  and  $\phi(y)$  to have sign-changing coefficients.

REMARK 3.5. As mentioned earlier, since Theorem 3.3 is true when  $\nu = 2$ , the results in this paper provide, so far as the author is aware, new results both for the fractional BVP and for the integer-order BVP. In particular, Theorem 3.3 extends the results in [6, 12, 21, 22, 23], among others. We shall illustrate this explicitly in the sequel.

#### 4. NUMERICAL EXAMPLES

We now present two examples illustrating the sorts of boundary conditions that can be treated by Theorem 3.3.

EXAMPLE 4.1. Put  $\nu := \frac{3}{2}$ ,  $b := 20$ ,  $\psi(y) := \frac{1}{10}y\left(\frac{7}{2}\right) - \frac{1}{15}y\left(\frac{13}{2}\right)$ , and  $\phi(y) := -\frac{1}{14}y\left(\frac{5}{2}\right) + \frac{1}{5}y\left(\frac{9}{2}\right)$ . Then it is easy to check numerically that both  $\psi$  and  $\phi$  satisfy each of the hypotheses of Theorem 3.3. Therefore, any boundary value problem of the form

$$(4.1) \quad -\Delta^{3/2}y(t) = f\left(t + \frac{1}{2}, y\left(t + \frac{1}{2}\right)\right),$$

$$(4.2) \quad y\left(-\frac{1}{2}\right) = \frac{1}{10}y\left(\frac{7}{2}\right) - \frac{1}{15}y\left(\frac{13}{2}\right),$$

$$(4.3) \quad y\left(\frac{43}{2}\right) = -\frac{1}{14}y\left(\frac{5}{2}\right) + \frac{1}{5}y\left(\frac{9}{2}\right),$$

where  $f(t, y)$  satisfies conditions (F1) and (F2), will have at least one positive solution.

REMARK 4.2. Note, importantly, that in Example 4.1 neither  $\psi$  nor  $\phi$  is nonnegative for all  $y \geq 0$ , and each has sign-changing coefficients. In particular, this example could not be treated by other results on fractional difference equations such as [6], for instance.

EXAMPLE 4.3. Let us now consider an example in which  $\nu = 2$ ; this will illustrate explicitly in what way our results extend even the results on the integer-order problem. To this end, put  $\nu := 2$ ,  $b := 20$ ,  $\psi(y) := \frac{3}{10}y(3) - \frac{1}{20}y(5)$ , and  $\phi(y) = \frac{1}{3}y(7) - \frac{1}{10}y(17)$ . Once again, routine numerical calculations reveal that conditions (G1)–(G3) are satisfied.

Indeed, we note that  $0 \leq \frac{3}{10} - \frac{1}{20} + \frac{1}{3} - \frac{1}{10} < \frac{1}{2}$ .

On the other hand, we find that  $\psi(\alpha) \approx 0.220$ ,  $\phi(\alpha) \approx 0.205$ ,  $\psi(\beta) \approx 0.030$ ,  $\phi(\beta) \approx 0.029$ , so that each of these is greater than or equal to zero. Finally, the remainder of condition (G2) can be easily checked numerically. Therefore, it follows that any boundary value problem of the form

$$(4.4) \quad -\Delta^2y(t) = f(t + 1, y(t + 1)),$$

$$(4.5) \quad y(0) = \frac{3}{10}y(3) - \frac{1}{20}y(5),$$

$$(4.6) \quad y(22) = \frac{1}{3}y(7) - \frac{1}{10}y(17),$$

where  $f(t, y)$  satisfies conditions (F1) and (F2), will have at least one positive solution.

REMARK 4.4. Observe that even though Example 4.3 is an integer-order problem, it still could not be handled by other recent works on problem (1.1) in case  $\nu = 2$ . In particular, this example illustrates that our results extend and generalize other recent results on this problem such as [12, 21, 22, 23], as previously noted.

## 5. CONCLUSIONS

In this paper we have considered a discrete FBVP with nonlocal conditions. In particular, we have shown how the recent ideas of INFANTE and WEBB [20] may be used to deduce the existence of at least one positive solution to problem (1.1)–(1.3) even when neither  $\psi$  nor  $\phi$  is nonnegative for all  $y \geq 0$ . By means of two numerical examples, we have illustrated how these results are new not only in the fractional-order case, but also in the integer-order case. This extends several recent works on this type of problem such as [6, 12, 21, 22, 23].

## REFERENCES

1. R. AGARWAL, M. MEEHAN, D. O'REGAN: *Fixed Point Theory and Applications*. Cambridge University Press, Cambridge, 2001.
2. G. A. ANASTASSIOU: *Foundations of nabla fractional calculus on time scales and inequalities*. *Comput. Math. Appl.*, **59** (2010), 3750–3762.
3. G. A. ANASTASSIOU: *Principles of delta fractional calculus on time scales and inequalities*. *Math. Comput. Modelling*, **52** (2010), 556–566.
4. F. M. ATICI, P. W. ELOE: *Initial value problems in discrete fractional calculus*. *Proc. Amer. Math. Soc.*, **137** (2009), 981–989.
5. F. M. ATICI, P. W. ELOE: *Discrete fractional calculus with the nabla operator*. *Electron. J. Qual. Theory Differ. Equ.*, (Spec. Ed. I) **3** (2009), 1–12.
6. F. M. ATICI, P. W. ELOE: *Two-point boundary value problems for finite fractional difference equations*. *J. Difference Equ. Appl.*, (2010), doi: 10.1080/10236190903029241.
7. F. M. ATICI, S. ŞENGÜL: *Modeling with fractional difference equations*. *J. Math. Anal. Appl.*, **369** (2010), 1–9.
8. N. R. O. BASTOS et al.: *Discrete-time fractional variational problems*. *Signal Process.*, **91** (2011), 513–524.
9. N. R. O. BASTOS et al.: *Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform*. *Int. J. Math. Comput.*, **11** (2011), 1–9.
10. N. R. O. BASTOS et al.: *Necessary optimality conditions for fractional difference problems of the calculus of variations*. *Discrete Cont. Dyn. Syst.*, **29** (2011), 417–437.

11. M. BENCHOHRA, S. HAMANI, S. K. NTOUYAS: *Boundary value problems for differential equations with fractional order and nonlocal conditions*. *Nonlinear Anal.*, **71** (2009), 2391–2396.
12. W. S. CHEUNG, J. REN, P. J. Y. WONG, D. ZHAO: *Multiple positive solutions for discrete nonlocal boundary value problems*. *J. Math. Anal. Appl.*, **330** (2007), 900–915.
13. C. S. GOODRICH: *Continuity of solutions to discrete fractional initial value problems*. *Comput. Math. Appl.*, **59** (2010), 3489–3499.
14. C. S. GOODRICH: *Solutions to a discrete right-focal fractional boundary value problem*. *Int. J. Difference Equ.*, **5** (2010), 195–216.
15. C. S. GOODRICH: *Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions*. *Comput. Math. Appl.*, **61** (2011), 191–202.
16. C. S. GOODRICH: *Existence of a positive solution to a first-order  $p$ -Laplacian BVP on a time scale*. *Nonlinear Anal.*, **74** (2011), 1926–1936.
17. C. S. GOODRICH: *Existence of a positive solution to a system of discrete fractional boundary value problems*. *Appl. Math. Comput.*, **217** (2011), 4740–4753.
18. C. S. GOODRICH: *On a discrete fractional three-point boundary value problem*. *J. Difference Equ. Appl.*, doi: 10.1080/10236198.2010.503240.
19. C. S. GOODRICH: *Some new existence results for fractional difference equations*. *Int. J. Dyn. Syst. Differ. Equ.*, **3** (2011), 145–162.
20. G. INFANTE, J. R. L. WEBB: *Positive solutions of nonlocal boundary value problems: a unified approach*. *J. London Math. Soc.*, **74** (3) (2006), 673–693.
21. E. KAUFMANN: *Positive solutions of a three-point boundary-value problem on a time scale*. *Electron. J. Differential Equations*, (82) 2003, 11 pp.
22. R. MA, Y. RAFFOUL: *Positive solutions of three-point nonlinear discrete second order boundary value problem*. *J. Difference Equ. Appl.*, **10** (2004), 129–138.
23. Y. TIAN, D. MA, W. GE: *Multiple positive solutions of four point boundary value problems for finite difference equations*. *J. Difference Equ. Appl.*, **12** (2006), 57–68.

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