

SOME CONGRUENCES FOR OVERPARTITIONS

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Abstract. We present simple proofs of some congruences recently given by D.Q.J. Dou and B.L.S. Lin for overpartitions.

1. Introduction

Let $\bar{p}(n)$ denote the number of overpartitions of n , that is, the number of partitions of n in two colours, the parts in one colour being distinct, while the parts in the other are unrestricted. Alternatively, $\bar{p}(n)$ can be regarded as the number of partitions of n in which odd parts come in two colours, while even parts come in one.

The generating function for $\bar{p}(n)$ is

$$\sum_{n \geq 0} \bar{p}(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} = \frac{1}{(q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}} = \frac{1}{\phi(-q)} \quad (1.1)$$

where it is understood that

$$(a; q)_{\infty} = \prod_{n \geq 1} (1 - aq^{n-1}) \quad (1.2)$$

and

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}. \quad (1.3)$$

Recently, D.Q.J. Dou and B.L.S. Lin [2] showed that

$$\bar{p}(80n + r) \equiv 0 \pmod{5} \quad (1.4)$$

for $r = 8, 52, 68$ and 72 .

The object of this note is to give proofs of these congruences using only the simplest formulas from q -series.

2. Preliminaries

We require the following definitions and relations.

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2}, \quad (2.1)$$

$$\phi(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}}, \quad (2.2)$$

$$\psi(q) = \sum_{n \geq 0} q^{(n^2+n)/2} = \sum_{-\infty}^{\infty} q^{2n^2+n} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}, \quad (2.3)$$

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \quad (2.4)$$

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2, \quad (2.5)$$

$$\phi(q)\psi(q^2) = \psi(q)^2, \quad (2.6)$$

$$(q; q)_{\infty}^5 \equiv (q^5; q^5)_{\infty} \pmod{5} \quad (2.7)$$

and

$$(q; q)_{\infty} = \sum_{-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} \quad (2.8)$$

has no terms in which the power of q is 3 or 4 $\pmod{5}$.

All these definitions and relations are standard in the theory of q -series. See, for instance, [1] or [3].

We also require the following remarkable result.

$$\phi(q)^8 - q\phi(q)^4\psi(q^2)^4 + q^2\psi(q^2)^8 \equiv 1 \pmod{5}. \quad (2.9)$$

We will start with a proof of (2.9). Let $F(q) = \phi(q)^8 - q\phi(q)^4\psi(q^2)^4 + q^2\psi(q^2)^8$. We have

$$\begin{aligned} F(q) &= (\phi(q)^2)^4 - q(\phi(q)^2)^2\psi(q^2)^4 + q^2\psi(q^2)^8 \\ &= (\phi(q^2)^2 + 4q\psi(q^4)^2)^4 - q(\phi(q^2)^2 + 4q\psi(q^4)^2)^2\phi(q^2)^2\psi(q^4)^2 \\ &\quad + q^2\phi(q^2)^4\psi(q^4)^4 \\ &= \phi(q^2)^8 + 15q\phi(q^2)^6\psi(q^4)^2 + 89q^2\phi(q^2)^4\psi(q^4)^4 + 240q^3\phi(q^2)^2\psi(q^4)^6 \\ &\quad + 256q^4\psi(q^4)^8 \end{aligned} \quad (2.10)$$

Modulo 5, (2.10) yields

$$F(q) \equiv \phi(q^2)^8 - q^2\phi(q^2)^4\psi(q^4)^4 + q^4\psi(q^4)^8 = F(q^2). \quad (2.11)$$

It follows by iteration of (2.11) that for any n ,

$$F(q) \equiv F(q^{2^n}) = 1 + 0q + 0q^2 + \cdots + 0q^{2^n-1} + \cdots, \quad (2.12)$$

from which it follows that

$$F(q) \equiv 1.$$

□

3. The Congruences

We have

$$\begin{aligned}
\sum_{n \geq 0} \bar{p}(n)q^n &= \frac{1}{\phi(-q)} = \frac{\phi(q)}{\phi(-q^2)^2} = \frac{\phi(q)\phi(q^2)^2}{\phi(-q^4)^4} \\
&= \frac{(\phi(q^4) + 2q\psi(q^8))(\phi(q^4)^2 + 4q^2\psi(q^8)^2)}{\phi(-q^4)^4} \\
&= \frac{\phi(q^4)^3 + 2q\phi(q^4)^2\psi(q^8) + 4q^2\phi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3}{\phi(-q^4)^4}. \quad (3.1)
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n \geq 0} \bar{p}(4n)q^n &= \frac{\phi(q)^3}{\phi(-q)^4} = \frac{\phi(q)^7}{\phi(-q^2)^8} = \frac{\phi(q)^7\phi(q^2)^8}{\phi(-q^4)^{16}} = \frac{\phi(q)^7(\phi(q^2)^2)^4}{\phi(-q^4)^{16}} \\
&= \frac{(\phi(q^4) + 2q\psi(q^8))^7(\phi(q^4)^2 + 4q^2\psi(q^8)^2)^4}{\phi(-q^4)^{16}} \\
&= \frac{1}{\phi(-q^4)^{16}} (\phi(q^4)^{15} + 14q\phi(q^4)^{14}\psi(q^8) + 100q^2\phi(q^4)^{13}\psi(q^8)^2 \\
&\quad + 504q^3\phi(q^4)^{12}\psi(q^8)^3 + 2000q^4\phi(q^4)^{11}\psi(q^8)^4 + 6496q^5\phi(q^4)^{10}\psi(q^8)^5 \\
&\quad + 17728q^6\phi(q^4)^9\psi(q^8)^6 + 41344q^7\phi(q^4)^8\psi(q^8)^7 + 82688q^8\phi(q^4)^7\psi(q^8)^8 \\
&\quad + 141824q^9\phi(q^4)^6\psi(q^8)^9 + 207872q^{10}\phi(q^4)^5\psi(q^8)^{10} \\
&\quad + 256000q^{11}\phi(q^4)^4\psi(q^8)^{11} + 258048q^{12}\phi(q^4)^3\psi(q^8)^{12} \\
&\quad + 204800q^{13}\phi(q^4)^2\psi(q^8)^{13} + 114688q^{14}\phi(q^4)\psi(q^8)^{14} + 32768q^{15}\psi(q^8)^{15}). \quad (3.2)
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n \geq 0} \bar{p}(16n + 4)q^n &= \frac{1}{\phi(-q)^{16}} (14\phi(q)^{14}\psi(q^2) + 6496q\phi(q)^{10}\psi(q^2)^5 \\
&\quad + 141824q^2\phi(q)^6\psi(q^2)^9 + 204800q^3\phi(q)^2\psi(q^2)^{13}) \quad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n \geq 0} \bar{p}(16n + 8)q^n &= \frac{1}{\phi(-q)^{16}} (100\phi(q)^{13}\psi(q^2)^2 + 17728q\phi(q)^9\psi(q^2)^6 \\
&\quad + 207872q^2\phi(q)^5\psi(q^2)^{10} + 114688q^3\phi(q)\psi(q^2)^{14}). \quad (3.4)
\end{aligned}$$

Modulo 5, (3.3) and (3.4) yield

$$\begin{aligned}
\sum_{n \geq 0} \bar{p}(16n+4)q^n &\equiv \frac{4\phi(q)^{14}\psi(q^2) + q\phi(q)^{10}\psi(q^2)^5 + 4q^2\phi(q)^6\psi(q^2)^9}{\phi(-q)^{16}} \\
&\equiv 4 \frac{\phi(q)^6\psi(q^2)}{\phi(-q)^{16}} (\phi(q)^8 - q\phi(q)^4\psi(q^2)^4 + q^2\psi(q^2)^8) \\
&\equiv 4 \frac{\phi(q)^6\psi(q^2)}{\phi(-q)^{16}} \quad \text{by (2.9)} \\
&= 4 \frac{(q^2; q^2)_{\infty}^{45}}{(q; q)_{\infty}^{45}(q^4; q^4)_{\infty}^{10}} (q; q)_{\infty} \\
&\equiv 4 \frac{(q^{10}; q^{10})_{\infty}^9}{(q^5; q^5)_{\infty}^9 (q^{20}; q^{20})_{\infty}^2} (q; q)_{\infty} \tag{3.5}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n \geq 0} \bar{p}(16n+8)q^n &\equiv \frac{3q\phi(q)^9\psi(q^2)^6 + 2q^2\phi(q)^5\psi(q^2)^{10} + 3q^3\phi(q)\psi(q^2)^{14}}{\phi(-q)^{16}} \\
&\equiv 3q \frac{\phi(q)\psi(q^2)^6}{\phi(-q)^{16}} (\phi(q)^8 - q\phi(q)^4\psi(q^2)^4 + q^2\psi(q^2)^8) \\
&\equiv 3q \frac{\phi(q)\psi(q^2)^6}{\phi(-q)^{16}} \\
&= 3q \frac{(q^2; q^2)_{\infty}^{15}(q^4; q^4)_{\infty}^{10}}{(q; q)_{\infty}^{35}} (q; q)_{\infty} \\
&\equiv 3q \frac{(q^{10}; q^{10})_{\infty}^3 (q^{20}; q^{20})_{\infty}^2}{(q^5; q^5)_{\infty}^7} (q; q)_{\infty}. \tag{3.6}
\end{aligned}$$

It follows from (3.5) and (2.8) that

$$\bar{p}(16(5n+3)+4) = \bar{p}(80n+52) \equiv 0 \pmod{5}$$

and

$$\bar{p}(16(5n+4)+4) = \bar{p}(80n+68) \equiv 0 \pmod{5}$$

and from (3.6) and (2.8) that

$$\bar{p}(16(5n+4)+8) = \bar{p}(80n+72) \equiv 0 \pmod{5}$$

and

$$\bar{p}(16(5n)+8) = \bar{p}(80n+8) \equiv 0 \pmod{5}.$$

These are the desired results. \square

References

- [1] B.C. Berndt, *Number Theory in the Spirit of Ramanujan*, Student Mathematical Library, 34 AMS, Providence Rhode Island, 2000.
- [2] D.Q.J. Dou and B.L.S. Lin, *New Ramanujan type congruences modulo 5 for overpartitions*, Ramanujan Journal, to appear.
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