NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 42 (2012), 17-26

WEIGHTED SHARP FUNCTION ESTIMATE AND BOUNDEDNESS FOR COMMUTATOR ASSOCIATED WITH SINGULAR INTEGRAL OPERATOR WITH GENERAL KERNEL

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(Received 26 February, 2011)

Abstract. In this paper, we establish a weighted sharp maximal function estimate for the commutator associated with the singular integral operator with general kernel. As an application, we obtain the weighted boundedness of the commutators on Lebesgue and Morrey spaces.

1. Introduction and Results

As the development of singular integral operators (see [10][23]), their commutators have been well studied. In [6][21][22], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [3]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [12][18], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1][11], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In this paper, we will study some singular integral operators as following(see [2][14]).

Definition 1. Let $T : S \to S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y)f(y)dy$$

2010 *Mathematics Subject Classification* 42B20, 42B25.
Key words and phrases: Commutator, Singular integral operator, Sharp maximal function, Morrey space.

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for every bounded and compactly supported function f, where K satisfies: there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\int_{2|y-z|<|x-y|} (|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|)dx \leq C,$$

and

$$\left(\int_{2^k|z-y|\leq|x-y|<2^{k+1}|z-y|} (|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|)^q dy\right)^{1/q} \leq C_k(2^k|z-y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Let b be a locally integrable function on R^n. The commutator related to T is defined by

$$T_b(f)(x) = \int_{R^n} (b(x) - b(y))K(x,y)f(y)dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1 with $C_j = 2^{-j\delta}$(see [6][10][23]).

Definition 2. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for} \quad t \geq 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n. Set, for $1 \leq p < \infty$,

$$||f||_{L^{p,\varphi}(w)} = \sup_{x\in R^n,\ d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p w(y)dy\right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized weighted Morrey space is defined by

$$L^{p,\varphi}(R^n, w) = \{f \in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\delta}(R^n, w)$, which is the classical Morrey spaces (see [19][20]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, which is the weighted Lebesgue spaces (see [10]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [4][7][8][15][16][19]).

It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [21-22]). In [22], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The main purpose of this paper is to prove a weighted sharp inequality for the commutator.

As the application, we obtain the weighted L^p-norm inequality and Morrey spaces boundedness for the commutator.

Now, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

where, and in what follows, $f_Q=|Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [10][23])

$$M^{\#}(f)(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-c|dy.$$

Let

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy.$$

For $\eta>0$, let $M_\eta^{\#}(f)(x)=M^{\#}(|f|^\eta)^{1/\eta}(x)$ and $M_\eta(f)(x)=M(|f|^\eta)^{1/\eta}(x)$.

For $0<\eta<n$, $1\le r<\infty$ and the non-negative weight function w, set

$$M_{\eta,w,r}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{w(Q)^{1-r\eta/n}}\int_Q|f(y)|^r w(y)dy\right)^{1/r}.$$

The A_p weight is defined by (see [10])

$$A_p=\left\{w\in L^1_{loc}(R^n):\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}<\infty\right\},$$

$1<p<\infty$,

and

$$A_1=\{w\in L^p_{loc}(R^n):M(w)(x)\le Cw(x),a.e.\}.$$

The $A(p,r)$ weight is defined by (see [17]), for $1<p,r<\infty$,

$$A(p,r)=$$
$$\left\{w>0:\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)^r dx\right)^{1/r}\left(\frac{1}{|Q|}\int_Q w(x)^{-p/(p-1)}dx\right)^{(p-1)/p}<\infty\right\}.$$

Given a weight function w. For $1\le p<\infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$||f||_{L^p(w)}=\left(\int_{R^n}|f(x)|^p w(x)dx\right)^{1/p}<\infty.$$

For the non-negative weight function w and $0 < \beta < 1$, the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions b such that

$$||b||_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |b(y) - b_Q| dy < \infty.$$

Remark. (1). It has been known that, for $b \in Lip_\beta(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^k Q}| \leq Ck||b||_{Lip_\beta(w)} w(x) w(2^k Q)^{\beta/n}.$$

(2). Let $b \in Lip_\beta(w)$ and $w \in A_1$. By [9], we know that spaces $Lip_\beta(w)$ coincide and the norms $||b||_{Lip_\beta(w)}$ are equivalent with respect to different values $1 \leq p \leq \infty$(see [9][11]).

We shall prove the following theorems.

Theorem 1. Let T be the singular integral operator as Definition 1, the sequence $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \beta < 1$, $0 < \eta < 1$, $q' < s < \infty$ and $b \in Lip_\beta(w)$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M_\eta^\#(T_b(f))(\tilde{x}) \leq C||b||_{Lip_\beta(w)} w(\tilde{x}) \left(M_{\beta,w,s}(f)(\tilde{x}) + M_{\beta,w,s}(T(f))(\tilde{x})\right).$$

Theorem 2. Let T be the singular integral operator as Definition 1, the sequence $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$ and $b \in Lip_\beta(w)$. Then T_b is bounded from $L^p(w)$ to $L^r(w^{1-r})$.

Theorem 3. Let $w \in A_1$, $0 < \beta < \min(1, n/q')$, $0 < D < 2^n$, $1 < p < n/\beta$, $1/r = 1/p - \beta/n$, $L^{p,\varphi}(R^n, w)$ be the weighted Morrey space as Definition 2, T be the singular integral operator as Definition 1, the sequence $\{kC_k\} \in l^1$ and $b \in Lip_\beta(w)$. Then T_b is bounded from $L^{p,\varphi}(w)$ to $L^{r,\varphi}(w^{1-r})$.

2. Proof of Theorem

To prove the theorems, we need the following lemma.

Lemma 1.([2]) Let T be the singular integral operator as Definition 1. Then T is bounded on $L^p(w)$ for $w \in A_p$ with $1 < p < \infty$, and weak (L^1, L^1) bounded.

Lemma 2.(see [24]) Let $0 < p, \eta < \infty$ and $w \in \cup_{1 \leq r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,

$$\int_{R^n} M_\eta(f)(x)^p w(x) dx \leq C \int_{R^n} M_\eta^\#(f)(x)^p w(x) dx.$$

Lemma 3.(see [11][17]) Suppose that $1 \leq s < p < n/\eta$, $1/r = 1/p - \eta/n$ and $w \in A(p, r)$. Then

$$||M_{\eta,w,s}(f)||_{L^r(w)} \leq C||f||_{L^p(w)}.$$

Lemma 4.(see [9][11]) For any cube Q, $b \in Lip_\beta(w)$, $0 < \beta < 1$ and $w \in A_1$, we have

$$\sup_{x\in Q} |b(x) - b_Q| \le C||b||_{Lip_\beta(w)} w(Q)^{1+\beta/n}|Q|^{-1}.$$

Lemma 5.([10]) If $w \in A_p$, then $w\chi_Q \in A_p$ for $1 \le p < \infty$ and any cube Q.

Lemma 6. Let $1 < q < \infty$, $0 < \eta < \infty$, $0 < D < 2^n$, $w \in A_1$ and $L^{r,\varphi}(R^n, w)$ be the weighted Morrey space as Definition 2. Then, for any smooth function f for which the left-hand side is finite,

$$||M_\eta(f)||_{L^{r,\varphi}(w^{1-r})} \le C||M_\eta^\#(f)||_{L^{r,\varphi}(w^{1-r})}.$$

Proof. Notice that $w^{1-r} \in A_1$ and $w^{1-r}\chi_Q \in A_1$ for any cube $Q = Q(x, d)$ by [5] and Lemma 4, thus, for $f \in L^{r,\varphi}(R^n, w^{1-r})$ and any cube Q, we have, by Lemma 2,

$$\begin{aligned} \int_Q M_\eta(f)(x)^r w^{1-r}(x)dx &= \int_{R^n} M_\eta(f)(x)^r w^{1-r}(x)\chi_Q(x)dx \\ \le\ & C\int_{R^n} M_\eta^\#(f)(x)^r w^{1-r}(x)\chi_Q(x)dx \\ =\ & C\int_Q M_\eta^\#(f)(x)^r w^{1-r}(x)dx, \end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)}\int_{Q(x,d)} M_\eta(f)(x)^r w^{1-r}(x)dx\right)^{1/r} \le C\left(\frac{1}{\varphi(d)}\int_{Q(x,d)} M_\eta^\#(f)(x)^r w^{1-r}(x)dx\right)^{1/r}$$

and

$$||M_\eta(f)||_{L^{r,\varphi}(w^{1-r})} \le C||M_\eta^\#(f)||_{L^{r,\varphi}(w^{1-r})}.$$

This finishes the proof.

Lemma 7. Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$, T be the singular integral operator as Definition 1 and $L^{p,\varphi}(R^n, w)$ be the weighted Morrey space as Definition 2. Then

$$||T(f)||_{L^{p,\varphi}(w)} \le C||f||_{L^{p,\varphi}(w)}.$$

Lemma 8. Let $0 < D < 2^n$, $1 \le s < p < n/\eta$, $1/r = 1/p - \eta/n$, $w \in A(p, r)$ and $L^{p,\varphi}(R^n, w)$ be the weighted Morrey space as Definition 2. Then

$$||M_{\eta,w,s}(f)||_{L^{r,\varphi}(w)} \le C||f||_{L^{p,\varphi}(w)}.$$

The proofs of two lemmas are similar to that of Lemma 6 by Lemma 1 and 3, we omit the details.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\begin{aligned} & \left(\frac{1}{|Q|}\int_Q ||T_b(f)(x)|^\eta - |T((b_{2Q}-b)f_2)(x_0)|^\eta|\, dx\right)^{1/\eta} \\ \leq \; & C||b||_{Lip_\beta(w)} w(\tilde{x}) \left(M_{\beta,w,s}(f)(\tilde{x}) + M_{\beta,w,s}(T(f))(\tilde{x})\right). \end{aligned}$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$T_b(f)(x) = (b(x) - b_{2Q})T(f)(x) - T((b - b_{2Q})f_1)(x) - T((b - b_{2Q})f_2)(x).$$

Then

$$\begin{aligned} & |T_b(f)(x) - T((b_{2Q}-b)f_2)(x_0)| \leq |T_b(f)(x) - T((b_{2Q}-b)f_2)(x_0)| \\ \leq \; & |(b(x) - b_{2Q})T(f)(x)| + |T((b - b_{2Q})f_1)(x)| \\ & \qquad + |T((b - b_{2Q})f_2)(x) - T((b - b_{2Q})f_2)(x_0)| \\ = \; & A(x) + B(x) + C(x) \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{1}{|Q|}\int_Q ||T_b(f)(x)|^\eta - |T((b_{2Q}-b)f_2)(x_0)|^\eta|\, dx\right)^{1/\eta} \\ \leq \; & \left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - T((b_{2Q}-b)f_2)(x_0)|^\eta dx\right)^{1/\eta} \\ \leq \; & C\left(\frac{1}{|Q|}\int_Q |(b(x)-b_{2Q})T(f)(x)|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q |T((b-b_{2Q})f_1)(x)|^\eta dx\right)^{1/\eta} \\ & + C\left(\frac{1}{|Q|}\int_Q |T((b-b_{2Q})f_2)(x) - T((b-b_{2Q})f_2)(x_0)|^\eta dx\right)^{1/\eta} \\ = \; & I_1 + I_2 + I_3. \end{aligned}$$

For I_1, by Hölder's inequality, we obtain

$$\begin{aligned} I_1 \leq \; & \frac{C}{|Q|}\int_Q |(b(x)-b_{2Q})|w(x)^{-1/s}|T(f)(x)|w(x)^{1/s}dx \\ \leq \; & C\left(\frac{1}{|2Q|}\int_{2Q} |b(x)-b_{2Q}|^{s'}w(x)^{1-s'}dx\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^s w(x)dx\right)^{1/s} \\ \leq \; & C||b||_{Lip_\beta(w)}\frac{w(Q)}{|Q|}M_{\beta,w,s}(T(f))(\tilde{x}) \\ \leq \; & C||b||_{Lip_\beta(w)}w(\tilde{x})M_{\beta,w,s}(T(f))(\tilde{x}). \end{aligned}$$

For I_2, by the weak (L^1, L^1) boundedness of T and Kolmogoro's inequality, we get, similar to the proof of I_1,

$$I_2 \leq \frac{C}{|Q|}\int_{2Q}|(b(x)-b_{2Q})f(x)|dx \leq C||b||_{Lip_\beta(w)}w(\tilde{x})M_{\beta,w,s}(f)(\tilde{x}).$$

For I_3, we have

$$\begin{aligned}
I_3 &\leq \frac{C}{|Q|}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}||f(y)||K(x,y)-K(x_0,y)|dydx\\
&\leq \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\int_{2^kd\leq|y-x_0|<2^{k+1}d}|K(x,y)-K(x_0,y)||b(y)-b_{2^{k+1}Q}||f(y)|dydx\\
&\quad+\frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\int_{2^kd\leq|y-x_0|<2^{k+1}d}|K(x,y)-K(x_0,y)||b_{2^{k+1}Q}-b_{2Q}||f(y)|dydx\\
&= I_3^{(1)}+I_3^{(2)}.
\end{aligned}$$

For $I_3^{(1)}$, recalling that $s > q'$ and noting that $w \in A_1 \subset A_p$ for any $p > 1$, choose $u > 1$ such that $1/q+1/s+1/u = 1$, then by Hölder's inequality and the condition of A_p, we obtain

$$\begin{aligned}
I_3^{(1)} &\leq \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\sup_{y\in 2^{k+1}Q}|b(y)-b_{2^{k+1}Q}|\int_{2^kd\leq|y-x_0|<2^{k+1}d}|K(x,y)-K(x_0,y)|\\
&\quad\times|f(y)|w(y)^{1/s}w(y)^{-1/s}dydx\\
&\leq \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\sup_{y\in 2^{k+1}Q}|b(y)-b_{2^{k+1}Q}|\left(\int_{2^kd\leq|y-x_0|<2^{k+1}d}|K(x,y)-\right.\\
&\quad\left.K(x_0,y)|^qdy\right)^{1/q}\left(\int_{2^{k+1}Q}|f(y)|^sw(y)dy\right)^{1/s}\left(\int_{2^{k+1}Q}w(y)^{-u/s}dy\right)^{1/u}dx\\
&\leq C\sum_{k=1}^\infty k||b||_{Lip_\beta(w)}w(\tilde{x})w(2^{k+1}Q)^{\beta/n}C_k(2^kd)^{-n/q'}w(2^{k+1}Q)^{1/s-\beta/n}\\
&\quad\times M_{\beta,w,s}(f)(\tilde{x})\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}w(y)dy\right)^{1/s}\\
&\quad\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}w(y)^{-u/s}dy\right)^{1/u}\frac{|2^{k+1}Q|^{1/s+1/u}}{w(2^{k+1}Q)^{1/s}}\\
&\leq C||b||_{Lip_\beta(w)}M_{\beta,w,s}(f)(\tilde{x})\sum_{k=1}^\infty C_k\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\\
&\leq C||b||_{Lip_\beta(w)}M_{\beta,w,s}(f)(\tilde{x})w(\tilde{x})\sum_{k=1}^\infty C_k\\
&\leq C||b||_{Lip_\beta(w)}w(\tilde{x})M_{\beta,w,s}(f)(\tilde{x}).
\end{aligned}$$

For $I_3^{(2)}$, similar to the proof of $I_3^{(1)}$, choose $v > 1$ such that $1/q + 1/s + 1/v = 1$, by Hölder's inequality and the condition of A_p, we get

$$
\begin{aligned}
I_3^{(2)} &\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty} |b_{2^{k+1}Q} - b_{2Q}| \int_{2^k d\leq |y-x_0|<2^{k+1}d} |K(x,y)-K(x_0,y)| \\
&\quad \times |f(y)|w(y)^{1/s}w(y)^{-1/s}dydx \\
&\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty} |b_{2^{k+1}Q} - b_{2Q}| \left(\int_{2^k d\leq |y-x_0|<2^{k+1}d} |K(x,y)-K(x_0,y)|^q dy\right)^{1/q} \\
&\quad \times \left(\int_{2^{k+1}Q} |f(y)|^s w(y)dy\right)^{1/s} \left(\int_{2^{k+1}Q} w(y)^{-v/s}dy\right)^{1/v} dx \\
&\leq C\sum_{k=1}^{\infty} k||b||_{Lip_\beta(w)} w(\tilde{x}) w(2^{k+1}Q)^{\beta/n} C_k (2^k d)^{-n/q'} w(2^{k+1}Q)^{1/s-\beta/n} \\
&\quad \times M_{\beta,w,s}(f)(\tilde{x}) \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q} w(y)dy\right)^{1/s} \\
&\quad \times \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q} w(y)^{-v/s}dy\right)^{1/v} \frac{|2^{k+1}Q|^{1/s+1/v}}{w(2^{k+1}Q)^{1/s}} \\
&\leq C||b||_{Lip_\beta(w)} M_{\beta,w,s}(f)(\tilde{x}) \sum_{k=1}^{\infty} kC_k \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \\
&\leq C||b||_{Lip_\beta(w)} M_{\beta,w,s}(f)(\tilde{x}) w(\tilde{x}) \sum_{k=1}^{\infty} kC_k \\
&\leq C||b||_{Lip_\beta(w)} w(\tilde{x}) M_{\beta,w,s}(f)(\tilde{x}).
\end{aligned}
$$

These complete the proof of Theorem 1.

Proof of Theorem 2. Choose $q' < s < p$ in Theorem 1 and notice $w^{1-r} \in A_1$, we have, by Lemma 1-3,

$$
\begin{aligned}
&||T_b(f)||_{L^r(w^{1-r})} \\
&\quad \leq \|M_\eta(T_b(f))\|_{L^r(w^{1-r})} \leq C\|M_\eta^{\#}(T_b(f))^{\#}\|_{L^r(w^{1-r})} \\
&\quad \leq C||b||_{Lip_\beta(w)}(\|M_{\beta,w,s}(T(f))w\|_{L^r(w^{1-r})} + \|M_{\beta,w,s}(f)w\|_{L^r(w^{1-r})}) \\
&\quad \leq C||b||_{Lip_\beta(w)}(\|M_{\beta,w,s}(T(f))\|_{L^r(w)} + \|M_{\beta,w,s}(f)\|_{L^r(w)}) \\
&\quad \leq C||b||_{Lip_\beta(w)}(\|T(f)\|_{L^p(w)} + \|f\|_{L^p(w)}) \\
&\quad \leq C||b||_{Lip_\beta(w)}\|f\|_{L^p(w)}.
\end{aligned}
$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Choose $q' < s < p$ in Theorem 1 and notice $w^{1-r} \in A_1$, we have, by Lemma 6-8,

$$\begin{aligned}
||T_b(f)||_{L^{r,\varphi}(w^{1-r})} &\leq \|M_\eta(T_b(f))\|_{L^{r,\varphi}(w^{1-r})} \leq C\|M_\eta^{\#}(T_b(f))^{\#}\|_{L^{r,\varphi}(w^{1-r})} \\
&\leq C||b||_{Lip_\beta(w)}(\|M_{\beta,w,s}(T(f))w\|_{L^{r,\varphi}(w^{1-r})} + \|M_{\beta,w,s}(f)w\|_{L^{r,\varphi}(w^{1-r})}) \\
&\leq C||b||_{Lip_\beta(w)}(\|M_{\beta,w,s}(T(f))\|_{L^{r,\varphi}(w)} + \|M_{\beta,w,s}(f)\|_{L^{r,\varphi}(w)}) \\
&\leq C||b||_{Lip_\beta(w)}(\|T(f)\|_{L^{p,\varphi}(w)} + \|f\|_{L^{p,\varphi}(w)}) \\
&\leq C||b||_{Lip_\beta(w)}\|f\|_{L^{p,\varphi}(w)}.
\end{aligned}$$

This completes the proof of Theorem 3.

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