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Simulation of Higher-Order Electrical Circuits with Stochastic Parameters via SDEs

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Abstract—**The paper deals with a technique for the simulation of higher-order electrical circuits with parameters varying randomly. The principle consists in the utilization of the theory of stochastic differential equations (SDE), namely the vector form of the ordinary SDEs. Random changes of both excitation voltage and some parameters of passive circuit elements are considered, and circuit responses are analyzed. The voltage and/or current responses are computed and represented in the form of the sample means accompanied by their confidence intervals to provide reliable estimates. The method is applied to analyze responses of the circuit models of optional orders, specially those consisting of a cascade connection of the RLGC networks. To develop the model equations the state-variable method is used, afterwards a corresponding vector SDE is formulated and a stochastic Euler numerical method applied. To verify the results the deterministic responses are also computed by the help of the PSpice simulator or the numerical inverse Laplace transforms (NILT) procedure in MATLAB®, while removing random terms from the circuit model.**

Index Terms—**circuit noise, circuit simulation, differential equations, stochastic processes, stochastic systems.**

I. Introduction

Stochastic differential equations (SDE) find their usage in various fields of engineering, wherever random effects are necessary to be considered while modelling real physical systems [1-4]. Such approach lies in replacing parameters in a deterministic model by random processes, and then in utilizing SDEs to characterize the response of such a model. In the field of electrical engineering it can cover a number of random processes occurring in electrical systems, see [5-7] to mention a few of them. Some attention is still paid to the first-order RL and RC circuits, e.g. [8-11], where suitable numerical techniques and different types of the noise have been studied. From practical point of view, however, the second-order RLC and RLGC networks play major role as they can serve as the building blocks of more complex models of physical systems. Some proposals for the SDE simulations related to random voltage excitations of RLGC networks have been discussed in [12], modelling of higher-order RLGC-based circuits in [13]. In [14], the application of the vector SDEs to the RLGC circuits under stochastic voltage excitations or stochastic resistance has been studied.

The current paper generalizes the above mentioned cases towards higher-order circuits, being formed by a cascade connection of a number of RLGC networks (see Fig. 1), under various parameters varying stochastically. Such a circuit structure was chosen because it can be utilized to model interconnects in high-speed electronic systems, i.e. it can represent a lumped-parameter model of transmission lines (TL) when choosing the elements in Fig. 1 properly [15]. Such a circuit model can be described by ordinary differential equations (ODE), and consequently, by ordinary SDEs in case of the model parameters varying randomly. It has its origin in fluctuations of the material properties of interconnects physical implementations [16].

The paper is organized as follows. In chap. II, a circuit math model based on a state-variable method is developed, in case of deterministic parameters. Then, formulae covering the effects of a stochastic drive voltage, series resistances, and shunt conductance are respectively derived in chaps III, IV, and V. Chapter VI deals with SDE numerical techniques applied, chap. VII with statistical estimates of the solutions. Some simulations based on MATLAB® are shown in chap. VII, chap. VIII discusses results and outlines future works.

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II. State–Variable Description

The solution of the electrical circuit in Fig. 1 will first be formulated for deterministic parameters, by using a classical state-variable method [17], resulting in a first-order vector ODE. Afterwards a vector SDE will be determined while replacing respective parameters by their noisy counterparts.

To better explain a solution we consider a two-sectional model, for $m = 2$, leading to (1)

$$\left[\begin{array}{ccc:cc} C_1 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & 0 & L_2 \end{array}\right] \cdot \frac{d}{dt} \left[\begin{array}{c} v_1(t) \\ v_2(t) \\ v_3(t) \\ \hdashline i_1(t) \\ i_2(t) \end{array}\right] = -\left[\begin{array}{ccc:cc} G_1+G_S & 0 & 0 & 1 & 0 \\ 0 & G_2 & 0 & -1 & 1 \\ 0 & 0 & G_3+G_L & 0 & -1 \\ \hdashline -1 & 1 & 0 & R_1 & 0 \\ 0 & -1 & 1 & 0 & R_2 \end{array}\right] \cdot \left[\begin{array}{c} v_1(t) \\ v_2(t) \\ v_3(t) \\ \hdashline i_1(t) \\ i_2(t) \end{array}\right] + \left[\begin{array}{c} G_S v_S(t) \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \end{array}\right] \tag{1}$$

While 2 sections led to 5 state variables, m sections will give $2m+1$ ones. Thus a general description is

$$\mathbf{M}\frac{d\mathbf{x}(t)}{dt} = -\left(\mathbf{H}+\mathbf{P}\right)\mathbf{x}(t) + \mathbf{P}\mathbf{u}(t) \tag{2}$$

with the individual terms as follows. The vector

$$\mathbf{x}(t) = \left[\mathbf{v}_C^{\mathrm{T}}(t), \mathbf{i}_L^{\mathrm{T}}(t)\right]^{\mathrm{T}} \tag{3}$$

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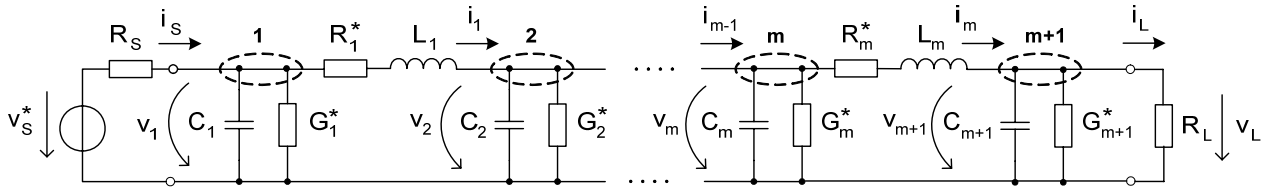


Figure 1. Cascade higher-order electrical circuit.

consists of the unknown state variables, namely subvectors $\mathbf{v}_C(t)$ and $\mathbf{i}_L(t)$ hold m+1 capacitor voltages and m inductor currents, respectively. The matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \tag{4}$$

is composed of diagonal matrices $\mathbf{C} = \mathrm{diag}(C_1,C_2,...,C_{m+1})$, $\mathbf{L} = \mathrm{diag}(L_1,L_2,...,L_m)$, and the zero matrices $\mathbf{0}$. The matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{G} & \mathbf{E} \\ -\mathbf{E}^{\mathrm{T}} & \mathbf{R} \end{bmatrix} \tag{5}$$

can similarly be created by blocks $\mathbf{G} = \mathrm{diag}(G_1,G_2,...,G_{m+1})$, $\mathbf{R} = \mathrm{diag}(R_1,R_2,...,R_m)$, the $(m+1) \times m$ bidiagonal matrix $\mathbf{E}$ containing elements ±1, see (1) for details. The matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{Y}_{\mathrm{E}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{6}$$

depends on external elements and contains an $(m+1)$-order diagonal matrix $\mathbf{Y}_E = \mathrm{diag}(G_S,0,...,0,G_L)$, where $G_S = R_S^{-1}$ and $G_L = R_L^{-1}$. Finally, a column vector

$$\mathbf{u}(t) = \left[\mathbf{v}_{\mathrm{E}}^{\mathrm{T}}(t), \mathbf{0}\right]^{\mathrm{T}} \tag{7}$$

contains the $(m+1)$-order vector $\mathbf{v}_E(t) = [v_S(t),0,...,0]^T$. If we introduce denotations

$$-\mathbf{M}^{-1}\left(\mathbf{H}+\mathbf{P}\right) = \mathbf{A} \ , \quad \mathbf{M}^{-1}\mathbf{P} = \mathbf{B} \tag{8}$$

the equation (2) results in the form

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{9}$$

with a solution

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}\cdot(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \ . \tag{10}$$

III. Stochastic Excitation Voltage

Here the Itô stochastic calculus [1] is considered to get responses to an input source influenced by random effects. Instead of $v_S(t)$ a non-deterministic version is used as

$$v_{\mathrm{S}}^{*}(t) = v_{\mathrm{S}}(t) + \alpha\xi(t) \tag{11}$$

with a noise term $\xi(t)$, being reasonably considered as a stochastic process called a "white noise process", and with α as a constant expressing its intensity. Then, taking into account (7) and substituting (11), (9) results in

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\begin{bmatrix} \mathbf{v}_{\mathrm{E}}(t) \\ \mathbf{0} \end{bmatrix} + \mathbf{B}\begin{bmatrix} \boldsymbol{\alpha}_{\mathrm{E}} \\ \mathbf{0} \end{bmatrix}\xi(t) \tag{12}$$

$\boldsymbol{\alpha}_E = [\alpha,0,\ldots,0]^T$ as an $(m+1)$-order noise intensity vector.

Equation (12) is multiplied by dt and then $\xi(t)dt$ replaced by $dW(t)$, where $W(t)$ is so-called Wiener process [1-3]. By this we get a vector linear stochastic differential equation with an additive noise

$$d\mathbf{X}(t) = \left(\mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{u}(t)\right)dt + \mathbf{b}dW(t) \tag{13}$$

where $\mathbf{b} = \mathbf{B}\cdot[\boldsymbol{\alpha}_E^{\mathrm{T}},\mathbf{0}]^T$. Note that the stochastic solution $\mathbf{X}(t)$ in (13) is marked by a capital letter to distinguish it from a deterministic solution (10). Considering a multidimensional Itô formula its formal analytical form is

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{b}dW(\tau) \tag{14}$$

with the right term called as a Itô integral. The $\mathbf{X}(t)$ is a random process and for its expectation we have for $t > t_0$

$$E\left[\mathbf{X}(t)\right] = e^{\mathbf{A}t}E\left[\mathbf{X}(t_0)\right] + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{15}$$

while the expectation of the Itô integral is zero. It is evident that for a constant initial value $\mathbf{X}(t_0)$, the expectation of the stochastic solution agrees to the deterministic solution (10).

IV. Stochastic Series Resistances

In this section the responses to a deterministic excitation voltage but under series resistances influenced by random effects are considered. Instead of R_k the non-deterministic versions are used

$$R_k^{*}(t) = R_k + \alpha_k\xi_k(t) \tag{16}$$

$k = 1,2,\ldots,m$, with noise terms $\xi_k(t)$, which are considered as the white noise processes, and with α_k constants as their intensities. Now taking into account (5) and (8), and substituting (16), (9) can be arranged into a form

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) - \mathbf{M}^{-1}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha}_{\mathrm{R}}\boldsymbol{\xi}_{\mathrm{R}}(t) \end{bmatrix}\mathbf{x}(t) \tag{17}$$

with $\boldsymbol{\alpha}_R = \mathrm{diag}(\alpha_1,\alpha_2,...,\alpha_m)$ and $\boldsymbol{\xi}_R = \mathrm{diag}(\xi_1(t),\xi_2(t),...,\xi_m(t))$. The last equation is multiplied by dt and then $\xi_k(t)dt$ replaced by $dW_k(t)$, where $W_k(t)$ mark respective Wiener processes. If we define, based on these operations, a matrix

$$\mathbf{D}_{\mathrm{W}}(t) = -\mathbf{M}^{-1}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha}_{\mathrm{R}}d\mathbf{W}_{\mathrm{R}}(t) \end{bmatrix} \tag{18}$$

with $d\mathbf{W}_R(t) = \mathrm{diag}(dW_1(t),dW_2(t),...,dW_m(t))$, then a vector linear SDE with a multiplicative noise can be formulated as

$$d\mathbf{X}(t) = \left(\mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{u}(t)\right)dt + \mathbf{D}_{\mathrm{W}}(t)\mathbf{X}(t) \ , \tag{19}$$

with the expectation given again by (15). The right part of (19) is a linear SDE with several multiplicative noises. Its

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exact analytical solution can be got by generalizing results shown in [14].

V. Stochastic Shunt Conductance

Similarly, responses to a deterministic excitation voltage but under shunt conductance influenced by random effects can be considered. Herein, instead of the original G_k, their non-deterministic versions are used

$$G_k^*(t) = G_k + \alpha_k \xi_k(t) \tag{20}$$

$k = 1,2,\ldots,m+1$, with noise terms $\xi_k(t)$ considered as white noise processes again, and with α_k as their intensities. Taking into account (5) and (8), and substituting (20), (9) can be arranged into a form

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) - \mathbf{M}^{-1}\begin{bmatrix} \boldsymbol{\alpha}_{\mathrm{G}}\boldsymbol{\xi}_{\mathrm{G}}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{x}(t) \tag{21}$$

$\boldsymbol{\alpha}_{\mathrm{G}} = \mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_{m+1})$, and $\boldsymbol{\xi}_{\mathrm{G}} = \mathrm{diag}(\xi_1(t),\xi_2(t),\ldots,\xi_{m+1}(t))$. Equation (21) is multiplied by dt and then $\xi_k(t)dt$ replaced by $dW_k(t)$, with $W_k(t)$ as corresponding Wiener processes. If we then define a matrix

$$\mathbf{D}_{\mathrm{W}}(t) = -\mathbf{M}^{-1}\begin{bmatrix} \boldsymbol{\alpha}_{\mathrm{G}} d\mathbf{W}_{\mathrm{G}}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{22}$$

with $d\mathbf{W}_{\mathrm{G}}(t) = \mathrm{diag}(dW_1(t),dW_2(t),\ldots,dW_{m+1}(t))$, formally the same form of the vector linear SDE with a multiplicative noise is obtained as in (19).

Note that in case of necessity to consider simultaneously both stochastic series resistances and shunt conductances the matrix $\mathbf{D}_{\mathrm{w}}(t)$ in (19) would have a form

$$\mathbf{D}_{\mathrm{W}}(t) = -\mathbf{M}^{-1}\begin{bmatrix} \boldsymbol{\alpha}_{\mathrm{G}} d\mathbf{W}_{\mathrm{G}}(t) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha}_{\mathrm{R}} d\mathbf{W}_{\mathrm{R}}(t) \end{bmatrix} \tag{23}$$

with properties mentioned above. By selecting components of the noise intensity vectors $\boldsymbol{\alpha}_{\mathrm{R}}$ and $\boldsymbol{\alpha}_{\mathrm{G}}$ the noise behaviour of the circuit model is controlled.

VI. SDE Numerical Techniques

While implementing a numerical method from the theory of ODEs, some care is required because according to the definition of the Itô integral, the stochastic integral must always be evaluated at lower endpoint of the discretization subinterval. To numerically solve our SDEs we utilize stochastic Euler schemes [3] being consistent with the Itô stochastic calculus. Let us consider solutions $\mathbf{X}(t)$ on the interval $t_0 \le t \le T$, in conjunction with an equidistant discretisation $t_n = t_0 + nh$, with $h = (T - t_0)/N = t_{n+1} - t_n$, for $n = 0,\ldots,N\text{-}1$, and with the corresponding discretization of the Wiener processes $\Delta W^n = W(t_{n+1}) - W(t_n) = \int_{t_n}^{t_{n+1}} dW(\tau)$. To apply any stochastic numerical scheme, first we have to generate the random increments of the Wiener processes as independent Gauss random variables, with $E[\Delta W^n] = 0$ and $E[(\Delta W^n)^2] = h$. Then, based on [3], generalized stochastic Euler schemes can be formulated as

$$\begin{aligned} \mathbf{X}^{n+1} &= \left[\mathbf{I} - \gamma\mathbf{A}h\right]^{-1} \times \\ &\left[\mathbf{X}^n + \gamma\mathbf{B}\mathbf{u}^{n+1}h + (1-\gamma)\left(\mathbf{A}\mathbf{X}^n + \mathbf{B}\mathbf{u}^n\right)h + \mathbf{b}\Delta W^n\right] \end{aligned} \tag{24}$$

for the SDE with an additive noise (13), and

$$\begin{aligned} \mathbf{X}^{n+1} &= \left[\mathbf{I} - \gamma\mathbf{A}h\right]^{-1} \times \\ &\left[\mathbf{X}^n + \gamma\mathbf{B}\mathbf{u}^{n+1}h + (1-\gamma)\left(\mathbf{A}\mathbf{X}^n + \mathbf{B}\mathbf{u}^n\right)h + \mathbf{D}_{\mathrm{w}}^n\mathbf{X}^n\right] \end{aligned} \tag{25}$$

for the SDE with the multiplicative noises (19), with $\mathbf{I}$ as an identity matrix, being both consistent with the Itô stochastic calculus. In the above formulae $\gamma \in \langle 0;1 \rangle$, and can be used to determine a degree of implicitness. Thus for $\gamma = 0$ we get explicit Euler schemes whilst for $\gamma = 1$ we get fully implicit ones. Another special case, namely $\gamma = ½$, gives generalized trapeizoidal schemes.

As computer simulations revealed, the explicit Euler schemes satisfy only for relatively low orders of the model in Fig. 1 to obtain stable solutions. Hereafter implicit Euler schemes are therefore used for practical simulations.

VII. SDE Solution Statistical Estimates

An important goal is to approximate an expectation $\mu = E[X]$ and a standard deviation $\sigma = \sqrt{E[(X-\mu)^2]}$ of SDE solutions, and to determine respective confidence intervals to get statistically reliable results. In practice it can be estimated by sample statistics based on a finite number of realizations [18]. In case of the additive noise (section III) a resultant stochastic process $\mathbf{X}(t)$ holds a normal distribution, $N(\mu,\sigma^2)$ [9]. That is why a common procedure to determine confidence intervals can be used, based on the student t distribution. To simplify notation we explain it for a scalar stochastic process $X(t)$, or one item of a vector $\mathbf{X}(t)$. Then, denoting $X^n = X(t_n)$, the sample mean is gained by

$$\bar{X}^n = \frac{1}{K}\sum_{k=1}^{K} X_k^n \tag{26}$$

and the sample standard deviation by

$$s^n = \sqrt{\frac{1}{K-1}\sum_{k=1}^{K}\left(X_k^n - \bar{X}^n\right)^2} \tag{27}$$

with K as the number of all realizations, and $n = 0,\ldots,N\text{-}1$. Then, based on a student-t distribution with $K - 1$ degrees of freedom, a two-sided $100\cdot(1-\alpha)\%$ confidence interval can be determined as

$$\left(\bar{X}^n - t_{1-\alpha/2,K-1}\frac{s^n}{\sqrt{K}},\ \bar{X}^n + t_{1-\alpha/2,K-1}\frac{s^n}{\sqrt{K}}\right) \tag{28}$$

In case of the multiplicative noise (sections IV and V), however, a presumption about a normal distribution of the resultant stochastic process $\mathbf{X}(t)$ is not fulfilled any more, an application of the above procedure would not be correct. As is proposed e.g. in [3] this can be resolved based on a central limit theorem as follows. All the K realizations are grouped into L batches containing M random processes each, i.e. $LM = K$, and particular sample means are computed as

$$\bar{X}_l^n = \frac{1}{M}\sum_{m=1}^{M} X_{l,m}^n \tag{29}$$

for $l = 1,2,\ldots,L$. In compliance with a central limit theorem such the sample means can roughly be considered as normal distributed, for batch size $M \ge 15$ at least [3], and a student-t

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distribution can again be used to state confidence intervals. Finally, we compute resultant sample statistics

$$\bar{X}^n = \frac{1}{L}\sum_{l=1}^{L}\bar{X}_l^n = \frac{1}{LM}\sum_{l=1}^{L}\sum_{m=1}^{M}X_{l,m}^n \tag{30}$$

$$\tilde{s}^n = \sqrt{\frac{1}{L-1}\sum_{l=1}^{L}\left(\bar{X}_l^n - \bar{X}^n\right)^2} \tag{31}$$

and the confidence interval is calculated by

$$\left(\bar{X}^n - t_{1-\alpha/2,L-1}\frac{\tilde{s}^n}{\sqrt{L}},\ \bar{X}^n + t_{1-\alpha/2,L-1}\frac{\tilde{s}^n}{\sqrt{L}}\right) \tag{32}$$

We can see, the sample mean (30) equals that in (26), but confidence interval (32) generally differs from (28).

VIII. Computer Simulation Examples

The cascade higher-order electrical circuit in Fig. 1 can play a role of a lumped-parameter model of a generally loss nonuniform TL, if right choosing its parameters [15]. As an example, let us consider an uniform TL of a length $l = 0.4$ m, and with per-unit-length parameters $R_0 = 0.1$ Ω/m, $L_0 = 494.6$ nH/m, $G_0 = 0.1$ S/m, and $C_0 = 62.8$ pF/m [19]. The TL is divided on $m = 128$ parts, parameters of its model are $R_k = R_0 l/m$, $L_k = L_0 l/m$, $G_k = G_0 l/m$, and $C_k = C_0 l/m$, $\forall k$. The exceptions are for shunt conductance and capacitances at the model boundaries, i.e. G_1, G_{m+1}, and C_1, C_{m+1}, that are considered as halfway to ensure the TL model is formed by a cascade connection of m identical Π networks. The terminal resistances are $R_S = 50$ Ω and $R_L = 500$ Ω, the drive voltage source is of the unit-step waveform, $v_S(t) = \underline{1}(t)$.

To verify the results, deterministic solutions for voltages and currents on the TL are depicted in Fig. 2, showing their overall distributions along the wire.

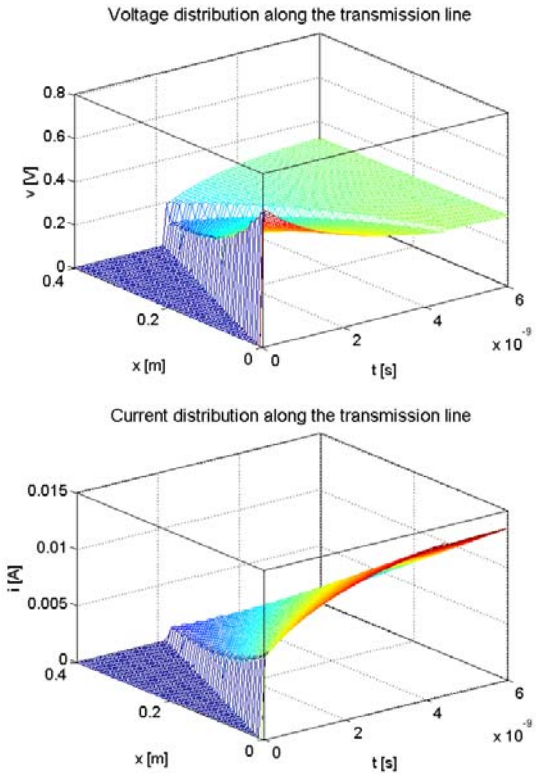


Figure 2. Voltage and current distributions along the TL.

They have been computed via a continuous Laplace-domain model [20] to ensure higher accuracy, and then processed by a numerical inverse Laplace transforms (NILT) procedure in Matlab [21]. Selected waveforms for the beginning ($x = 0$) and the end ($x = l$) of the TL are shown in Fig. 3 which will be compared to the sample means of stochastic solutions.

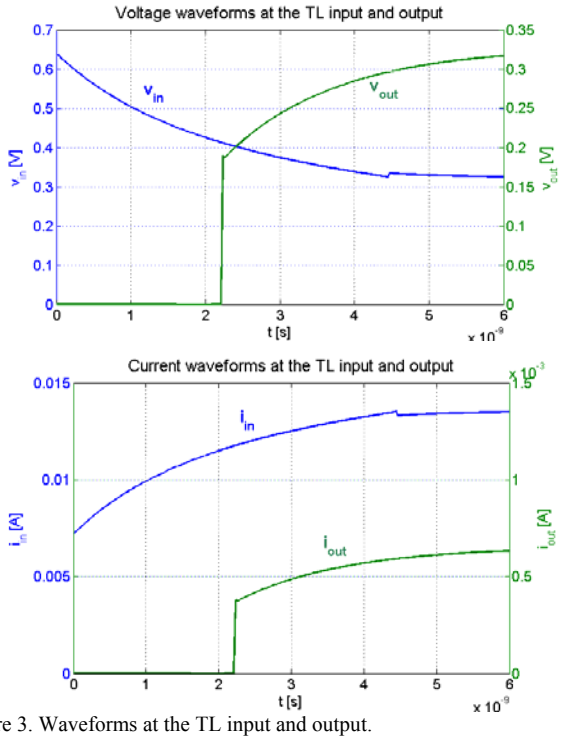


Figure 3. Waveforms at the TL input and output.

We consider a stochastic excitation voltage (11), with a noise intensity $\alpha = 10^{-6}$, while taking TL model parameters deterministic. Some simulated results are shown in Fig. 4.

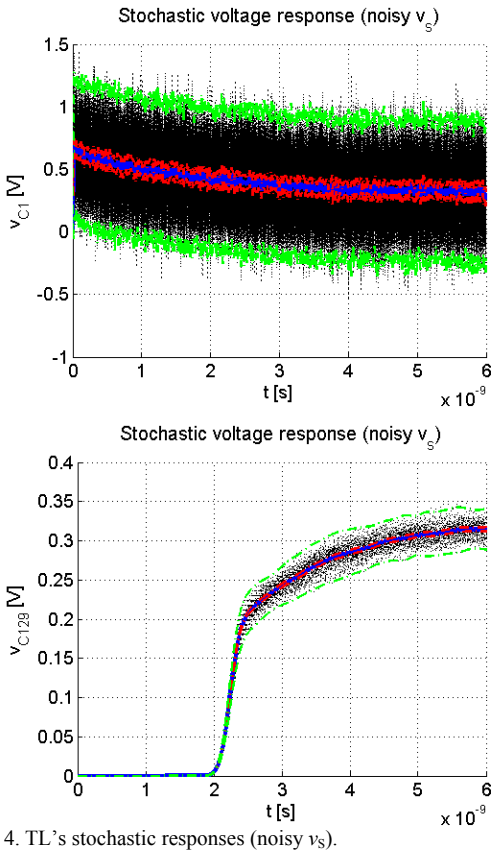


Figure 4. TL's stochastic responses (noisy v_S).

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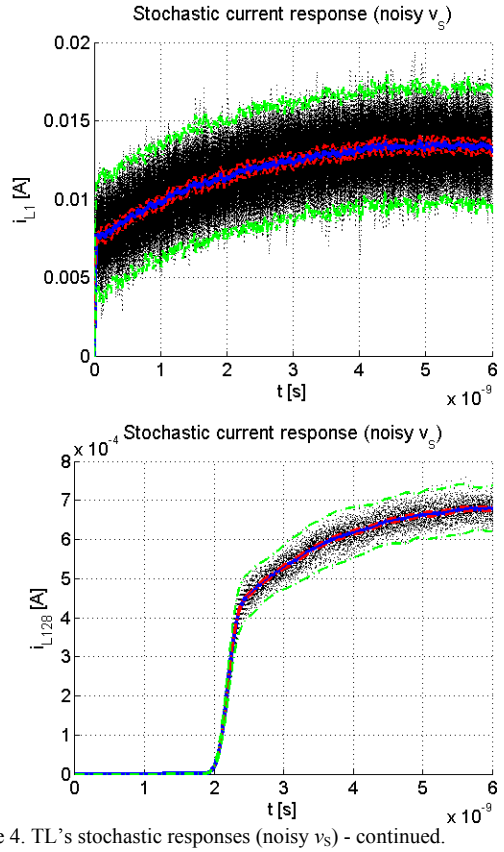


Figure 4. TL's stochastic responses (noisy v_S) - continued.

The sample means (solid blue lines) are supplemented by their 99% confidence intervals (dashed red lines), based on (26)-(28), as in this case the resultant stochastic processes remain Gaussian (a case with an additive noise). To evaluate overall dispersion the 99% confidence intervals are also calculated for the individual stochastic realizations (dashed green lines), from the set of $K = 100$ samples.

We now consider stochastic series resistances (16), with a noise intensities $\alpha_k = 10^{-5}$, $k = 1,2,\ldots,m$, under the excitation voltage and the remaining TL parameters deterministic. The simulated results are depicted in Fig. 5. In this case a whole number of realizations was $K = 225$ to enable dividing them into at least $L = 15$ batches with $M = 15$ samples each, for applying (29)-(32) error-free. As was already remarked, the resultant stochastic processes are not Gaussian, and (28) would not lead to correct estimates. The dashed green lines express again confidence intervals for individual samples, but due to the non-Gaussian distribution, a 99% reliability does not express a safe value.

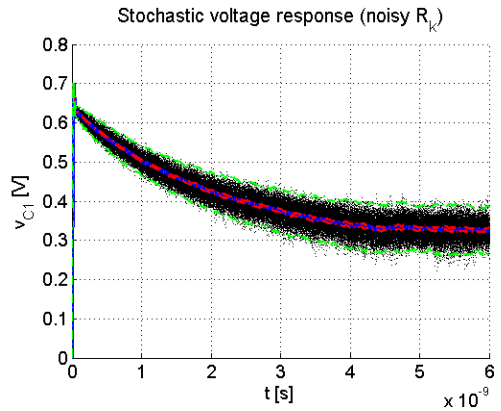


Figure 5. TL's stochastic responses (noisy R_k).

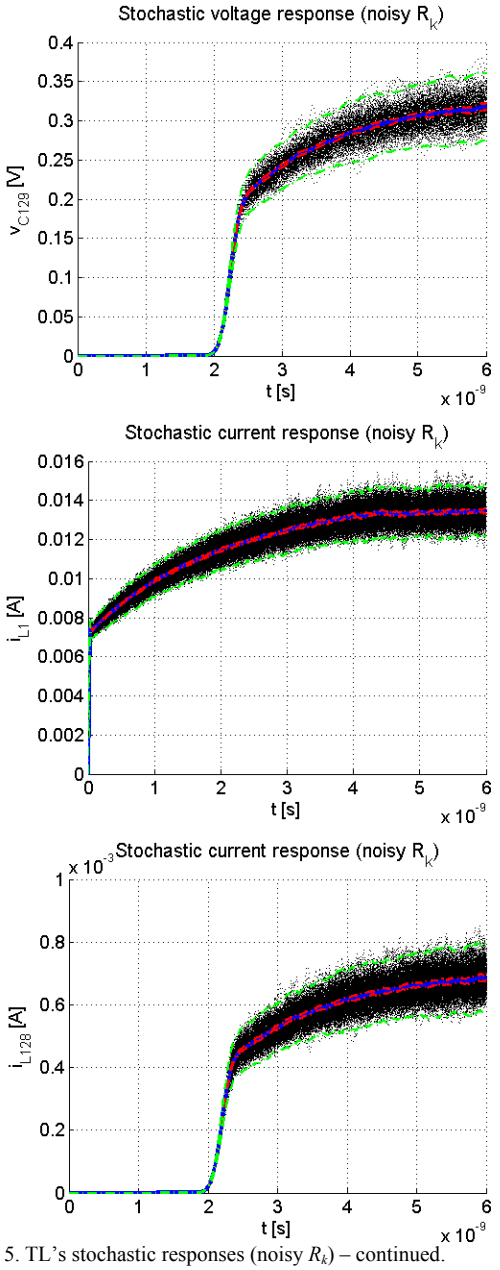


Figure 5. TL's stochastic responses (noisy R_k) – continued.

Finally, we consider stochastic shunt conductances (20), with the noise intensities $\alpha_k = 10^{-9}$, $k = 1,2,\ldots,m+1$, under all remaining TL's model parameters deterministic, see Fig. 6. It should be noticed that stochastic processes $dW_k(t)$, needed for all numerical solutions, are generated by the MATLAB® function for normally distributed random numbers.

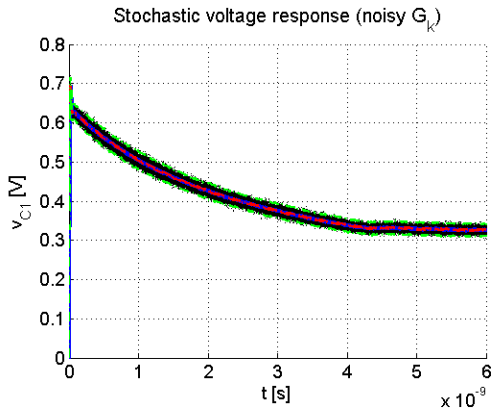


Figure 6. TL's stochastic responses (noisy G_k).

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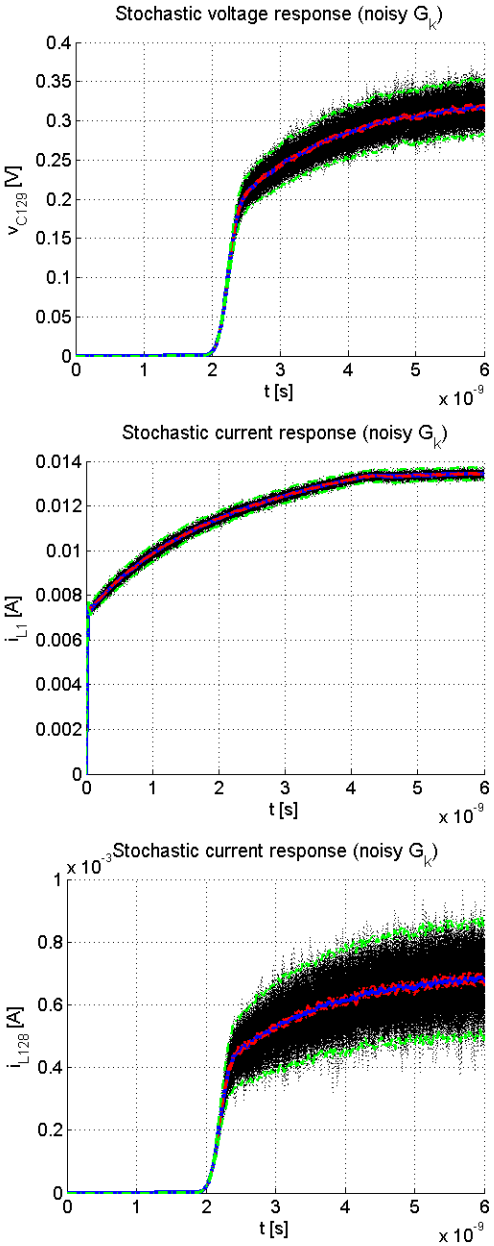


Figure 6. TL's stochastic responses (noisy G_k) - continued.

IX. Conclusion

It follows from the simulation results in Fig.s 4-6 that the sample means well coincide with deterministic solutions in Fig. 3. It is also interesting to observe that rather distant noise intensities enter into the computation for comparable dispersions of stochastic processes that can help to evaluate componets of the TL model with significant noise effects. The cascade lumped-parameter TL models can find their importance at the interconnects simulations, mostly in the high-speed electronic systems. For transmission paths on semiconductor substrates, e.g. small random changes of their parameters occur due to a number of various physical effects [15], [16]. Therefore the SDEs approach has its real basis and can be interesting alternative or extension to other probabilistic techniques introduced in a circuits theory, e.g. [22], [23]. Further research will mainly be focused on others stochastic model parameters and their simultaneous actions, and on generalization towards multiconductor transmission lines (MTL) [24], [25]. Besides, in our future works, we also purpose to include some measuring test on TLs specimens.

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