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Decoupling zeros of positive continuous-time linear systems

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Abstract. Necessary and sufficient conditions for the reachability and observability of the positive continuous-time linear systems are established. Definitions of the input-decoupling zeros, output-decoupling zeros and input-output decoupling zeros are proposed. Some properties of the decoupling zeros are discussed.

Key words: decoupling zeros; positive; continuous-time; linear system; observability; reachability.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models [1–19]. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive linear theory is given in the monographs [2, 3].

The notions of controllability and observability and the decomposition of linear systems have been introduced by Kalman [13, 14]. Those notions are the basic concepts of the modern control theory [1, 2, 4, 12, 15, 19]. They have been also extended to positive linear systems [2, 3].

The reachability and controllability to zero of standard and positive fractional discrete-time linear systems have been investigated in [9] and controllability and observability of electrical circuits in [6, 8, 10]. The decomposition of positive discrete-time linear systems has been addressed in [5]. The notion of decoupling zeros of standard linear systems have been introduced by Rosenbrock [15]. The zeros of linear standard discrete-time system have been addressed in [18] and zeros of positive continuous-time and discrete-time linear systems have been defined in [16, 17]. The decoupling zeros of positive discrete-time linear systems have been introduced in [7].

In this paper the notions of decoupling zeros is extended for positive continuous-time linear systems.

The paper is organized as follows. In Sec. 2 the basic definitions and theorems concerning reachability and observability of positive discrete-time linear systems are given. The decomposition of the pair (A, B) and (A, C) of positive linear system is addressed in Sec. 3. The main result of the paper is given in Sec. 4 where the definitions of the decoupling-zeros are proposed. Concluding remarks are given in Sec. 5.

The following notation is used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Reachability and observability of positive continuous-time linear systems

2.1. Reachability of positive systems. Consider the linear continuous-time system.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned} \tag{1}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$, $m\le n$.

Definition 1. [2, 3] The linear system (1) is called (internally) positive if $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p, t\ge 0$ for any $x(0)=x_0\in\Re_+^n$ and every $u(t)\in\Re_+^m, t\ge 0$.

Theorem 1. [2, 3] The system (1) is positive if and only if

$$A\in M_n,\quad B\in\Re_+^{n\times m},\quad C\in\Re_+^{p\times n},\quad D\in\Re_+^{p\times m} \tag{2}$$

Definition 2. The positive system (1) (or positive pair (A, B)) is called reachable in time t_f if for any given final state $x_f\in\Re_+^n$ there exists an input $u(t)\in\Re_+^m$, $t\in[0,t_f]$ which steers the state of the system from zero state ($x(0)=0$) to state $x_f\in\Re_+^n$, i.e. $x(t_f)=x_f$.

A column $a\in\Re_+^n$ (row $a^T\in\Re_+^n$) is called monomial if only one its entry is positive and the remaining entries are zero. A real matrix $A\in\Re_+^{n\times n}$ is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

Theorem 2. The positive system (1) is reachable in time t_f if and only if the matrix $A\in M_n$ is diagonal and the matrix $B\in\Re_+^{n\times n}$ is monomial.

Proof. Sufficiency. It is well-known [3] that if $A\in M_n$ is diagonal then $e^{At}\in\Re_+^{n\times n}$ is also diagonal and if $B\in\Re_+^{n\times m}$ is monomial then $BB^T\in\Re_+^{n\times n}$ is also monomial. In this

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case the matrix

$$R_f = \int_0^{t_f} e^{A\tau} BB^T e^{A^T\tau} d\tau \in \Re_+^{n\times n} \tag{3}$$

is also monomial and $R_f^{-1} \in \Re_+^n$. The input

$$u(t) = B^T e^{A^T(t_f-t)} R_f^{-1} x_f \in \Re_+^{n\times n} \text{ for } t \in [0, t_f], \tag{4}$$

steers the state $x(t)$ of the system from $x(0) = x_0 = 0$ to the state $x(t_f) = x_f$ since

$$\begin{aligned} x(t_f) &= \int_0^{t_f} e^{A(t_f-\tau)} Bu(\tau) d\tau \\ &= \int_0^{t_f} e^{A(t_f-\tau)} BB^T e^{A^T(t_f-\tau)} d\tau R_f^{-1} x_f \\ &= \int_0^{t_f} e^{A\tau} BB^T e^{A^T\tau} d\tau R_f^{-1} x_f = x_f. \end{aligned} \tag{5}$$

Necessity. From Cayley-Hamilton theorem we have

$$e^{At} = \sum_{k=0}^{n-1} c_k(t) A^k, \tag{6}$$

where $c_k(t)$, $k = 0, 1, \ldots, n-1$ are some nonzero functions of time depending on the matrix A. Substitution of (6) into

$$\int_0^{t_f} e^{A(t_f-\tau)} Bu(\tau) d\tau \tag{7}$$

yields

$$x_f = \begin{bmatrix} B & AB & \ldots & A^{q-1}B \end{bmatrix} \begin{bmatrix} v_0(t_f) \\ v_1(t_f) \\ \vdots \\ v_{n-1}(t_f) \end{bmatrix}, \tag{8}$$

where

$$v_k(t_f) = \int_0^{t_f} c_k(\tau) u(t_f - \tau) d\tau, \quad k = 0, 1, \ldots, n-1. \tag{9}$$

For given $x_f \in \Re_+^n$ it is possible to find nonnegative $v_k(t_f)$ for $k = 0, 1, \ldots, n-1$ if and only if the matrix

$$\begin{bmatrix} B & AB & \ldots & A^{q-1}B \end{bmatrix} \tag{10}$$

has n linearly independent monomial columns and this takes place only if the matrix $[B,\ A]$ contains n linearly independent columns [3]. Note that for the nonnegative $v_k(t_f)$, $k = 0, 1, \ldots, n-1$ it is possible to find a nonnegative input $u(t) \in \Re_+^m$, $t \in [0, t_f]$ only if the matrix $B \in \Re_+^{n\times n}$ is monomial and the matrix $A \in M_n$ is diagonal.

Example 1. Consider the positive system (1) with the matrices

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{11}$$

In this case using (8), (9) and (11) we obtain

$$\begin{aligned} x_f &= \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} \int_0^{t_f} c_0(\tau) u(t_f-\tau) d\tau \\ \int_0^{t_f} c_1(\tau) u(t_f-\tau) d\tau \\ \int_0^{t_f} c_2(\tau) u(t_f-\tau) d\tau \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\cdot \begin{bmatrix} \int_0^{t_f} c_0(\tau) \begin{bmatrix} u_1(t_f-\tau) \\ u_2(t_f-\tau) \end{bmatrix} d\tau \\ \int_0^{t_f} c_1(\tau) \begin{bmatrix} u_1(t_f-\tau) \\ u_2(t_f-\tau) \end{bmatrix} d\tau \\ \int_0^{t_f} c_2(\tau) \begin{bmatrix} u_1(t_f-\tau) \\ u_2(t_f-\tau) \end{bmatrix} d\tau \end{bmatrix}. \end{aligned} \tag{12}$$

The matrix

$$\begin{bmatrix} B_1 & B_2 & AB_1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{13}$$

is monomial and from (12) we have

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \int_0^{t_f} c_0(\tau) \begin{bmatrix} u_1(t_f-\tau) \\ u_2(t_f-\tau) \end{bmatrix} d\tau \\ \int_0^{t_f} c_1(\tau) u_1(t_f-\tau) d\tau \end{bmatrix} = x_f \tag{14a}$$

and

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \int_0^{t_f} c_1(\tau) u_2(t_f-\tau) d\tau \\ \int_0^{t_f} c_2(\tau) \begin{bmatrix} u_1(t_f-\tau) \\ u_2(t_f-\tau) \end{bmatrix} d\tau \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{14b}$$

It is easy to see that does not exist $\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \in \Re_+^2$ satisfying (14) for any given $x_f \in \Re_+^3$, since $c_k(t)$ for k = 0,1,2 are nonzero.

2.2. Observability of positive systems. Consider the positive system.

$$\dot{x}(t) = Ax(t), \tag{15a}$$

$$y(t) = Cx(t), \tag{15b}$$

where $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$ and $A \in M_n$, $C \in \Re_+^{p\times n}$.

Definition 3. The positive system (15) is called observable if knowing the output $y(t) \in \Re_+^p$ and its derivatives $y^{(k)}(t) = \dfrac{d^k y(t)}{dt^k} \in \Re_+^p$, k = 1,2,…,n – 1 it is possible to find the initial values $x_0 = x(0) \in \Re_+^n$ of $x(t) \in \Re_+^n$.

Theorem 3. The positive system (15) is observable if and only if the matrix $A \in M_n$ is diagonal and the matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{16}$$

has n linearly independent monomial rows.

Proof. Substituting of the solution

$$x(t) = e^{At}x_0 \tag{17}$$

of the equation (15a) into (15b) yields

$$y(t) = Ce^{At}x_0. \tag{18}$$

From (18) we have

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} e^{At}x_0. \tag{19}$$

It is possible to find from (19) $e^{At}x_0 \in \Re_+^n$ if and only if the matrix (16) has n linearly independent monomial rows. From the equality $e^{At}e^{-At} = I_n$ it follows that the matrix $e^{At} \in \Re_+^{n\times n}$ for $A \in M_n$ if and only if it is diagonal. Therefore, it is possible to find $x_0 \in \Re_+^n$ from the equation (19) if and only if the matrix $A \in M_n$ is diagonal and the matrix (16) has n linearly independent rows.

Theorem 4. The positive system (15) is observable if the matrix

$$O_p = e^{A^T t}C^T Ce^{At} \tag{20}$$

is monomial.

Proof. Premultiplying (18) by $e^{A^T t}C^T$ we obtain

$$e^{A^T t}C^T Ce^{At}x_0 = e^{A^T t}C^T y(t). \tag{21}$$

If the matrix (20) is monomial then $O_p^{-1} = [e^{A^T t}C^T Ce^{At}]^{-1} \in \Re_+^{n\times n}$ and from (21) we have

$$x_0 = [e^{A^T t}C^T Ce^{At}]^{-1}e^{A^T t}C^T y(t) \in \Re_+^n \tag{22}$$

since $e^{A^T t}C^T y(t) \in \Re_+^n$ for $y(t) \in \Re_+^p$.

Note that the matrix (20) can be monomial only if the matrix C is monomial.

3. Decomposition of the pairs (A,B) and (A,C)

3.1. Decomposition of the pair (A,B). Consider the pair (A,B) with A being diagonal.

$$A = \text{diag}\,[a_{11}, a_{22}, ..., a_{n,n}] \in M_n \tag{23a}$$

and the matrix B with m linearly independent columns $B_1, B_2, ..., B_m$

$$B = [\; B_1 \quad B_2 \quad \dots \quad B_m \;]. \tag{23b}$$

By Theorem 2 the pair (23) is unreachable if $m < n$.

It is shown that in this case the pair can be decomposed into the reachable pair $(\overline{A}_1, \overline{B}_1)$ and unreachable pair $(\overline{A}_2, \overline{B}_2 = 0)$.

Theorem 5. For the unreachable pair (23) ($m < n$) there exists a monomial matrix $P \in \Re_+^{n\times n}$ such that the pair (A, B) can be reduced to the form

$$\begin{aligned} \overline{A} &= PAP^{-1} = \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & \overline{A}_2 \end{bmatrix}, \\ \overline{B} &= PB = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}, \end{aligned} \tag{24}$$

where $\overline{A}_1 = \text{diag}\,[\overline{a}_{11}, \overline{a}_{22}, ..., \overline{a}_{n_1,n_1}] \in M_{n_1}$, $\overline{A}_2 = \text{diag}\,[\overline{a}_{n_1+1,n_1+1}, ..., \overline{a}_{n,n}] \in M_{n_2}$, $\overline{B}_1 \in \Re_+^{n_1\times m}$, $n = n_1 + n_2$, the pair $(\overline{A}_1, \overline{B}_1)$ is reachable and the pair $(\overline{A}_2, \overline{B}_2 = 0)$ is unreachable.

Proof. Performing on the matrix B the following elementary row operations:

1. interchange the i-th and j-th rows, denoted by $L[i, j]$,
2. multiplication of i-th rows by positive number c, denoted by $L[i \times c]$,

we may reduce the matrix B to the form $\begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}$, where $\overline{B}_1 \in \Re_+^{n_1\times m}$ is monomial with positive entries equal to 1. Performing the same elementary row operations on the identity matrix I_n we obtain the desired monomial matrix P. It is well-known [3] that $P^{-1} \in \Re_+^{n\times n}$ and for diagonal matrix A we have $\overline{A} = PAP^{-1} = \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & \overline{A}_2 \end{bmatrix}$.

Example 2. Consider the unreachable pair (23) with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}. \tag{25}$$

Performing on the matrix B the following elementary row operations $L[1,3]$, $L[1 \times 1/2]$, $L[2,3]$, $L[2 \times 1/3]$ we obtain

$$\overline{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{26}$$

Performing the same elementary row operations on the identity matrix I_3 we obtain the desired monomial matrix

$$P = \begin{bmatrix} 0 & 0 & 1/2 \\ 1/3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{27}$$

and

$$PB = \begin{bmatrix} 0 & 0 & 1/2 \\ 1/3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \overline{B} = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix},$$

$$\overline{A} = PAP^{-1} = \begin{bmatrix} 0 & 0 & 1/2 \\ 1/3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \overline{A}_1 & 0 \\ 0 & \overline{A}_2 \end{bmatrix}. \tag{28}$$

The positive pair

$$\overline{A}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \overline{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{29}$$

is reachable.

3.2. Decomposition of the pair (A,C).

Let the observability matrix

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \Re_+^{pn \times n} \tag{30}$$

of the positive unobservable system has $n_1 < n$ linearly independent monomial rows.

If the conditions

$$Q_k A Q_j^T = 0 \quad \text{for} \quad k = 1, 2, ..., \widehat{n}_1 \quad \text{and} \quad j = \widehat{n}_1 + 1, ..., n \tag{31}$$

are satisfied then there exists the monomial matrix [5, 6]

$$Q^T = \Big[Q_{j_1}^T \ \dots \ Q_{j_1 \overline{d}_1}^T \ Q_{j_2}^T \ \dots \ Q_{j_2 \overline{d}_2}^T \ \dots \ Q_{j_l \overline{d}_l}^T \ Q_{n_1+1}^T \ \dots \ Q_n^T \Big] \in \Re_+^{n \times n}, \tag{32a}$$

where

$$Q_{j_1} = C_{j_1}, ..., Q_{j_1 \overline{d}_1} = C_{j_1} A^{\overline{d}_1 - 1}, Q_{j_2} = C_{j_2}, ..., Q_{j_2 \overline{d}_2} = C_{j_2} A^{\overline{d}_2 - 1}, ..., Q_{j_l \overline{d}_l} = C_{j_l} A^{\overline{d}_l - 1} \tag{32b}$$

and $\overline{d}_j$, $\quad j = 1, ..., l$ are some natural numbers.

Theorem 6. Let the positive system (15) be unobservable and let there exist the monomial matrix (32). Then the pair (A,C) of the system can be reduced by the use of the matrix (32) to the form

$$\widehat{A} = QAQ^{-1} = \begin{bmatrix} \widehat{A}_1 & 0 \\ 0 & \widehat{A}_2 \end{bmatrix},$$

$$\widehat{C} = CQ^{-1} = [\ \widehat{C}_1 \quad 0\ ] \tag{33}$$

$$\widehat{A}_1 \in \Re_+^{n_1 \times n_1}, \qquad \widehat{A}_2 \in \Re_+^{n_2 \times n_2},$$

$$(n_2 = n - n_1), \qquad \widehat{C}_1 \in \Re_+^{p \times n_1},$$

where the pair $(\widehat{A}_1, \widehat{C}_1)$ is observable and the pair $(\widehat{A}_2, \widehat{C}_2 = 0)$ is unobservable.

Proof is given in [5].

Example 3. Consider the unobservable pair

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad C = [\ 0 \quad 0 \quad 1\ ]. \tag{34}$$

In this case the observability matrix

$$Q_3 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

has only one monomial row $Q_1 = C$, i.e. $n_1 = 1$ and the conditions (31) are satisfied for $Q_2 = [\ 1 \quad 0 \quad 0\ ]$ and $Q_3 = [\ 0 \quad 1 \quad 0\ ]$ since $Q_1 A Q_j^T = 0$ for $j = 2, 3$. The matrix has the form

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{36}$$

Using (33) and (36) we obtain

$$\widehat{A} = QAQ^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \widehat{A}_1 & 0 \\ 0 & \widehat{A}_2 \end{bmatrix}, \tag{37}$$

$$\widehat{C} = CQ^{-1} = [\ 0 \quad 0 \quad 1\ ] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = [\ 1 \quad 0 \quad 0\ ] = [\ \widehat{C}_1 \quad 0\ ],$$

where

$$\widehat{A}_1 = [-1], \qquad \widehat{A}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad \widehat{C}_1 = [1]. \quad (38)$$

The pair $(\widehat{A}_1, \widehat{C}_1)$ is observable and the pair $(\widehat{A}_2, 0)$ is unobservable.

Note that the Singular Value Decomposition (SVD) can be applied to compute the decomposition of the pairs (A,B) and (A,C).

4. Decoupling zeros of the positive systems

It is well-known [15] that for standard linear systems the input-decoupling zeros are the eigenvalues of the matrix $\overline{A}_2$ of the unreachable (uncontrollable) part $(\overline{A}_2, \overline{B}_2 = 0)$.

In a similar way we define the input-decoupling zeros of the positive continuous-time linear systems.

Definition 4. Let $\overline{A}_2$ be the matrix of unreachable part of the system (1). The zeros $s_{i1}, s_{i2}, ..., s_{i\overline{n}_2}$ of the characteristic polynomial

$$\det[I_{\overline{n}_2} s - \overline{A}_2] = z^{\overline{n}_2} + \overline{a}_{\overline{n}_2 - 1} z^{\overline{n}_2 - 1} + ... + \overline{a}_1 z + \overline{a}_0 \quad (39)$$

of the matrix $\overline{A}_2$ are called the input-decoupling zeros of the positive system (1).

The list of the input-decoupling zeros are denoted by $Z_i = \{s_{i1}, s_{i2}, ..., s_{i\overline{n}_2}\}$.

Theorem 7. The state vector $x(t)$ of the positive system (1) is independent of the input-decoupling zeros for any input $u(t)$ and zero initial conditions.

Proof. From (1) for zero initial conditions $x(0) = 0$ we have

$$X(s) = [I_n s - A]^{-1} B U(s), \quad (40)$$

where $X(s)$ and $U(s)$are Laplace transforms of $x(t)$ and $u(t)$, respectively. Taking into account (24) we obtain

$$\begin{aligned} X(s) &= [I_n s - P^{-1}\overline{A}P]^{-1} P^{-1} B U(s) \\ &= P^{-1}[I_n s - \overline{A}]^{-1} \overline{B} U(s) \\ &= P^{-1} \begin{bmatrix} I_{\overline{n}_1} s - \overline{A}_1 & 0 \\ 0 & I_{\overline{n}_2} s - \overline{A}_2 \end{bmatrix}^{-1} \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix} U(s) \\ &= P^{-1} \begin{bmatrix} [I_{\overline{n}_1} s - \overline{A}_1]^{-1} \overline{B}_1 \\ 0 \end{bmatrix} U(s). \end{aligned} \quad (41)$$

From (41) it follows that $X(s)$ is independent of the matrix $\overline{A}_2$ and of the input-decoupling zeros for any input $u(t)$.

Example 4. (continuation of Example 2) In Example 2 it was shown that for the unreachable pair (23) the matrix $\overline{A}_2$ has the form $\overline{A}_2 = [-2]$. Therefore, the positive system (23) with (25) has one input-decoupling zero $s_{i1} = -2$.

For standard continuous-time linear systems the output-decoupling zeros are defined as the eigenvalues of the matrix of the unobservable part of the system. In a similar way we define the output-decoupling zeros of the positive continuous-time linear systems.

Definition 5. Let $\widehat{A}_2$ be the matrix of unobservable part of the system (15). The zeros $s_{o1}, s_{o2}, ..., s_{o\widehat{n}_2}$ of the characteristic polynomial

$$\det[I_{\widehat{n}_2} z - \widehat{A}_2] = z^{\widehat{n}_2} + \widehat{a}_{\widehat{n}_2 - 1} z^{\widehat{n}_2 - 1} + ... + \widehat{a}_1 z + \widehat{a}_0 \quad (42)$$

of the matrix $\widehat{A}_2$ are called the output-decoupling zeros of the positive system (15).

The list of the output-decoupling zeros is denoted by $Z_o = \{s_{o1}, s_{o2}, ..., s_{o\widehat{n}_2}\}$.

Theorem 8. The output vector $y(t)$ of the positive system (15) is independent of the output-decoupling zeros for any input $\overline{u}(t) = Bu(t)$ and zero initial conditions.

Proof is similar to the proof of Theorem 7.

Example 5. (continuation of Example 3) In Example 3 it was shown that the matrix $\widehat{A}_2$ of the positive unobservable pair (34) has the form

$$\widehat{A}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad (43)$$

and the positive system has two output-decoupling zero $s_{o1} = -1$, $s_{o2} = -2$.

Following the same way as for standard continuous-time linear systems we define the input-output decoupling zeros of the positive systems as follows.

Definition 6. Zeros $s_{io}^{(1)}, s_{io}^{(2)}, ..., s_{io}^{(k)}$ which are simultaneously the input-decoupling zeros and the output-decoupling zeros of the positive system are called the input-output decoupling zeros of the positive system, i.e.

$$\begin{gathered} s_{io}^{(j)} \in Z_i \quad \text{and} \quad s_{io}^{(j)} \in Z_o \\ \text{for} \quad j = 1, 2, \ldots, k; \;\; k \le \min(\overline{n}_2, \widehat{n}_2). \end{gathered} \quad (44)$$

The list of input-output decoupling zeros is denoted by $Z_{io} = \{z_{io}^{(1)}, z_{io}^{(2)}, ..., z_{io}^{(k)}\}$.

Example 6. Consider the positive system with the matrices A, B, C given by (25) and (36). In Example 4 it was shown that the positive system has one input-decoupling zero $s_{i1} = -2$ and in Example 5 that the system has two output-decoupling zeros $s_{o1} = -1$, $s_{o2} = -2$. Therefore, by Definition 6 the positive system has one input-output decoupling zero $s_{io}^{(1)} = -2$.

5. Concluding remarks

New necessary and sufficient conditions for the reachability and observability of the positive continuous-time linear systems have been established. The definitions of the input-decoupling zeros, output-decoupling zeros and input-output decoupling zeros of the positive systems have been proposed. Some properties of the new decoupling zeros have been discussed. The considerations have been illustrated by numerical examples of positive continuous-time linear systems. An open problem is an extension of these considerations to fractional discrete-time and continuous-time positive linear systems [11].

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