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Stabilization of positive descriptor fractional discrete-time linear system with two different fractional orders by decentralized controller

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Abstract. Positive descriptor fractional discrete-time linear systems with fractional different orders are addressed in the paper. The decomposition of the regular pencil is used to extend necessary and sufficient conditions for positivity of the descriptor fractional discrete-time linear system with different fractional orders. A method for finding the decentralized controller for the class of positive systems is proposed and its effectiveness is demonstrated on a numerical example.

Key words: positive, fractional, different order, noncomensurate, descriptor, stabilization, decentralized controller.

1. Introduction

Decentralized (state-feedback) controller for linear time-invariant systems allows stabilization of unstable but controllable systems. This problem has been considered in many papers and books [1–5]. Linear time-invariant (LTI) system theory deals with numerous types of such systems e.g. positive [6–9], descriptor [5, 10–13] and/or fractional [4, 14–16].

LTI systems for which inputs, state variables and outputs take only non-negative values are called (internally) positive systems. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of art in the positive systems theory is given in the monographs [8, 9].

Recently the fractional systems have drawn more attention since the fractional differential equations were used by engineers for modelling different processes [17, 18]. From the mathematical point of view, the fractional calculus is well-known [4, 14–16, 19], yet there are still areas in this field which have not been comprehensively addressed, e.g. the descriptor systems, systems with delays or systems with different fractional orders (non-commensurate) [20–22].

A solution to the state equation of descriptor fractional linear systems with regular pencils has been given in [12, 13, 23]. A comparison of three different methods for finding the solution of descriptor fractional discrete-time linear systems can be found in [24] and the solution to the descriptor fractional discrete-time linear systems with two different fractional orders has been introduced in [25]. Stability of positive fractional discrete-time linear systems have been addressed in [26–28] and the decentralized stabilization of fractional positive descriptor discrete-time linear systems in [3].

In this paper the decentralized stabilization problem for positive descriptor fractional discrete-time linear systems with two different fractional order will be formulated and solved.

The paper is organized as follows. In Section 2 basic information on the positive fractional discrete-time linear systems with different fractional orders is recalled. Descriptor fractional discrete-time linear systems with different fractional orders are addressed in Section 3, where the decomposition and positivity conditions are presented. The main idea of the paper is presented in Section 4, where the solution to decentralized stabilization of positive descriptor fractional discrete-time linear systems with different fractional orders is given and illustrated by numerical example. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, Z_+ – the set of nonnegative integers, M_n – the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Positive fractional different orders discrete-time linear systems

Consider the fractional discrete-time linear system with two different fractional orders α and β of the form

$$\begin{aligned}\Delta^{\alpha}x_1(k+1)&=A_{11}x_1(k)+A_{12}x_2(k)+B_1u(k),\\ \Delta^{\beta}x_2(k+1)&=A_{21}x_1(k)+A_{22}x_2(k)+B_2u(k),\end{aligned} \tag{1}$$

where $k \in Z_+$, $x_1(k) \in \Re^{n_1}$ and $x_2(k) \in \Re^{n_2}$ are the state vectors, $u(k) \in \Re^m$ is the input vector and $A_{ij} \in \Re^{n_i\times m}$, $B_i \in \Re^{n_i\times m}$; $i, j = 1, 2$, $n = n_1 + n_2$.

The fractional difference of α (β) order is defined by [4]

$$\Delta^{\alpha}x(k)=\sum_{j=0}^{k}(-1)^j\binom{\alpha}{j}x(k-j), \tag{2a}$$

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$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j=0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for} \quad j=1,2,... \end{cases} \tag{2b}$$

Using (2), we can write the equation (1) in the matrix form

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \\ &+ \sum_{j=2}^{k+1} \begin{bmatrix} c_{\alpha,j} I_{n_1} & 0 \\ 0 & c_{\beta,j} I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k-j+1) \\ x_2(k-j+1) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k), \end{aligned} \tag{3}$$

where $A_{1\alpha} = A_{11} + I_{n_1}\alpha$, $A_{2\beta} = A_{22} + I_{n_2}\beta$, $c_{\alpha,0} = 1$,

Definition 1. [4] The fractional system (1) is called (internally) positive if $x_1(k) \in \Re_+^{n_1}$, $x_2(k) \in \Re_+^{n_2}$ $k \in Z_+$. for all $x_1(0) \in \Re_+^{n_1}$, $x_2(0) \in \Re_+^{n_2}$ and every $u(k) \in \Re_+^m$, $k \in Z_+$.

Theorem 1. [4] The fractional system (1) for $0 < \alpha, \beta < 1$ is positive if and only if (iff)

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \in \Re_+^{n \times n}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{n \times m}. \tag{4}$$

3. Positive descriptor fractional different orders discrete-time linear systems

Consider the descriptor fractional discrete-time linear system with two different fractional orders

$$\begin{aligned} E_1 \Delta^\alpha x_1(k+1) &= A_{11} x_1(k) + A_{12} x_2(k) + B_1 u(k), \\ E_2 \Delta^\beta x_2(k+1) &= A_{21} x_1(k) + A_{22} x_2(k) + B_2 u(k), \end{aligned} \tag{5}$$

where $k \in Z_+$, $x_1(k) \in \Re_+^{n_1}$ and $x_2(k) \in \Re_+^{n_2}$ are the state vectors, $u(k) \in \Re^m$ is the input vector and $E_i \in \Re^{n_i \times n_i}$ $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$; $i, j = 1, 2$.

The state-space solution to this class of fractional systems can be found in [25].

Using the definition of fractional derivative (2), the fractional system (5) can be written as the descriptor system with delays

$$E \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = A \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \sum_{j=2}^{k+1} c_j \begin{bmatrix} x_1(k-j+1) \\ x_2(k-j+1) \end{bmatrix} + Bu(k) \tag{6a}$$

where

$$\begin{aligned} E &= \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \; A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix}, \\ B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; c_j = \begin{bmatrix} c_{\alpha,j} E_1 & 0 \\ 0 & c_{\beta,j} E_2 \end{bmatrix} \end{aligned} \tag{6b}$$

and $A_{1\alpha} = A_{11} + E_1\alpha$, $A_{2\beta} = A_{22} + E_2\beta$.

Further, we will consider the descriptor system with regular pencil

$$\det E = 0 \tag{7a}$$

and

$$\det \left[\begin{bmatrix} E_1 z_1 & 0 \\ 0 & E_2 z_2 \end{bmatrix} - \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \right] \neq 0 \tag{7b}$$

for some $z \in C$ (the field of complex numbers), where matrices E_1, E_2 contain only n_1^1, n_2^1 linearly independent columns, respectively.

It is well-known [4] that every descriptor system with regular pencil can be decomposed e.g. by the use of the Weierstrass-Kronecker decomposition theorem.

Lemma 1. If (7a) and (7b) hold for the system with two different fractional orders (6), then there exist nonsingular matrices

$$\begin{aligned} P &= \text{blockdiag } (P_1, P_2) \in \Re^{n \times n}, \\ Q &= \text{blockdiag } (Q_1, Q_2) \in \Re^{n \times n} \end{aligned} \tag{8}$$

such that

$$\begin{aligned} &P \left[\begin{bmatrix} E_1 z_1 & 0 \\ 0 & E_2 z_2 \end{bmatrix} - \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \right] Q = \\ &= \begin{bmatrix} \overline{E}_1 z_1 & 0 \\ 0 & \overline{E}_2 z_2 \end{bmatrix} - \begin{bmatrix} \overline{A}_{1\alpha} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{2\beta} \end{bmatrix} \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\overline{E}_1 = P_1 E_1 Q_1 = \begin{bmatrix} I_{n_1^1} & 0 \\ 0 & 0 \end{bmatrix}, \; \overline{E}_2 = P_2 E_2 Q_2 = \begin{bmatrix} I_{n_2^1} & 0 \\ 0 & 0 \end{bmatrix}, \\ &\overline{A}_{1\alpha} = P_1 A_{1\alpha} Q_1 = \begin{bmatrix} \overline{A}_{1\alpha}^{11} & \overline{A}_{12}^{11} \\ \overline{A}_{21}^{11} & \overline{A}_{2\alpha}^{11} \end{bmatrix}, \\ &\overline{A}_{12} = P_1 A_{12} Q_2 = \begin{bmatrix} \overline{A}_{11}^{12} & \overline{A}_{12}^{12} \\ \overline{A}_{21}^{12} & \overline{A}_{22}^{12} \end{bmatrix}, \\ &\overline{A}_{21} = P_2 A_{21} Q_1 = \begin{bmatrix} \overline{A}_{11}^{21} & \overline{A}_{12}^{21} \\ \overline{A}_{21}^{21} & \overline{A}_{22}^{21} \end{bmatrix}, \\ &\overline{A}_{2\beta} = P_2 A_{2\beta} Q_2 = \begin{bmatrix} \overline{A}_{1\beta}^{22} & \overline{A}_{12}^{22} \\ \overline{A}_{21}^{22} & \overline{A}_{2\beta}^{22} \end{bmatrix}, \\ &\overline{B}_1 = P_1 B_1 = \begin{bmatrix} \overline{B}_1^1 \\ \overline{B}_2^1 \end{bmatrix}, \overline{B}_2 = P_2 B_2 = \begin{bmatrix} \overline{B}_1^2 \\ \overline{B}_2^2 \end{bmatrix}, \end{aligned} \tag{10}$$

where $\text{rank} E_1 = n_1^1$, $\text{rank} E_2 = n_2^1$, $n_1 = n_1^1 + n_2^1$, $n_2 = n_2^1 + n_2^2$, $n = n_1 + n_2$.

The matrices P and Q, which decompose the system (6), can be found by the use of many different method, see e.g. [1, 29].

Premultiplying the state equation (6a) by the matrix $P \in \Re^{n \times n}$ and introducing the new state vector

$$Q^{-1}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \end{bmatrix} = \begin{bmatrix} \bar{x}_1^1(k) \\ \bar{x}_1^2(k) \\ \bar{x}_2^1(k) \\ \bar{x}_2^2(k) \end{bmatrix}, \quad \bar{x}_1^1(k) \in \Re^{n_1^1}, \tag{11}$$

$\bar{x}_1^2(k) \in \Re^{n_1^2}$, $\bar{x}_2^1(k) \in \Re^{n_2^1}$ $\bar{x}_2^2(k) \in \Re^{n_2^2}$

for $k \in Z_+$ we obtain

$$\begin{aligned} PEQQ^{-1}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= PAQQ^{-1}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \\ &+ \sum_{j=2}^{k+1} c_j PEQQ^{-1}\begin{bmatrix} x_1(k-j+1) \\ x_2(k-j+1) \end{bmatrix} + PBu(k). \end{aligned} \tag{12}$$

Taking into consideration (10, 11) and recombining the state equation (12) we obtain two subsystems – the standard dynamical subsystem

$$\begin{aligned} \begin{bmatrix} \bar{x}_1^1(k+1) \\ \bar{x}_2^1(k+1) \end{bmatrix} &= \begin{bmatrix} \bar{A}_{1\alpha}^{11} & \bar{A}_{11}^{12} \\ \bar{A}_{11}^{21} & \bar{A}_{1\beta}^{22} \end{bmatrix}\begin{bmatrix} \bar{x}_1^1(k) \\ \bar{x}_2^1(k) \end{bmatrix} + \begin{bmatrix} \bar{A}_{12}^{11} & \bar{A}_{12}^{12} \\ \bar{A}_{12}^{21} & \bar{A}_{12}^{22} \end{bmatrix}\begin{bmatrix} \bar{x}_1^2(k) \\ \bar{x}_2^2(k) \end{bmatrix} \\ &+ \sum_{j=2}^{k+1} c_j \begin{bmatrix} \bar{x}_1^1(k-j+1) \\ \bar{x}_2^1(k-j+1) \end{bmatrix} + \begin{bmatrix} \bar{B}_1^1 \\ \bar{B}_2^1 \end{bmatrix} u(k) \end{aligned} \tag{13}$$

and the static (algebraic) subsystem

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \bar{A}_{21}^{11} & \bar{A}_{21}^{12} \\ \bar{A}_{21}^{21} & \bar{A}_{21}^{22} \end{bmatrix}\begin{bmatrix} \bar{x}_1^1(k) \\ \bar{x}_2^1(k) \end{bmatrix} + \\ &+ \begin{bmatrix} \bar{A}_{2\alpha}^{11} & \bar{A}_{22}^{12} \\ \bar{A}_{22}^{21} & \bar{A}_{2\beta}^{22} \end{bmatrix}\begin{bmatrix} \bar{x}_1^2(k) \\ \bar{x}_2^2(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_1^2 \\ \bar{B}_2^2 \end{bmatrix} u(k). \end{aligned} \tag{14}$$

The subsystems (13) and (14) can be written as

$$\begin{aligned} \tilde{x}_1(k+1) &= \tilde{A}_{11}\tilde{x}_1(k) + \tilde{A}_{12}\tilde{x}_2(k) \\ &+ \sum_{j=2}^{i+1} c_j \tilde{x}_1(k-j+1) + \tilde{B}_1 u(k), \end{aligned} \tag{15a}$$

$$0 = \tilde{A}_{21}\tilde{x}_1(k) + \tilde{A}_{22}\tilde{x}_2(k) + \tilde{B}_2 u(k), \tag{15b}$$

where

$$\tilde{A}_{11} = \begin{bmatrix} \bar{A}_{1\alpha}^{11} & \bar{A}_{11}^{12} \\ \bar{A}_{11}^{21} & \bar{A}_{1\beta}^{22} \end{bmatrix} \in \Re^{\tilde{n}_1 \times \tilde{n}_1}, \quad \tilde{A}_{12} = \begin{bmatrix} \bar{A}_{12}^{11} & \bar{A}_{12}^{12} \\ \bar{A}_{12}^{21} & \bar{A}_{12}^{22} \end{bmatrix} \in \Re^{\tilde{n}_1 \times \tilde{n}_2},$$

$$\tilde{A}_{21} = \begin{bmatrix} \bar{A}_{21}^{11} & \bar{A}_{21}^{12} \\ \bar{A}_{21}^{21} & \bar{A}_{21}^{22} \end{bmatrix} \in \Re^{\tilde{n}_2 \times \tilde{n}_1}, \quad \tilde{A}_{22} = \begin{bmatrix} \bar{A}_{2\alpha}^{11} & \bar{A}_{22}^{12} \\ \bar{A}_{22}^{21} & \bar{A}_{2\beta}^{22} \end{bmatrix} \in \Re^{\tilde{n}_2 \times \tilde{n}_2},$$

$$\tilde{B}_1 = \begin{bmatrix} \bar{B}_1^1 \\ \bar{B}_2^1 \end{bmatrix} \in \Re^{\tilde{n}_1 \times m}, \quad \tilde{B}_2 = \begin{bmatrix} \bar{B}_1^2 \\ \bar{B}_2^2 \end{bmatrix} \in \Re^{\tilde{n}_2 \times m},$$

$$\tilde{x}_1(k) = \begin{bmatrix} \bar{x}_1^1(k) \\ \bar{x}_2^1(k) \end{bmatrix} \in \Re^{\tilde{n}_1}, \quad \tilde{x}_2(k) = \begin{bmatrix} \bar{x}_1^2(k) \\ \bar{x}_2^2(k) \end{bmatrix} \in \Re^{\tilde{n}_2}, \tag{16}$$

$$\tilde{n}_1 = n_1^1 + n_2^1, \quad \tilde{n}_2 = n_1^2 + n_2^2, \quad n = \tilde{n}_1 + \tilde{n}_2.$$

The fact, that matrices E_1, E_2 contains respectively only n_1^1, n_2^2 linearly independent columns and the rest are zero columns, imply that the matrices $Q_1 \in \Re_+^{n_1 \times n_1}$, $Q_2 \in \Re_+^{n_2 \times n_2}$ are a permutation matrices (monomial matrices) and $Q^{-1} \in \Re_+^{n \times n}$, since Q is block diagonal (8). Therefore, if $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in \Re_+^n$, $k \in Z_+$ then $Q^{-1}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in \Re_+^n$, $k \in Z_+$ and $\tilde{x}_1(k) \in \Re_+^{\tilde{n}_1}$, $\tilde{x}_2(k) \in \Re_+^{\tilde{n}_2}$ for $k \in Z_+$, since $\bar{x}_1(k) \in \Re_+^{n_1}$, $\bar{x}_2(k) \in \Re_+^{n_2}$ for $k \in Z_+$.

Theorem 2. The descriptor fractional system (5) for $0 < \alpha$, $\beta < 1$ is positive iff $\tilde{A}_{22} \in M_{n_2}$ is asymptotically stable Metzler matrix and

$$\begin{aligned} &\tilde{A}_{11} \in \Re_+^{\tilde{n}_1 \times \tilde{n}_1}, \ \tilde{A}_{12} \in \Re_+^{\tilde{n}_1 \times \tilde{n}_2}, \ \tilde{A}_{21} \in \Re_+^{\tilde{n}_2 \times \tilde{n}_1}, \\ &\tilde{B}_1 \in \Re_+^{\tilde{n}_1 \times m}, \ \tilde{B}_2 \in \Re_+^{\tilde{n}_2 \times m}. \end{aligned} \tag{17}$$

Proof. It is well-known [4] that if $\tilde{A}_{22} \in M_{n_2}$ is asymptotically stable Metzler matrix then $-\tilde{A}_{22}^{-1} \in \Re_+^{n_2 \times n_2}$. In this case from (15b) and definition of positive system we have

$$\tilde{x}_2(k) = -\tilde{A}_{22}^{-1}(\tilde{A}_{21}\tilde{x}_1(k) + \tilde{B}_2 u(k)) \in \Re_+^{\tilde{n}_2}, \ k \in Z_+ \tag{18}$$

iff $\tilde{A}_{21} \in \Re_+^{\tilde{n}_2 \times \tilde{n}_1}$ and $\tilde{B}_2 \in \Re_+^{\tilde{n}_2 \times m}$.

Next, the substitution of (18) into (15a) yields

$$\tilde{x}_1(k+1) = \hat{A}_{11}\tilde{x}_1(k) + \sum_{j=2}^{k+1} c_j \tilde{x}_1(k-j+1) + \hat{B}_1 u(k), \quad k \in Z_+ \tag{19}$$

where

$$\hat{A}_{11} = \tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}, \ \hat{B}_1 = \tilde{B}_1 - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{B}_2. \tag{20}$$

It is well-known [4] that $C_j > 0$ for $j = 2, 3, \dots$ and $0 < \alpha, \beta < 1$. In this case, from (19) and definition of positive system we have $\tilde{x}_1(k) \in \Re_+^{\tilde{n}_1}$, $k \in Z_+$ for $\tilde{x}_1(0) \in \Re_+^{\tilde{n}_1}$, $u(k) \in \Re_+^m$, $k \in Z_+$ iff $\hat{A}_{11} \in \Re_+^{\tilde{n}_1 \times \tilde{n}_1}$ and $\hat{B}_1 \in \Re_+^{\tilde{n}_1 \times m}$. Further, from (20) we have that the matrices $\hat{A}_{11}$ and $\hat{B}_1$ are positive if $-\tilde{A}_{22}^{-1} \in \Re_+^{\tilde{n}_2 \times \tilde{n}_2}$, $\tilde{A}_{11} \in \Re_+^{\tilde{n}_1 \times \tilde{n}_1}$, $\tilde{A}_{12} \in \Re_+^{\tilde{n}_1 \times \tilde{n}_2}$, $\tilde{A}_{21} \in \Re_+^{\tilde{n}_2 \times \tilde{n}_1}$, $\tilde{B}_1 \in \Re_+^{\tilde{n}_1 \times m}$, $\tilde{B}_2 \in \Re_+^{\tilde{n}_2 \times m}$.

From (18) it follows that $\tilde{x}_2(k) \in \Re_+^{\tilde{n}_2}$, $k \in Z_+$ iff $\tilde{x}_1(k) \in \Re_+^{\tilde{n}_1}$ and $\tilde{B}_2 u(k) \in \Re_+^m$ for $k \in Z_+$. □

4. Decentralized stabilization of positive descriptor fractional linear systems

For positive system (5) we are looking for a gain matrix

$$K = NG^{-1},\ G = \mathrm{diag}[g_1,\dots,g_n],\quad g_k > 0,\ k = 1,\dots,n,\ N \in \Re^{m\times n} \tag{21}$$

such that the close-loop system matrix

$$A_c = A + BK \in M_n \tag{22}$$

is asymptotically stable.

We choose matrices N and G such that

$$AG + BN \in M_n \text{ and } (AG + BN)G^{-1} < 0. \tag{23}$$

Then, if (23) holds then the matrix (22) is asymptotically stable Metzler matrix (see. [30]).

Matrices G and N can be computed by the use of linear matrix inequalities method or linear programming, see e.g. [29].

Definition 2. [3] The positive system (3) (or equivalently the pair (A, B) defined by (4)) is called stabilizable by the state feedback if there exists a gain matrix (21) such that the closed-loop system matrix (22) is asymptotically stable.

Definition 3. [3] The matrix $C \in \Re^{n_2\times n_1}$ satisfying the equality

$$\tilde{x}_2 = C\tilde{x}_1, \tag{24}$$

is called contracting matrix if

$$\|\tilde{x}_2\| < \|\tilde{x}_1\|, \tag{25}$$

where the norm of $\tilde{x}_2$, $(\tilde{x}_1)$ is defined as

$$\|\tilde{x}_2\| = \sum_{j=1}^{\tilde{n}_2} \left|\tilde{x}_2^j\right|,\quad \tilde{x}_2 = [\tilde{x}_2^1 \quad \tilde{x}_2^2 \quad \dots \quad \tilde{x}_2^{n_2}]^T. \tag{26}$$

Remark 1. For positive systems the controllability of the pair (A, B) is not sufficient for the stabilization of close-loop system with Metzler matrix. The pair (A, B) should be stabilizable.

Consider the fractional system (15) with decentralized controller

$$u(k) = \begin{bmatrix} \tilde{K}_1 & 0 \\ 0 & \tilde{K}_2 \end{bmatrix}\begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix},\ \tilde{K}_1 \in \Re^{1\times\tilde{n}_1},\ \tilde{K}_2 \in \Re^{1\times\tilde{n}_2}, \tag{27}$$

and the closed-loop system has the form

$$\begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} + \tilde{B}_1\tilde{K}_1 & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} + \tilde{B}_2\tilde{K}_2 \end{bmatrix}\begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \begin{bmatrix} \sum_{j=2}^{k+1} c_j\tilde{x}_1(k-j+1) \\ 0 \end{bmatrix},\ k \in Z_+. \tag{28}$$

The close-loop system (28) is called (internally) positive if

$$\tilde{x}_i(k) \in \Re_+^{\tilde{n}_i},\ i = 1, 2;\ k \in Z_+ \quad \text{for } \tilde{x}_i(0) \in \Re_+^{\tilde{n}_i},\ i = 1, 2. \tag{29}$$

The positive close-loop system (28) is called asymptotically stable if

$$\lim_{k\to\infty} \tilde{x}_i(k) = 0 \quad \text{for all } \tilde{x}_i(0) \in \Re_+^{n_i},\ i = 1, 2. \tag{30}$$

Now, we have to find $\tilde{K}_1$ and $\tilde{K}_2$, which stabilize the descriptor system and do not violate its positivity.

From (28) we have

$$\tilde{x}_2(k) = -\hat{A}_{22}^{-1}\tilde{A}_{21}\tilde{x}_1(k) \in \Re_+^{\tilde{n}_2} \quad \text{for } -\hat{A}_{22}^{-1}\tilde{A}_{21} \in \Re_+^{\tilde{n}_2\times\tilde{n}_2}, \tag{31}$$

where $\hat{A}_{22} = \tilde{A}_{22} + \tilde{B}_2\tilde{K}_2$.

Taking under consideration (21–23), we compute (using e.g. LMI) $\tilde{K}_2$ so that $\hat{A}_{22} \in M_{\tilde{n}_2}$ is asymptotically stable and $-\hat{A}_{22}^{-1}\tilde{A}_{21}$ is contracting matrix.

Substituting (31) into (28) we obtain

$$\tilde{x}_1(k+1) = (\hat{A}_{11} + \tilde{B}_1\tilde{K}_1)\tilde{x}_1(k) + \sum_{j=2}^{k+1} c_j\tilde{x}_1(k-j+1), \tag{32}$$

where

$$\hat{A}_{11} = \tilde{A}_{11} - \tilde{A}_{12}\hat{A}_{22}^{-1}\tilde{A}_{21} \in \Re_+^{\tilde{n}_1\times\tilde{n}_1} \text{ for } \tilde{A}_{11} \in \Re_+^{\tilde{n}_1\times\tilde{n}_1}. \tag{33}$$

It is well-known [13] that $\sum_{j=2}^{\infty} c_j = 0$ and $\sum_{j=2}^{\infty} c_j = \begin{bmatrix} 1-\alpha & 0 \\ 0 & 1-\beta \end{bmatrix}$.

In this case we compute the matrix $\tilde{K}_1$ so that the matrix

$$\hat{A}_{11} + \begin{bmatrix} I_{\tilde{n}_1}(1-\alpha) & 0 \\ 0 & I_{\tilde{n}_2}(1-\beta) \end{bmatrix} + \tilde{B}_1\tilde{K}_1 \in \Re_+^{\tilde{n}_1\times\tilde{n}_1} \tag{34}$$

is asymptotically stable.

If the condition (34) is satisfied then from (32) it follows that $\tilde{x}_1(k) \in \Re_+^{\tilde{n}_1}$, $k \in Z_+$ and

$$\lim_{k\to\infty} \tilde{x}_1(k) = 0, \tag{35}$$

This imply $\tilde{x}_2(k) \in \Re_+^{\tilde{n}_2}$, $k \in Z_+$ and

$$\lim_{k\to\infty} \tilde{x}_2(k) = 0. \tag{36}$$

Therefore, it was proven that:

Theorem 3. The positive descriptor fractional discrete-time linear system with two different fractional orders (6) can be stabilized by the decentralized controller (27) iff the pairs $(\tilde{A}_{22}, \tilde{B}_2)$, $(\hat{A}_{11}, \tilde{B}_1)$ are stabilizable and $-\hat{A}_{22}^{-1}\tilde{A}_{21}$ is the contracting matrix.

Example 1. Find the solution of the descriptor fractional linear system (6) with the fractional orders $\alpha = 0.5$, $\beta = 0.6$ and the matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.05 \\ 0 \\ -0.05 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.05 \\ 0 \\ 0.05 \end{bmatrix},$$
$$A_{11} = \begin{bmatrix} -0.45 & 0.1 & 0.05 \\ 0.06 & -0.5 & 0 \\ -0.05 & 0 & -0.05 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.2 & 0.1 & 0.4 \\ 0.05 & 0.1 & 0 \\ 0 & 0 & -0.4 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 0.22 & 0.07 & 0.25 \\ 0.46 & 0.08 & 0.15 \\ 0.1 & 0.35 & 0.85 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.13 & -0.55 & 0.14 \\ -0.54 & 0.02 & 0.07 \\ 0.11 & 0.23 & 0.11 \end{bmatrix}. \tag{37}$$

It is easy to check that the matrices (37) satisfies the assumptions (7a, 7b). In this case the matrices P and Q have the form

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{38}$$

and the decomposition (10) is given by

$$\bar{E}_1 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{E}_2 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0.05 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 \\ 0.05 \\ 0.05 \end{bmatrix},$$
$$\bar{A}_{1\alpha} = \begin{bmatrix} 0 & 0.06 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0.05 & 0.05 \end{bmatrix}, \quad \bar{A}_{12} = \begin{bmatrix} 0.05 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \tag{39}$$
$$\bar{A}_{21} = \begin{bmatrix} 0.08 & 0.46 & 0.15 \\ 0.07 & 0.22 & 0.25 \\ 0.35 & 0.1 & 0.85 \end{bmatrix}, \quad \bar{A}_{2\beta} = \begin{bmatrix} 0.06 & 0.02 & 0.07 \\ 0.13 & 0.05 & 0.14 \\ 0.11 & 0.23 & 0.11 \end{bmatrix}.$$

Using (16) we have

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 0.06 & 0.05 & 0.1 \\ 0.1 & 0 & 0.2 & 0.1 \\ 0.08 & 0.46 & 0.06 & 0.02 \\ 0.08 & 0.22 & 0.13 & 0.05 \end{bmatrix}, \quad \tilde{A}_{12} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \\ 0.15 & 0.07 \\ 0.25 & 0.14 \end{bmatrix},$$
$$\tilde{A}_{21} = \begin{bmatrix} 0 & 0.05 & 0 & 0 \\ 0.35 & 0.1 & 0.11 & 0.23 \end{bmatrix}, \quad \tilde{A}_{22} = \begin{bmatrix} 0.05 & 0.4 \\ 0.85 & 0.11 \end{bmatrix}, \tag{40}$$
$$\tilde{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.05 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0.05 \\ 0.05 \end{bmatrix}.$$

Now, taking under consideration (21–23) we obtain

$$\hat{A}_{22} = \tilde{A}_{22} + \tilde{B}_2\tilde{K}_2 = (\tilde{A}_{22}\tilde{G}_2 + \tilde{B}_2\tilde{N}_2)\tilde{G}_2^{-1} \in M_2. \tag{41}$$

We assume, that our desired matrix should have the form

$$\hat{A}_{22} = \begin{bmatrix} -0.75 & 0.05 \\ 0.05 & -0.24 \end{bmatrix} \in M_2. \tag{42}$$

since it is asymptotically stable Meltzer matrix and

$$-\hat{A}_{22}^{-1}\tilde{A}_{21} = \begin{bmatrix} 0.1 & 0.1 & 0.03 & 0.06 \\ 1.48 & 0.44 & 0.46 & 0.97 \end{bmatrix} \tag{43}$$

is contracting matrix, e.g. by Definition 3 for vector $\tilde{x}_1 = [1 \;\; 1 \;\; 1 \;\; 1]^T$ we have $\|\tilde{x}_1\| = 4$, $\|\tilde{x}_2\| = 3.64$.

Lets take $\tilde{G}_2 = \mathrm{diag}[1, 1]$. Using one of the well-known methods (in this case Symbolic Math Toolbox), we compute

$$\tilde{K}_2 = \tilde{N}_2\tilde{G}_2^{-1} = [-16 \quad -7] \tag{44}$$

which satisfy (41).

Unlike the matrix (41) which should be Metzler, the matrix (34) need to be positive. In this case, using similar approach, we compute $\tilde{K}_1 = [-4 \;\; -4 \;\; -2 \;\; -5]$ for which

$$\hat{A}_{11} + \begin{bmatrix} I_{\tilde{n}_1}(1-\alpha) & 0 \\ 0 & I_{\tilde{n}_2}(1-\beta) \end{bmatrix} + \tilde{B}_1\tilde{K}_1 = \\ = \begin{bmatrix} 0.65 & 0.1 & 0.1 & 0.2 \\ 0.1 & 0.5 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.35 \end{bmatrix}. \tag{45}$$

is positive and asymptotically stable matrix, since its eigenvalues are $\lambda = [0.99 \;\; 0.54 \;\; 0.19 \;\; 0.28]$.

Inversing recombination and decomposition on the matrix $\tilde{K} = \text{blockdiag}(\tilde{K}_1, \tilde{K}_2)$ we can find the gain matrix of decentralized controller for descriptor fractional discrete-time linear system with two different fractional orders (5) described by matrices (37) of the form

$$K = \begin{bmatrix} -4 & -4 & -16 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -2 & -7 \end{bmatrix}. \tag{46}$$

5. Concluding remarks

The positive fractional discrete-time linear systems with two different fractional orders were analyzed. Based on the decomposition of the regular pencil, necessary and sufficient conditions for the positivity were extended to the descriptor fractional discrete-time linear system with two different fractional orders. A method for finding the decentralized controller for the class of positive systems was proposed and its effectiveness demonstrated on a numerical example. An extension of these considerations to the systems consisting of n subsystems with different fractional orders is possible.

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