NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 38 (2008), 239–244

A NOTE ON THE CERTAIN DIFFERENTIAL POLYNOMIALS

Junfeng Xu and Feng Lü

(Received July 2007)

Abstract. In this paper we consider the zeros of the differential polynomials $f^2 + af^{(k)} - c$ where k is a positive integer, and a, c are non-zero constants. Our result improves a result of H.H. Chen and corrects an error of Y. Xu whose result related to H.H. Chen.

1. Introduction and results

Let $\mathbb{C}$ be the open complex plane and $\mathcal{D} \in \mathbb{C}$ be a domain. Let f be a meromorphic function in the complex plane, we assumed that the reader is familiar with the notations of Nevanlinna theory(see, [**5, 12, 11**]). We say that f is a Yosida function if there exists a positive number M such that $f^\sharp(z) \leq M$ for all $z \in C$, where

$$f^\sharp(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denotes the spherical derivative.

One of the most important results in the value distribution theory is the following theorem of Hayman.

Theorem A. *If g is a transcendental meromorphic function, then either g itself assumes every finite complex value infinitely often, or $g^{(k)}$ assumes every finite non-zero value infinitely often for any positive integer k.*

As a consequence of Theorem A, we have

Theorem B. *If f is a transcendental integral function, then $f^2 + af'$ has infinitely many zeros for finite non-zero complex value a.*

In fact, for an integral function f, $g = 1/f$ has no zeros and the zeros of $g' - 1/a$ are zeros of $f^2 + af'$.

Ye [**13**], Chen and Hua [**2**] independently proved that Theorem B can be generalized by substituting $f^{(k)}$ for f'. In 1996, Chen [**1**] proved a stronger conclusion for $k = 2$ in the case that f is not a Yosida function.

Theorem C. *Let f be a transcendental integral function. If f is not a Yosida function(in particular, if f is a function of order greater than 1), then for any finite non-zero complex number a and any positive integer k, $f^2 + af^{(k)}$ assumes every*

2000 *Mathematics Subject Classification* 30D35.
Key words and phrases: Meromorphic function; differential polynomials; Nevanlinna theory.

finite complex value infinitely often.

Xu [**10**] removed the restriction that f is not a Yosida function in Theorem C, but there is a gap in his proof(the formula (6) does not always hold). In fact, Doeringer [**4**] gave an example to show that there may exist some exceptional cases.

Example 1. If w ia a transcendental solution of $w^k = -2ac(w-c)$ with a, c non-zero constants, then $w^{(k)} + aw^2$ may omit altogether the value ac^2.

In this note, we shall prove that Doeringer's example is unique in some sense.

Theorem 1.1. *Let f be a transcendental integral function, for any finite non-zero complex number a and any positive integer k. Set $F = f^2 + af^{(k)} - c$, then for any non-zero complex number c: either (i)*

$$T(r,f) \leq (k+1)\overline{N}(r, \frac{1}{F}) + S(r,f). \tag{1.1}$$

or (ii) $f = -d + Ce^{\lambda z}$, where C, λ are non-zero constants and λ satisfies the equation $az^k - 2d = 0$, d satisfies $d^2 - c = 0$.

Corollary 1.2. *Let f be a transcendental integral function, for any finite non-zero complex number a and any positive integer k. If f does not satisfy the case (ii) in Theorem 1.1, then $f^2 + af^{(k)} - c$ has infinitely many zeros.*

Remark 1. If $c = 0$, Chen ([**1**], Theorem 1) has proved that $f^2 + af^{(k)}$ has infinitely many zeros. If $c \neq 0$, we know there exists an additional condition f is not Yosida function(the order great than 1) in Theorem C provided that $f^2 + af^{(k)} - c$ has infinitely many zeros. Theorem 1.1 shows the condition of Theorem C is not necessary and gives a quantitative estimate in the case (i). We will give some examples to show our results sharp in some sense.

Example 2. If $f = -d + Ce^{\lambda z}$, where d, C, λ satisfy the condition of case (ii) in Theorem 1.1 and set $k = 1$, then

$$\begin{aligned} f^2 + af^{(k)} - c &= [d^2 - 2dCe^{\lambda z} + Ce^{2\lambda z}] + aCe^{\lambda z} - c \\ &= Ce^{2\lambda z} + (a\lambda - 2d)Ce^{\lambda z} + d^2 - c \\ &= Ce^{2\lambda z}. \end{aligned}$$

Obviously, $f^2 + af^{(k)} - c$ has no zero and does not satisfy the case (i).

Example 3. If $f = -d + Ce^{\lambda z}$, where d, C, λ don't satisfy the condition of case (ii) in Theorem 1.1 and set $k = 1$, then

$$\begin{aligned} f^2 + af^{(k)} - c &= [d^2 - 2dCe^{\lambda z} + Ce^{2\lambda z}] + aCe^{\lambda z} - c \\ &= Ce^{2\lambda z} + (a\lambda - 2d)Ce^{\lambda z} + d^2 - c. \end{aligned}$$

For $a\lambda - 2d \neq 0$, we can see $f^2 + af^{(k)} - c$ has infinitely many zeros and satisfy the inequality of (1.1), but the order of f not great than 1. The example also shows the condition of Theorem C is not necessary.

Remark 2. Chen's proof used the theory of normal family, here we will take the standard notation of Nevanlinna theory, and Wiman-Valiron theory (cf. [**7, 9**]). Some ideas come from the proof of Hua in [**8**].

2. Some Lemmas

If the coefficients of differential polynomials $M[f]$ are $a_j, j = 0, 1, \cdots, n$, which satisfy

$$m(r, a_j) = S(r, f), \tag{2.1}$$

then differential polynomials $M[f]$ is called a quasi-differential polynomials in f. The following Lemma is nothing but an easy variant of standard Clunie lemma ([**3**], Lemma 1).

Lemma 2.1. *Let f be a non-constant meromorphic in the complex plane, $Q_1[f], Q_2[f]$ are quasi-differential polynomials in f, satisfy $f^n Q_1[f] = Q_2[f]$, if the total degree of $Q_2 \leq n$, then*

$$m(r, Q_1[f]) = S(r, f).$$

Lemma 2.2 ([**5**])**.** *Suppose that $F(z)$ is meromorphic in a domain D and set*

$$f(z) = \frac{F'(z)}{F(z)};$$

Then we have for $n \geq 1$

$$\frac{F^{(n)}(z)}{F(z)} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4} f'^2 + P_{n-3}(f), \tag{2.2}$$

where $a_n = \frac{1}{6} n(n-1)(n-2)$, $b_n = \frac{1}{8} n(n-1)(n-2)(n-3)$, and $P_{n-3}(f)$ is a differential polynomial with constant coefficients ,which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

Lemma 2.3 (Wiman-Valiron [**6, 9**])**.** *Let $f(z)$ be a transcendental entire function and $0 < \delta < \frac{1}{4}$. Suppose that at the point z with $|z| = r$ the inequality*

$$|f(z)| > M(r, f)\nu(r, f)^{-\frac{1}{4}+\delta}.$$

hold. Then there exists a set E in R^+ and of finite logarithmic measure,i.e., $\int_E dt/t < \infty$, such that

$$\frac{f^{(m)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^m (1 + o(1)) \tag{2.3}$$

holds whenever m is a fixed nonnegative integer and $r \notin E$.

3. Proof of Theorem 1.1

Proof. We know that

$$F = f^2 + af^{(k)} - c. \tag{3.1}$$

By differentiating the equation (3.1), we get

$$2fA = Q \tag{3.2}$$

with

$$A = f' - \frac{F'}{2F} f, \quad Q = \frac{F'}{F} f^{(k)} - f^{(k+1)}, \tag{3.3}$$

where A and Q are the quasi-differential polynomials in f. Then $\deg Q \leq 1$. By applying Lemma 2.1 we get

$$m(r, A) = S(r, f).$$

Now put

$$g = \frac{1}{2}\frac{F'}{F}. \tag{3.4}$$

Then from (3.3) we obtain

$$f' = gf + A.$$

Differentiating this equality we have

$$\begin{aligned} f'' &= g'f + gf' + A' = g'f + g(gf + A) + A' \\ &= T_2(g)f + A_2, \end{aligned}$$

where $T_2(g) = g' + g^2$ and $A_2 = gA + A'$. By induction, we deduce that

$$f^{(k)} = T_k(g)f + A_k(g, A). \tag{3.5}$$

where $T_k(g) = g^k + \cdots + g^{(k-1)}$ by Lemma 2.2 and $A_k(g, A)$ is a differential polynomial in g and A which satisfy (2.1). Also, by (3.4) and calculation we obtain

$$T_k(g) = \frac{(F^{1/2})^{(k)}}{F^{1/2}}. \tag{3.6}$$

Substituting (3.5) into (3.1) and letting

$$d = \frac{1}{2}T_k(g), \quad h = f + d, \quad P[h] = A_k(g, A) - d^2 - c, \tag{3.7}$$

we have $m(r, d) = S(r, f)$, $m(r, P[h]) = S(r, f)$ and

$$F = h^2 + P[h]. \tag{3.8}$$

Differentiating (3.8) we get

$$hQ_1 = Q_2 \tag{3.9}$$

with

$$Q_1 = 2h' - \frac{F'}{F}h, \quad Q_2 = \frac{F'}{F}P[h] - P'[h].$$

If $Q_1 \equiv 0$, then there exists a constant b such that $F = bh^2 = b(f + d)^2$. This and (3.1) give

$$(1 - b)f^2 = 2bdf - af^{(k)} + c + bd^2.$$

If $b \neq 1$, then Lemma 2.1 gives $T(r, f) = m(r, f) = S(r, f)$, a contradiction. Thus $b = 1$ and

$$af^{(k)} - 2df - c - d^2 = 0. \tag{3.10}$$

From (3.10) we see that d is entire, which results in $T(r, d) = S(r, f)$. It follows from (3.6) and (3.7) that F has no zeros. Thus there exists an entire function $\alpha(z)$ such that $f + d = e^{\alpha}$. This and (3.10) imply

$$d^2 - ad^{(k)} - c = P(d, \alpha', \cdots, \alpha^{(k)})e^{\alpha},$$

where P is a differential polynomial in d and $\alpha^{(j)}$,$j = 0, 1, \cdots, k$. Since $T(r, d) = S(r, f) = S(r, e^{\alpha})$, we must have $d^2 - ad^{(k)} - c \equiv 0$. Thus

$$d = a\frac{d^{(k)}}{d} + \frac{c}{d}. \tag{3.11}$$

If d is transcendental, then by Lemma 2.3 we choose $r_n \notin E$ and z_n such that $r_n \to \infty (n \to \infty)$, $|z_n| = r_n$, $|d(z_n)| = M(r, d_n)$ and obtain

$$\frac{d^{(k)}(z_n)}{d(z_n)} = \Big(\frac{\nu(r_n, d)}{r_n}\Big)^k (1 + o(1)), \quad \lim_{n\to\infty} \frac{c}{d(z_n)} = 0.$$

From this and (3.11), we have

$$M(r, d_n) = \Big(\frac{\nu(r_n, d)}{r_n}\Big)^k (1 + o(1)), \quad r_n \to \infty. \tag{3.12}$$

Note that the fact that outside of r-values of finite logarithmic measure, we have

$$\nu(r, d) < (\log M(r, d))^2,$$

see Hayman [**6**], p. 344. From the fact and (3.12), we can get a contradiction.

If d is a non-constant polynomial, by a degree argument, we also get a contradiction. Thus we conclude that d is a constant satisfying $d^2 - c = 0$.

Solving the equation (3.10), we have

$$f + \frac{c + d^2}{2d} = \sum_{i=1}^{s} P_i(z) e^{\lambda_i z}, \tag{3.13}$$

where $1 \le s \le k$, $\lambda_i (i = 1, 2, \cdots, s)$ are some distinct roots of the characteristic equation $az^k - 2d = 0$, and $P_i(z) (\not\equiv 0) (i = 1, 2, \cdots, s)$ are polynomials.

From $d^2 = c$, we know (3.13) can be written into

$$f + d = \sum_{i=1}^{s} P_i(z) e^{\lambda_i z}, \tag{3.14}$$

Note that F has no zeros, this is, $f + d$ has no zeros. We know $s = 1$ and $P_1(z) \equiv C$, where C is a non-zero constant. Hence we obtain $f = -d + Ce^{\lambda z}$, where λ satisfies the equation $az^k - 2d = 0$ and d satisfies $d^2 - c = 0$.

If $Q_1 \not\equiv 0$. We see from $h = f + \frac{1}{2}\frac{F'}{F}$ that the poles of h only occur at zeros of F. The expressions of T_k and g yield

$$N(r, h) \le k\overline{N}(r, \frac{1}{F}). \tag{3.15}$$

Note that $T(r, F) \le 2T(r, f) + S(r, f)$ and $m(r, h) = m(r, f) + S(r, f)$, we deduce from (3.15) that $T(r, h) \le (2k + 1)T(r, f) + S(r, f)$, and so

$$S(r, h) = S(r, f).$$

Using (3.15) again we deduce from the expression of Q_1 and Q_2 that

$$N(r, Q_1) \le (k + 1)\overline{N}(r, \frac{1}{F}). \tag{3.16}$$

$$m(r, hQ_1) = m(r, Q_2) = S(r, f).$$

On the other hand, it follows from (3.9) and Lemma 2.1 that

$$m(r, Q_1) = S(r, f).$$

Thus by (3.16),

$$\begin{aligned} T(r,f) &= m(r,f) = m(h-d) \leq m(r,h) + S(r,f) \\ &= m(r,hQ_1) + m(r,\frac{1}{Q_1}) + S(r,f) \\ &= m(,hQ_1) + m(r,Q_1) + N(r,Q_1) - N(r,\frac{1}{Q_1}) + S(r,f) \\ &\leq N(r,Q_1) + S(r,f) \\ &\leq (k+1)\overline{N}(r,\frac{1}{F}) + S(r,f). \end{aligned}$$

This completes the proof of Theorem 1.1. □

Acknowledgement Supported by Specialized Research Fund for the Doctoral Program of Higher Education (No.20060422049) and NSF of China(10771121).

References

[1] H. H. Chen, *Yosida function and Picard values of integral functions and their derivatives*, Bull. Austral. Math. Soc. **54**(1996), 373–381.
[2] H. H. Chen and X. H. Hua, *Normal families of holomorphic functions*, J. Austral. Math. Soc. Ser. A. **59**(1995), 112–117.
[3] J. Clunie, *On integral and meromorphic functions*, J. London Math. Soc. **37**(1962), 17–27.
[4] W. Döringer, *Exceptional values of differential polynomials*, Pacific J. Math. **98** (1982), 55–62.
[5] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.
[6] W. K. Hayman, *The local growth of power series; a survey of Wiman-Valiron method*. Canada Math. Bull. **7**(1974), 317–358.
[7] Y. Z. He, Xiao, X. Z, *Algebroid Functions and Ordinary Differential Equations*, Science Press, Beijing, 1988.
[8] X. H. Hua, *Some extension of the Tumra-Clunie theorem*, Complex Variables Theory Appl. **16**(1991), 69–77.
[9] I. Laine, *Nevanlinna theory and complex differential equations*, Walter de Gruyter, Berlin-New York, 1993.
[10] Y. Xu, *On the value distribution of $f^2 + af^{(k)}$*, Bull. Malaysian. Math. Soc. **27**(2004), 67–71.
[11] C. C. Yang and H.X. Yi, *Uniqueness Theory of Meromorphic Functions*. New York, Dordrecht, Boston, London, 2003.
[12] L. Yang, *Value distribution theory*, Springer, Berlin, Heidelberg, New York, 1993.
[13] Y. Ye, *A new normal criterion and its applications*, Chinese Ann. Math. Ser. A. **12**(1991), 44–49.

Junfeng Xu
Department of Mathematics
Wuyi University
Jiangmen
Guangdong 529020
P.R. CHINA
xujunf@gmail.com

Feng lü]
Department of Mathematics
Shandong University
Jinan 250100
P.R. CHINA
lvfeng@mail.sdu.edu.cn