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DYNKIN DIAGRAMS OF BASIC LIE SUPERALGEBRAS

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Abstract. A basic Lie superalgebra $\mathfrak{g}$ has many Dynkin diagrams, due to different choices of simple root systems. For $\mathfrak{g}$ with rank 4 or lower, all its Dynkin diagrams have already been given explicitly. To deal with $\mathfrak{g}$ with rank higher than 4, this article introduces a combinatorial invariant on the Dynkin diagrams. It enables us to recognize $\mathfrak{g}$ from a given Dynkin diagram. In this way, we obtain all the Dynkin diagrams of any given basic Lie superalgebra.

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie superalgebra. We say that $\mathfrak{g}$ is *basic* if it has a nondegenerate invariant supersymmetric form, and its even part is reductive [**2**]. Once and for all, we assume that $\mathfrak{g}$ is not a Lie algebra, namely its odd part does not vanish. There are distinct Dynkin diagrams of $\mathfrak{g}$ which occur from different simple systems. The method of odd root reflection [**3**][**4**] allows one to produce all Dynkin diagrams of $\mathfrak{g}$ from a given one. It has been used to find all the Dynkin diagrams of $\mathfrak{g}$ with rank 4 or lower [**1**]. It remains to do the same for $\mathfrak{g}$ with rank higher than 4, and this is the purpose of this article. For the Dynkin diagrams of a given classical series, we introduce a combinatorial invariant to distinguish among the Lie superalgebras of that series. It enables us to recognize the Lie superalgebra from a given Dynkin diagram. We now discuss our result in more details.

Since we only need to consider $\mathfrak{g}$ with rank higher than 4, we may ignore $F(4)$, $G(3)$ and $D(2,1;\alpha)$. This leaves the classical series of basic Lie superalgebras $A(p,q)$, $B(p,q)$, $C(p)$ and $D(p,q)$ with $p-q\neq 1$ [**2**, p.43 Theorem 1], and their Dynkin diagrams with n vertices are all given in Figure 1 below [**5**]. There exist different conventions regarding the edges of the Dynkin diagrams, and the convention used here is the one in [**1**], which differs from [**2**, p.54 Table V]. The vertices are colored white, black or grey. Let $\otimes$ denote the grey vertices. Some vertices in Figure 1 appear as half white and half grey (resp. black), it means that they can

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be either white or grey (resp. black). The vertices which are entirely white or grey in Figure 1 can only accept their designated colors.

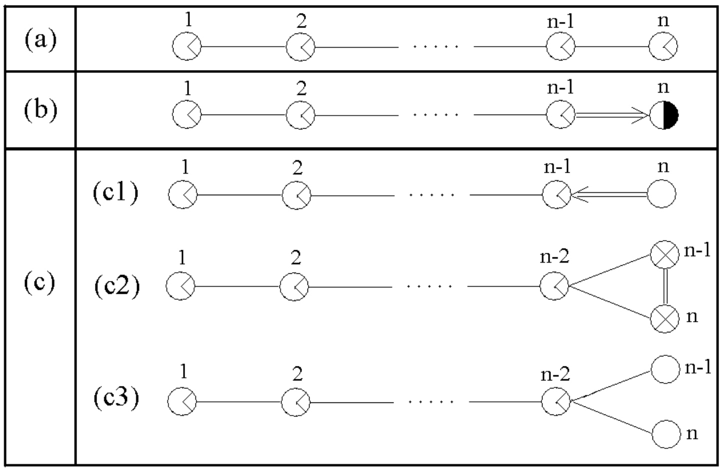


Figure 1

Two Dynkin diagrams x, y are said to be *related* if they represent the same basic Lie superalgebra, and we denote this by $x \sim y$. The diagrams in Figure 1 are placed into three sections (a), (b) and (c), so that diagrams of different sections are definitely not related.

In Figure 1, the numbers $1, 2, ..., n$ merely label the vertices. Given a diagram in section (a), (b), (c1) or (c2), we denote it by $(i_1, i_2, ..., i_r)$, where $1 \leq i_1 < ... < i_r \leq n$. It refers to the diagram whose vertices $i_1, ..., i_r$ are black or gray, with remaining vertices white. No vertex can simultaneously allow black and grey, so this notation does not cause confusion. More generally, write

$$(i_1, i_2, ..., i_r) \ , \ 0 \leq i_1 \leq ... \leq i_r \leq n, \tag{1.1}$$

where a grey or black vertex appears odd number of times, and a white vertex appears even number of times. The number 0 appears without any effect. This more general notation facilitates later computations.

For computational purpose, although a diagram in section (c3) has white vertices $n-1$ and n, we insert an extra n and write

$$(i_1, i_2, ..., i_r, n) \ , \ 0 \leq i_1 \leq ... \leq i_r < n. \tag{1.2}$$

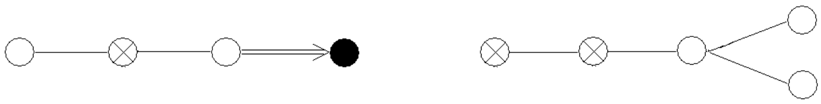
Other than vertex n, a grey or black vertex appears odd number of times, and a white vertex appears even number of times. The number 0 appears without any effect.

For a Dynkin diagram x expressed in (1.1) or (1.2), define $\phi(x) \in \mathbb{N}$ by

$$\phi(i_1, ..., i_r) = i_r - i_{r-1} + i_{r-2} - ... + (-1)^{r+1} i_1 = \sum_{a=1}^{r} (-1)^{r-a} i_a. \tag{1.3}$$

Despite the multiple expressions in (1.1) and (1.2), ϕ is well-defined.

For example, consider the two diagrams below. The left diagram is denoted by $(2,4)$, $(2,2,2,3,3,4)$ or other methods. Then $\phi(2,4) = 4-2 = 2$ and $\phi(2,2,2,3,3,4) = 4-3+3-2+2-2 = 2$. The right diagram is denoted by $(1,2,5)$, $(0,1,2,3,3,5)$ or other methods. Then $\phi(1,2,5) = 5-2+1 = 4$ and $\phi(1,2,5) = 5-3+3-2+1-0 = 4$.



The following is the Main Theorem of this article. It determines whether two Dynkin diagrams x and y in the same section of Figure 1 are related.

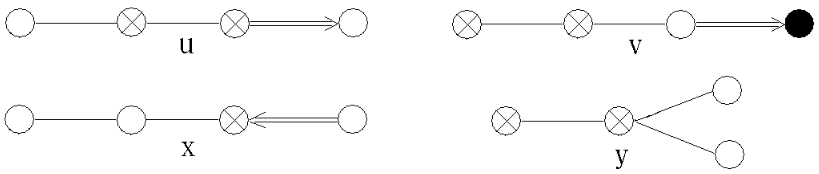
Main Theorem.
(a) *In Figure 1(a), $x \sim y$ if and only if $\phi(x) = \phi(y)$ or $\phi(x) + \phi(y) = n+1$.*
(b) *In Figure 1(b) and 1(c), $x \sim y$ if and only if $\phi(x) = \phi(y)$.*

The diagrams in Figure 1 are of types A, B, C and D. They correspond to the special linear and orthogonal-symplectic superalgebras as follows.

Figure 1(a)	:	$A(k-1, n-k)$	:	$\mathfrak{sl}(k, n+1-k)$
Figure 1(b)	:	$B(n-k, k)$	:	$\mathfrak{osp}(2n-2k+1, 2k)$
Figure 1(c)	:	$C(n), D(k, n-k)$	:	$\mathfrak{osp}(2, 2n-2), \mathfrak{osp}(2k, 2n-2k)$

Each basic Lie superalgebra $\mathfrak{g}$ has a Dynkin diagram with a single non-white vertex, known as the *distinguished diagram* [**2**, p.56 Table VI] [**5**]. The theorem leads to the following list of all Dynkin diagrams of each $\mathfrak{g}$, along with the distinguished diagrams.

Corollary of Main Theorem.
(a) $A(k-1, n-k)$: $\{x$ in Figure 1(a) ; $\phi(x) = k$ or $\phi(x) = n+1-k\}$.
Distinguished diagrams are (k) and $(n+1-k)$.
(b) $B(n-k, k)$: $\{x$ in Figure 1(b) ; $\phi(x) = k\}$.
Distinguished diagram is (k).
(c) $C(n)$: $\{x$ in Figure 1(c1) and 1(c2) ; $\phi(x) = 1\}$.
Distinguished diagram is (1) in Figure 1(c1).
(d) $D(k, n-k)$: $\{x$ in Figure 1(c) ; $\phi(x) = k\}$.
Distinguished diagrams are (k) in Figure 1(c1) and $(n-k, n)$ in Figure 1(c3).



To illustrate the Main Theorem and its corollary, consider the above diagrams. Diagrams u and v are not related because $\phi(u) = 3-2 = 1$ and $\phi(v) = 4-2+1 = 3$. Here u represents $B(2,2)$, and v represents $B(1,3)$. Their distinguished diagrams are respectively (1) and (3) of Figure 1(b). Diagrams x and y are related because $\phi(x) = 3$ and $\phi(y) = 4-2+1 = 3$. They are both represent $D(3,1)$.

This article is structured as follows. In Section 2, we recall the odd root reflections and study their effects on the Dynkin diagrams. In Section 3, we prove Main Theorem (a) for the diagrams of type A. In Section 4, we prove Main Theorem (b) for the diagrams of types B, C and D. The corollary follows immediately from the Main Theorem, so we omit its proof.

2. Odd Root Reflections

In this section, we recall the root systems of basic Lie superalgebras, and study the action of odd root reflections on their Dynkin diagrams.

Let $\mathfrak{g}$ be a basic Lie superalgebra, with Cartan subalgebra $\mathfrak{h}$ and root system $\Delta \subset \mathfrak{h}^*$. Fix a simple system $\Pi \subset \Delta$. Its Dynkin diagram has Π as vertices, and its edges depend on the pairings of roots under an invariant supersymmetric form [**1**, (2.18)]. The form is not positive definite, so roots may have zero length. Write $\Delta = \Delta_0 \cup \Delta_1$ for the even and odd roots. A vertex is white if it is an even root, is black if it is an odd root of non-zero length, and is grey if it is an odd root of zero length. We shall consider all the simple systems in $\mathfrak{h}^*$ in order to obtain all Dynkin diagrams of $\mathfrak{g}$. Let $\alpha \in \Pi$. If α is an even root, then its Weyl reflection R_α is an automorphism on Δ_0 and Δ_1, so Π and $R_\alpha\Pi$ produce the same Dynkin diagram. Therefore, to anticipate α to provide a different Dynkin diagram, we assume that α is an odd root. Define the *odd root reflection* R_α as follows. Given $\beta \in \Pi$, we let $R_\alpha(\beta) \in \Pi$ be

$$R_\alpha(\beta) = \begin{cases} \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha & \text{if } (\alpha,\alpha) \neq 0, \\ \beta + \alpha & \text{if } (\alpha,\alpha) = 0 \text{ and } (\alpha,\beta) \neq 0, \\ \beta & \text{if } (\alpha,\alpha) = (\alpha,\beta) = 0, \\ -\alpha & \text{if } \beta = \alpha. \end{cases} \tag{2.1}$$

Theorem 2.1. [**3**] *For any $\alpha \in \Pi$, $R_\alpha\Pi$ is again a simple system. For fixed Π, its orbit under such R_α exhausts all simple systems of $(\mathfrak{g},\mathfrak{h})$. Namely if $\Pi' \subset \Delta$ is another simple system, then there is a sequence $\Pi = \Pi_1 \longrightarrow \Pi_2 \longrightarrow \ldots \longrightarrow \Pi_a = \Pi'$ such that each $\Pi_i \longrightarrow \Pi_{i+1}$ is given by R_α for some odd root $\alpha \in \Pi_i$.*

The above method gives a practical way to find the Dynkin diagrams related to a given one. Since a simple system amounts to a Borel subalgebra, this method is used to find the Borel subalgebras of $\mathfrak{g}$ [**4**, p.850]. In fact it has been used to find all Dynkin diagrams of $\mathfrak{g}$ with rank ≤ 4 [**1**]. In this article, we work out this problem for higher rank $\mathfrak{g}$.

We shall use Theorem 2.1 to study the effect of R_α on the Dynkin diagram when α is a grey vertex. We can ignore R_α for black vertex α, because in that case 2α is even and so R_α is an ordinary reflection. If $\beta \neq \alpha$ is perpendicular to α, then β is not moved by R_α; otherwise β changes by a multiple of α. Hence the colors of the vertices that are not adjacent to α remain the same, as do the edges attached to these vertices. Therefore, it suffices to study the vertices which are adjacent to α, and this is described by the next proposition.

In Figure 2 below, some half white and half grey (resp. black) vertices have their white and grey (resp. black) parts reversed under R_α. It means that R_α turns a white vertex to grey (resp. black), and vice versa. Formula (2.1) leads to the following proposition.

Proposition 2.2. *Let α be an odd root. Then Figure 2 below reveals the effect of the odd root reflection R_α.*

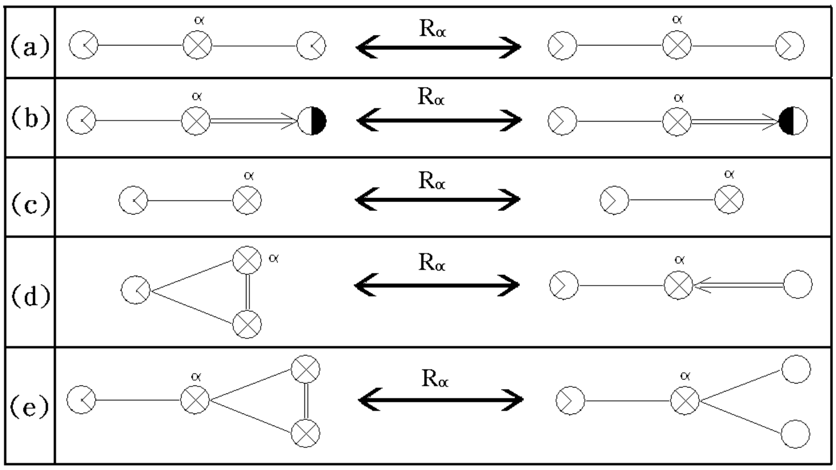


Figure 2

3. Special Linear Superalgebras

In this section, we prove Main Theorem (a) for the special linear superalgebras $\mathfrak{sl}(p,q)$. So throughout this section, all Dynkin diagrams are assumed to be in Figure 1(a) of the Introduction. Let $1,...,n$ be the vertex labelling in Figure 1(a). We denote a diagram by $(i_1,...,i_r)$, as in (1.1). Given a grey vertex i, let R_i be the odd root reflection (2.1). By Proposition 2.2 on Figure 2(a,c),

$$\begin{aligned}&R_{i_a}(i_1,...,i_r) = (i_1,...,i_{a-1},i_a-1,i_a,i_a+1,i_{a+1},...,i_r) \text{ for } i_a<n,\\ &R_n(i_1,...,i_{r-1},n) = (i_1,...,i_{r-1},n-1,n).\end{aligned} \tag{3.1}$$

For example, $R_4(1,4,5,7) = (1,3,4,5,5,7) = (1,3,4,7)$. Visually, R_4 shifts the grey pair $4,5$ leftward to $3,4$. We shall use this idea to shift the grey vertices. Let $\phi(x)$ be as defined in (1.3).

Proposition 3.1. *Let x be a Dynkin diagram, and i a grey vertex.*
(a) $\phi R_i(x) = \phi(x)$ *for* $i=1,...,n-1$*;*
(b) $\phi R_n(x) = n+1-\phi(x)$.

Proof: For part (a), let $x=(i_1,...,i_r)$ and let $i_a<n$. By (3.1),

$$\begin{aligned}\phi R_{i_a}(i_1,...,i_r) &= \phi(i_1,...,i_{a-1},i_a-1,i_a,i_a+1,i_{a+1},...,i_r)\\ &= \phi(i_{a+1},...,i_r)+(-1)^{r-a}((i_a+1)-i_a+(i_a-1))\\ &\quad +(-1)^{r-a+1}\phi(i_1,...,i_{a-1})\\ &= \phi(i_{a+1},...,i_r)+(-1)^{r-a}i_a+(-1)^{r-a+1}\phi(i_1,...,i_{a-1})\\ &= \phi(i_1,...,i_r).\end{aligned} \tag{3.2}$$

For part (b), let $x=(i_1,...,i_r)$ with $i_r=n$. By (3.1),

$$\begin{aligned}\phi R_n(i_1,...,i_{r-1},n) &= \phi(i_1,...,i_{r-1},n-1,n)\\ &= n-(n-1)+\phi(i_1,...,i_{r-1})\\ &= n+1-\phi(i_1,...,i_{r-1},n).\end{aligned}$$

This proves the proposition. □

Recall that diagrams x and y are said to be related if they represent the same $\mathfrak{g}$, and we denote this by $x\sim y$.

Corollary 3.2. *If $x\sim y$, then either $\phi(x)=\phi(y)$ or $\phi(x)+\phi(y)=n+1$.*

Proof: If $x\sim y$, then by Theorem 2.1, there exists a sequence of Dynkin diagrams $x=x_1\longrightarrow x_2\longrightarrow ...\longrightarrow x_a=y$ such that each $x_i\longrightarrow x_{i+1}$ is given by an odd root reflection. By Proposition 3.1, either $\phi(x_i)=\phi(x_{i+1})$ or $\phi(x_i)+\phi(x_{i+1})=n+1$. The corollary follows. □

To prove Main Theorem (a), it remains to obtain the converse of Corollary 3.2. This will be done by the next few statements.

Proposition 3.3.
(a) $(i_1, i_2, ..., i_r) \sim (i_1 - 1, i_2 - 1, i_3, ..., i_r)$;
(b) $(i_1, i_2, ..., i_r) \sim (i_2 - i_1, i_3, ..., i_r)$;
(c) $(i) \sim (n + 1 - i)$.

Proof: For part (a), apply $R_{i_1}, R_{i_1+1}, ..., R_{i_2-1}$ consecutively to $(i_1, i_2, ..., i_r)$. Namely,

$$\begin{array}{llr} (i_1, i_2, ..., i_r) & \longrightarrow (i_1 - 1, i_1, i_1 + 1, i_2, ..., i_r) & \text{by } R_{i_1} \\ & \longrightarrow ... \longrightarrow (i_1 - 1, i_2 - 2, i_2 - 1, i_2, i_3, ..., i_r) & \text{by } R_{i_1+1}, ..., R_{i_2-2} \\ & \longrightarrow (i_1 - 1, i_2 - 1, i_3, ..., i_r). & \text{by } R_{i_2-1} \end{array}$$

This proves part (a). It shifts the pair i_1 and i_2 one unit leftward. By applying part (a) inductively, we keep shifting the pair leftward and obtain

$$(i_1, i_2, ..., i_r) \sim (1, i_2 - i_1 + 1, i_3, ..., i_r).$$

Then apply $R_1, R_2, ..., R_{i_2-i_1}$ consecutively to $(1, i_2 - i_1 + 1, i_3, ..., i_r)$ and we obtain $(i_2 - i_1, i_3, ..., i_r)$. This proves part (b).

For part (c), we apply $R_i, R_{i+1}, ..., R_n$ to (i) and obtain $(i) \sim (i - 1, n)$. Then $(i - 1, n) \sim (n + 1 - i)$ by part (b). This proves part (c). □

Corollary 3.4. *Every diagram x satisfies $x \sim (\phi(x)) \sim (n + 1 - \phi(x))$.*

Proof: Given a diagram $x = (i_1, i_2, ..., i_r)$, Proposition 3.3(b) reduces its number of entries i_a by one. We apply Proposition 3.3(b) inductively until we obtain a distinguished diagram (i). So $x \sim (i)$. Clearly $\phi(i) = i$, so by Corollary 3.2, either $\phi(x) = i$ or $\phi(x) + i = n + 1$. Together with Proposition 3.3(c), we have $x \sim (\phi(x)) \sim (n + 1 - \phi(x))$. □

Proof of Main Theorem (a):

If $x \sim y$, then by Corollary 3.2,

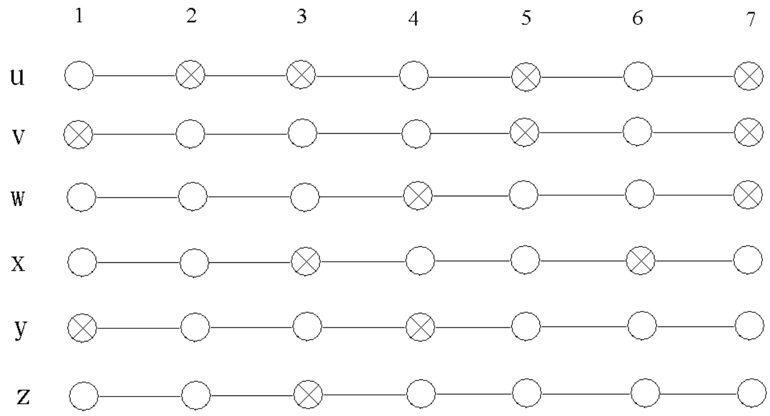
$$\phi(x) = \phi(y) \ \text{ or } \ \phi(x) + \phi(y) = n + 1. \tag{3.3}$$

Conversely, suppose that (3.3) holds. By Corollary 3.4,

$$x \sim (\phi(x)) \sim (n + 1 - \phi(x)) \ , \ y \sim (\phi(y)) \sim (n + 1 - \phi(y)) \, . \tag{3.4}$$

By (3.3) and (3.4), we get $x \sim y$. This proves Main Theorem (a). □

We give an example to illustrate the idea. Consider diagram $u = (2, 3, 5, 7)$ below. We explain how it is related to the distinguished diagram z.



For diagram u, we first move the grey pair $2, 3$ leftward by R_2, R_1. They become a grey vertex at 1, and we obtain v. We next handle the grey pair $1, 5$ of v. They become a grey vertex at 4 after $R_1, ..., R_4$, and we obtain w. Apply R_4, R_5, R_6 to w and we obtain x. Compare w with x, and we see that the grey pair has moved one unit leftward. We further move the grey pair leftward by R_3, R_4, R_5. Repeat this process inductively until the grey pair hits the left end, and we obtain y. Then apply R_1, R_2, R_3 to y, and we obtain z. In z, the location of the single grey vertex at 3 is not surprising because $\phi(u) = 7 - 5 + 3 - 2 = 3$.

4. Orthogonal-Symplectic Superalgebras

In this section, we prove Main Theorem (b) for Dynkin diagrams of the orthogonal-symplectic superalgebras $\mathfrak{osp}(p, q)$. So throughout this section, all Dynkin diagrams are assumed to be in Figure 1(b) or (c) of the Introduction. Express a Dynkin diagram by $(i_1, ..., i_r)$ as in (1.1) and (1.2). Given a grey vertex i, let R_i be the odd root reflection (2.1). By Proposition 2.2,

$$R_{i_a}(i_1, ..., i_r) = (i_1, ..., i_{a-1}, i_a - 1, i_a, i_a + 1, i_{a+1}, ..., i_r) \text{ for } i_a < n. \quad (4.1)$$

Here (4.1) follows from Figure 2, which covers all cases of R_i for $i < n$. We can ignore R_n because $R_n = 1$ in Figure 1(b) regardless of whether vertex n is white or black, and also $R_n = R_{n-1}$ in Figure 1(c2).

Let $\phi(x)$ be as defined in (1.3). Also, diagrams x and y which represent the same $\mathfrak{g}$ are denoted by $x \sim y$.

Proposition 4.1. *If α is a grey vertex of a Dynkin diagram x, then $\phi R_\alpha(x) = \phi(x)$. Hence if $x \sim y$, then $\phi(x) = \phi(y)$.*

Proof: Similar to the arguments in (3.2), formula (4.1) implies that $\phi R_\alpha(x) = \phi(x)$ for all R_α. This proves the first statement of this proposition.

If $x \sim y$, then by Theorem 2.1, they are related by $x = x_1 \longrightarrow x_2 \longrightarrow ... \longrightarrow x_a = y$, where each $x_i \longrightarrow x_{i+1}$ is an odd root reflection. The first statement of this proposition says that $\phi(x_i) = \phi(x_{i+1})$, hence $\phi(x) = \phi(y)$. This proves the proposition. □

Proposition 4.1 is analogous to Proposition 3.1 and Corollary 3.2. Similarly, the following Proposition 4.2 is analogous to Corollary 3.4.

Proposition 4.2. *Every diagram x in Figure 1(b) or (c) satisfies $x \sim (\phi(x))$, where $(\phi(x))$ is in Figure 1(b) or (c1).*

Proof: It suffices to show that x is equivalent to a distinguished diagram y. This is because if $y = (i)$, then Proposition 4.1 says that $\phi(x) = \phi(y) = i$.

We first consider a diagram in Figure 1(b). Suppose that vertex n is white. Treat its subdiagram of first $n-1$ vertices as a diagram of Figure 1(a), and apply Proposition 3.3(a) and (b) to obtain a diagram with a single grey vertex among the first $n-1$ vertices. The arguments of Proposition 3.3(a) and (b) do not involve R_{n-1}, so vertex n remains white.

Next suppose that diagram x of Figure 1(b) has black vertex n. If there is no grey vertex, we are done. If there are some grey vertices, use Proposition 3.3(a) and (b) to move a grey pair rightward, so that vertex $n-1$ becomes grey. Now apply R_{n-1}, and Figure 2(b) says that vertex n becomes white. Then proceed with the method of the previous paragraph. This proves the proposition for Figure 1(b).

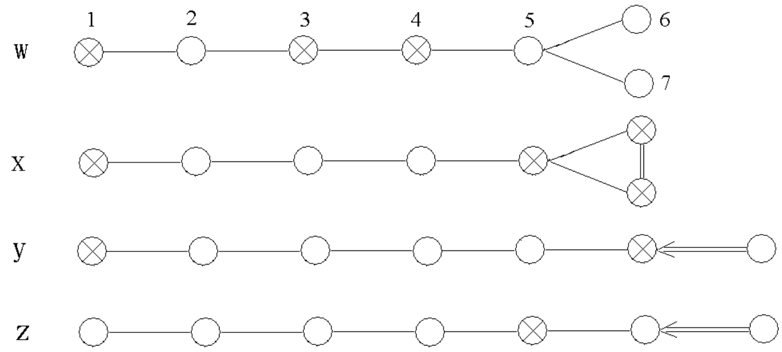
Finally we prove the proposition for Figure 1(c). Proposition 2.2 says that every diagram in (c3) is related to a diagram in (c2), and that every diagram in (c2) is related to a diagram in (c1). Given a diagram in (c1), we again treat its first $n-1$ vertices as a diagram of Figure 1(a), and use Proposition 3.3(a) and (b) to move its grey vertices leftward until we obtain a distinguished diagram. The proof follows.□

Incidently, other than the diagrams x with $\phi(x) = 1$ in (c1) and (c2), all the diagrams in part (c) are also related to a diagram in (c3) with a single grey vertex.

Proof of Main Theorem (b):

By Proposition 4.1, if $x \sim y$, then $\phi(x) = \phi(y)$. Conversely, suppose that $\phi(x) = \phi(y)$. By Proposition 4.2, $x \sim (i)$ and $y \sim (j)$ with $\phi(x) = i$ and $\phi(y) = j$. It follows that $i = j$, so $x \sim y$. □

We illustrate our results with an example. Consider the following diagram w in Figure 1(c3). We shall use the odd root reflections to transform it to diagrams in Figure 1(c1) and (c2).



By applying R_4, R_5 to diagram w, we obtain diagram x. By applying R_6 to diagram x, we obtain diagram y. By applying $R_1, R_2, ..., R_5$ to diagram y, we obtain diagram z. By (1.2), $\phi(w) = 7 - 4 + 3 - 1 = 5$, so indeed z has grey vertex at 5, as predicted by Proposition 4.2.

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