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Simple sufficient conditions for asymptotic stability of positive linear systems for any switchings

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Abstract. The asymptotic stability of positive switched linear systems for any switchings is addressed. Simple sufficient conditions for the asymptotic stability of positive switched continuous-time and discrete-time linear systems are established. It is shown that the positive switched continuous-time (discrete-time) system is asymptotically stable for any switchings if the sum of entries of every column of the matrices of subsystems is negative (less than 1).

Key words: positive, linear, switched system, asymptotic stability, sufficient conditions.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview [1–16] of state of the art in positive theory is given in the monographs [3, 6]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc..

A positive switched system consists of a collection of positive state space models and a switching function (signal) governing the switching among the models [7, 8, 10, 11]. The stability and stabilization of positive switched linear 1D system have been investigated in [1, 2, 4, 5, 10, 11, 13, 14, 16] and for positive 2D linear systems in [7, 8]. The copositive Lyapunov functions approach to switched linear systems has been applied in [2, 10, 11, 14].

The choice of the forms of Lyapunov functions for 2D Roesser model has been analyzed in [9].

In this paper new simple sufficient conditions for the asymptotic stability of positive switched linear systems for any switchings are established.

The paper is organized as follows. In Sec. 2 basic definitions and theorems concerning positive continuous-time and discrete-time systems are recalled and the formulation of the problem is given. The main result of the paper is presented in Sec. 3 where sufficient conditions for the asymptotic stability of positive switched continuous-time and discrete-time linear systems for any switchings are established. Concluding remarks are given in Sec. 4.

The following notation is used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Preliminaries and problem formulation

Consider the continuous-time linear systems

$$\dot{x}(t)=A_{\delta(t)}x(t), \tag{1}$$

where $x(t)\in\Re^n$ is the state vector, $A_{\delta(t)}\in\Re^{n\times n}$ and $\delta(t)$ is the switching function which takes its values in the finite set $S=\{1,2,...,N\}$, N is the number of subsystems. It is assumed that the state vector $x(t)\in\Re^n$ does not jump at the switching instants $0\le t_0<t_1<....$ When $t\in[t_k,t_{k+1})$, $k=0,1,...$ then $\delta(t_k)$-th system of (1) is active.

Definition 1. [3, 6] The continuous-time system (1) is called (internally) positive if $x(t)\in\Re_+^n$, $t\ge 0$ for any initial conditions $x(0)=x_0\in\Re_+^n$.

Theorem 1. [3, 6] The continuous-time system (1) is positive if and only if

$$A_{\delta(t)}\in M_n. \tag{2}$$

Definition 2. [3, 6] The positive continuous-time system (1) is called asymptotically stable if

$$\lim_{t\to\infty}x(t)=0 \quad \text{for all} \quad x_0\in\Re_+^n. \tag{3}$$

Theorem 2. [3, 6] The positive continuous-time system (1) is asymptotically stable if and only if one of the following conditions is satisfied:

1. the coefficient of the polynomial
$$\det[I_ns-A_{\delta(t)}]=s^n+a_{n-1}s^{n-1}+...+a_1s+a_0 \tag{4}$$
are positive, i.e. $a_k>0$, $k=0,1,...,n-1$.
2. there exists a strictly positive vector $\lambda>0$ (with all positive components) such that $A^T\lambda$ is a strictly negative vector, i.e.
$$A^T\lambda<0. \tag{5}$$

The positive system (1) will be called the positive switched continuous-time linear systems (shortly PSCLS).

Consider now the discrete-time linear system

$$x_{i+1}=\overline{A}_{\delta_i}x_i, \qquad i\in Z_+=\{0,1,...\}, \tag{6}$$

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where $x_i \in \Re^n$ is the state vector, $\overline{A}_{\delta_i} \in \Re^{n\times n}$ and δ_i is the switching function which takes its value in the finite set $S = \{1, 2, ..., N\}$, $N > 1$ is the number of subsystems. It is assumed that the state vector $x_i \in \Re^n$ does not jump at the switching instants $0 \le i_0 < i_1 < ....$ When $i \in [i_k, i_{k+1})$, $k = 0, 1, ...$ then k-th subsystem of (6) is active.

Definition 3. [3, 6] The discrete-time system (6) is called (internally) positive if $x_i \in \Re^n_+$, $i \in Z_+$ for any initial conditions $x_0 \in \Re^n_+$.

Theorem 3. [3, 6] The discrete-time system (6) is positive if and only if

$$\overline{A}_{\delta_i} \in \Re^{n\times n}_+. \tag{7}$$

Definition 4. [3, 6] The positive discrete-time system (6) is called asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \quad \text{for all} \quad x_0 \in \Re^n_+. \tag{8}$$

Theorem 4. [3, 6] The positive discrete-time system (6) is asymptotically stable if and only if one of the following conditions is satisfied:

1. the coefficient of the polynomial

$$\begin{aligned}\det[I_n(z+1) - \overline{A}_{\delta_i}] &= z^n + \overline{a}_{n-1}z^{n-1} + ... \\ &+ \overline{a}_1 z + \overline{a}_0\end{aligned} \tag{9}$$

   are positive, i.e. $\overline{a}_k > 0$, $k = 0, 1, ..., n-1$.
2. there exists a strictly positive vector $\lambda > 0$ (with all positive components) such that $(\overline{A}^T_{\delta_i} - I_n)\lambda$ is a strictly negative vector, i.e.

$$(\overline{A}^T_{\delta_i} - I_n)\lambda < 0. \tag{10}$$

The positive system (6) is called the positive switched discrete-time linear systems (shortly PSDLS).

It is assumed that that for both positive switched systems (1) and (6) the number of switchings is finite for any finite interval.

The problem under considerations for both the positive continuous-time (1) and discrete-time (6) systems can be stated as follows:

Find conditions under which the positive switched systems (1) and (6) are asymptotically stable for any switchings (finite number for any finite interval).

3. Problem solution

In this section simple sufficient conditions are established for the asymptotic stability of the positive switched systems (1) and (6) for any switchings finite in number for any finite interval.

It is easy to show that the positive switched system (1) and (6) are asymptotically stable for any switchings only if all subsystems are asymptotically stable. Therefore, it is assumed that the subsystems (1) and (6) are asymptotically stable.

3.1. Positive continuous-time linear systems.

Consider the positive continuous-time linear system

$$\dot{x}(t) = Ax(t), \tag{11}$$

where $x(t) \in \Re^n_+$, $t \ge 0$ and $A = [a_{ij}] \in M_n$.

Theorem 5. The positive continuous-time system (11) is asymptotically stable if

$$a_{ii} < 0 \quad \text{and} \quad a_{ii} + \sum_{\substack{j=1\\ j\ne i}}^{n} a_{ij} < 0 \quad \text{for } i = 1, 2, ..., n \tag{12a}$$

or

$$a_{jj} < 0 \quad \text{and} \quad a_{jj} + \sum_{\substack{i=1\\ i\ne j}}^{n} a_{ij} < 0 \quad \text{for } j = 1, 2, ..., n. \tag{12b}$$

Proof. The positive system (11) is asymptotically stable if the condition (12a) (or (12b)) is met since by Gershgorin's Circle Theorem [15] all discs centered at the point a_{ii} (a_{jj}) with the radii

$$r_i = \sum_{\substack{j=1\\ j\ne i}}^{n} a_{ij}, \quad i = 1, 2, ..., n \quad \left(r_j = \sum_{\substack{i=1\\ i\ne j}}^{n} a_{ij}, \quad j = 1, 2, ..., n \right)$$

are located in the left half of the complex plane.

Remark 1. Asymptotically stable Metzler matrices may satisfy only one of the conditions (12). For example the matrix

$$A = \begin{bmatrix} -0.8 & 1 \\ 0.21 & -1.2 \end{bmatrix} \in M_2 \tag{13}$$

satisfies only the condition (12b) since $a_{11} = -0.8$, $a_{21} = 0.21$ but it does not satisfy the condition (12a) since $a_{12} = 1$. The positive system (11) with (13) is asymptotically stable since the polynomial

$$\begin{aligned}\det[I_2 s - A] &= \begin{vmatrix} s+0.8 & -1 \\ -0.21 & s+1.2 \end{vmatrix} \\ &= s^2 + 2s + 0.75\end{aligned} \tag{14}$$

has all positive coefficients (the condition 1) of Theorem 2).

Theorem 6. Let the subsystems of (1) be asymptotically stable, i.e. $A_{\delta(t)}$ for $\delta(t) \in S = \{1, 2, ..., N\}$ be asymptotically stable Metzler matrices. The PSCLS (1) is asymptotically stable for any switchings if the sum of entries of every column of the matrices $A_{\delta(t)}$, $\delta(t) \in S$ is negative.

Proof. By Theorem 5 the subsystems of (1) are asymptotically stable since the matrices $A_{\delta(t)}$, $\delta(t) \in S$ satisfy the condition (12b). As a common Lyapunov function for all subsystems we choose

$$V(x(t)) = 1^T_n x(t), \tag{15}$$

where $1_n^T = [\ 1 \quad ... \quad 1\ ] \in \Re_+^n$. The function (15) is positive definite for all positive subsystems since $1_n^T x(t) > 0$ for any nonzero $x(t) \in \Re_+^n$, $t \geq 0$. From (15) and (1) we have

$$\dot{V}(x(t)) = 1_n^T \dot{x}(t) = 1_n^T A_{\delta(t)} x(t) < 0 \tag{16}$$

since by assumption the sum of entries of every column of the matrices $A_{\delta(t)}$, $\delta(t) \in S$ is negative, i.e. the row vector $1_n^T A_{\delta(t)}$ has all negative components. Therefore, the positive switched system (1) is asymptotically stable for any switchings.

Example 1. Consider the positive switched system (1) with two subsystems

$$A_1 = \begin{bmatrix} -0.8 & 0.5 \\ 0.4 & -0.7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ 0.2 & -1.1 \end{bmatrix}. \tag{17}$$

The switching function $\delta(t)$ is presented on Fig. 1.

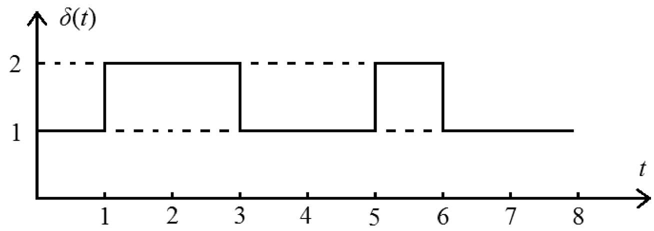

Fig. 1. Switching function $\delta(t)$ for the system (1) with matrices (17)

By Theorem 6 the positive switched system (1) with (17) is asymptotically stable for any switchings since the sum of entries of every column of the matrices (17) is negative. The same result can be obtained as follows. The matrices (17) are asymptotically stable Metzler matrices with the eigenvalues $s_{11} = -0.3$, $s_{12} = -1.2$ and $s_{21} = -0.6$, $s_{22} = -1.5$, respectively. The solution of the equation

$$\dot{x}(t) = A_1 x(t) = \begin{bmatrix} -0.8 & 0.5 \\ 0.4 & -0.7 \end{bmatrix} x(t) \tag{18}$$

has the form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{A_1 t} x_0 = \frac{1}{9} \begin{bmatrix} 4e^{-0.3t} + 5e^{-1.2t} & 5(e^{-0.3t} - e^{-1.2t}) \\ 4(e^{-0.3t} - e^{-1.2t}) & 5e^{-0.3t} + 4e^{-1.2t} \end{bmatrix} x_0 \tag{19}$$

and the solution of the equation

$$\dot{x}(t) = A_2 x(t) = \begin{bmatrix} -1 & 1 \\ 0.2 & -1.1 \end{bmatrix} x(t) \tag{20}$$

has the form

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = e^{A_2 t} x_0 = \frac{1}{9} \begin{bmatrix} 5e^{-0.6t} + 4e^{-1.5t} & 10(e^{-0.6t} - e^{-1.5t}) \\ 2(e^{-0.6t} - e^{-1.5t}) & 4e^{-0.6t} + 5e^{-1.5t} \end{bmatrix} x_0. \tag{21}$$

Taking into account the switching function $\delta(t)$, (19) and (21) we obtain

$$x(t) = \frac{1}{9} \begin{bmatrix} 4e^{-0.3t} + 5e^{-1.2t} & 5(e^{-0.3t} - e^{-1.2t}) \\ 4(e^{-0.3t} - e^{-1.2t}) & 5e^{-0.3t} + 4e^{-1.2t} \end{bmatrix} x_0 \quad \text{for} \quad 0 \leq t < 1, \tag{22a}$$

$$\begin{aligned} x(t) = \frac{1}{81} &\begin{bmatrix} 5e^{-0.6(t-1)} + 4e^{-1.5(t-1)} & 10(e^{-0.6(t-1)} - e^{-1.5(t-1)}) \\ 2(e^{-0.6(t-1)} - e^{-1.5(t-1)}) & 4e^{-0.6(t-1)} + 5e^{-1.5(t-1)} \end{bmatrix} \\ &\times \begin{bmatrix} 4e^{-0.3} + 5e^{-1.2} & 5(e^{-0.3} - e^{-1.2}) \\ 4(e^{-0.3} - e^{-1.2}) & 5e^{-0.3} + 4e^{-1.2} \end{bmatrix} x_0 \quad \text{for} \quad 1 \leq t < 3 \end{aligned} \tag{22b}$$

$$\begin{aligned} x(t) = \frac{1}{729} &\begin{bmatrix} 4e^{-0.3(t-3)} + 5e^{-1.2(t-3)} & 5(e^{-0.3(t-3)} - e^{-1.2(t-3)}) \\ 4(e^{-0.3(t-3)} - e^{-1.2(t-3)}) & 5e^{-0.3(t-3)} + 4e^{-1.2(t-3)} \end{bmatrix} \\ &\times \begin{bmatrix} 5e^{-1.2} + 4e^{-3} & 10(e^{-1.2} - e^{-3}) \\ 2(e^{-1.2} - e^{-3}) & 4e^{-1.2} + 5e^{-3} \end{bmatrix} \\ &\times \begin{bmatrix} 4e^{-0.3} + 5e^{-1.2} & 5(e^{-0.3} - e^{-1.2}) \\ 4(e^{-0.3} - e^{-1.2}) & 5e^{-0.3} + 4e^{-1.2} \end{bmatrix} x_0 \quad \text{for} \quad 3 \leq t < 5. \end{aligned} \tag{22c}$$

Form (22) it follows that the swichted system is asymptotically stable for any swichtings.

3.2. Positive discrete-time systems. Consider the positive discrete-time linear system

$$x_{i+1} = \overline{A}x_i, \tag{23}$$

where

$$x_i \in \Re_+^n, \;\; i \in Z_+ \quad \text{and} \quad \overline{A} \in \Re_+^{n\times n}.$$

Theorem 7. The positive discrete-time system (23) is asymptotically stable if

$$\sum_{j=1}^{n} a_{ij} < 1 \quad \text{for} \quad i = 1,2,...,n \tag{24a}$$

or

$$\sum_{i=1}^{n} a_{ij} < 1 \quad \text{for} \quad j = 1,2,...,n. \tag{24b}$$

Proof. The positive system (23) is asymptotically stable if the condition (24a) (or (24b)) is met since by Gershgorin's Circle Theorem [15] all discs centered at the points a_{ii} (a_{jj}) with the radii

$$r_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}, \;\; i = 1,2,...,n \left(r_j = \sum_{\substack{i=1 \\ i\neq j}}^{n} a_{ij}, \;\; j = 1,2,...,n \right)$$

are located in the unit circle.

Remark 2. Asymptotically stable matrices with nonnegative entries may satisfy only one of the conditions (24). For example the matrix

$$\overline{A} = \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.3 \end{bmatrix} \in \Re_+^{2\times 2} \tag{25}$$

satisfies only the condition (24a) since $a_{11} + a_{12} = 0.9$ and $a_{21} + a_{22} = 0.8$ but $a_{11} + a_{21} = 1$. The positive system (23) with (25) is asymptotically stable since the polynomial

$$\begin{aligned} \det[I_2(z+1) - \overline{A}] &= \begin{vmatrix} z+0.5 & -0.4 \\ -0.5 & z+0.7 \end{vmatrix} \\ &= z^2 + 1.2z + 0.15 \end{aligned} \tag{26}$$

has all positive coefficients (the condition 1) of Theorem 4).

Theorem 8. Let the subsystems (6) be asymptotically stable, i.e. $\overline{A}_{\delta_i} \in \Re_+^{n\times n}$ for $\delta_i \in S = \{1,2,...,N\}$ be asymptotically stable matrices. The PSDLS (6) is asymptotically stable for any switchings if the sum of entries of every column of the matrices $\overline{A}_{\delta_i}$, $\delta_i \in S$ is less than 1.

Proof. By Theorem 7 the subsystems of (6) are asymptotically stable since the matrices $\overline{A}_{\delta_i}$, $\delta_i \in S$ satisfy the condition (24a). As a common Lyapunov function for all subsystems we choose

$$V(x_i) = 1_n^T x_i, \tag{27}$$

where

$$1_n^T = [\;1 \quad ... \quad 1\;]^{\in}\Re_+^n.$$

The function (27) is positive definite for all positive subsystems since $1_n^T x_i > 0$ for any nonzero $x_i \in \Re_+^n$, $i \in Z_+$. From (27) and (6) we have

$$\begin{aligned} \Delta V(x_i) &= V(x_{i+1}) - V(x_i) = 1_n^T(x_{i+1} - x_i) \\ &= 1_n^T(\overline{A}_{\delta_i} - I_n)x_i < 0 \end{aligned} \tag{28}$$

since by assumption the sum of entries of every column of the matrices $\overline{A}_{\delta_i}$, $\delta_i \in S$ is less than 1, i.e. the row vector $1_n^T(\overline{A}_{\delta_i} - I_n)$ has all negative components. Therefore, the positive switched system (6) is asymptotically stable for any switchings.

4. Concluding remarks

Simple sufficient conditions for the asymptotic stability of positive switched continuous-time (Theorem 6) and discrete-time (Theorem 8) linear systems for any switchings have been established. It has been shown that the positive switched continuous-time (discrete-time) system is asymptotically stable for any switchings if the sum of entries of every column of the matrices of subsystem is negative (less than 1). Note that the well-known [10] condition that the matrices of subsystems commute is not necessary for the asymptotic stability of the positive switched systems for any switchings. The effectiveness of the presented sufficient conditions is demonstrated on a numerical example of continuous-time positive switched linear system. The considerations can be extended to the Lyapunov functions (15), where $1_n = \lambda$, where $\lambda = (A_{\delta(1)})^{-1}1_n \in \Re_+^n$ is a strictly positive vector.

Following [7, 8] the presented sufficient conditions can be extended to the positive switched 2D linear systems.

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REFERENCES

[1] A. Benzaouia and F. Tadeo, "Stabilization of positive switching linear discrete-time systems", *Int. J. Innovative Computing, Information and Control* 6 (4), 2427–2437 (2010).

[2] S. Bundfuss and M. Dur, "Copositive Lyapunov functions for switched systems over cones", *System and Control Letters* 58 (5), 342–345 (2009).

[3] L. Farina and S. Rinaldi, *Positive Linear Systems; Theory and Applications*, J. Wiley, New York, 2000.

[4] E. Fornasini and M.E. Valcher, "Stability and stabilizability criteria for discrete-time positive switched systems", *IEEE Trans. Autom. Control* 57 (5), 1208–1221 (2012).

[5] L. Gurvits, R. Shorten, and O. Mason, "On the stability of switched positive linear systems", *IEEE Trans. Autom. Control*. 52, 1099–1103 (2007).

[6] T. Kaczorek, *Positive 1D and 2D Systems*, Springer Verlag, London, 2002.

[7] T. Kaczorek, "Positive switched 2D linear systems described by the Roesser models", *Proc. 19th Int. Symp. Math. Theory of Network and Systems* 1, CD-ROM (2012).

[8] T. Kaczorek, "Positive switched 2D linear systems described by general model", *Acta Mechanica et Automatica* 4, 36–41 (2010).

[9] T. Kaczorek, "Choice of the forms of Lyapunov functions for 2D Roesser model", *Int. J. Apply. Math. and Comp. Sci*. 17, 471–475 (2007).

[10] D. Liberzon, *Switching in System and Control*, Birklauser, Berlin, 2003.

[11] X.W. Liu, "Stability analysis of switched positive systems: a switched linear copositive Lyapunov function method", *IEEE Trans. Circ. Sys. II, Express Brief* 56, 414–418 (2009).

[12] X.W. Liu, L. Wang, W.S. Yu, and S.M. Zhong, "Constrained control of positive discrete-time systems with delays", *IEEE Trans. Circ. Sys. II, Express Brief* 55, 193–197 (2008).

[13] X.W. Liu and C.Y. Dang, "Stability analysis of positive switched linear systems with delays", *IEEE Trans. Autom. Control* 56, 1684–1690 (2011).

[14] O. Mason and R. Shorten, "On linear copositive Lyapunov functions and the stability of switched positive linear systems", *IEEE Trans. Autom. Control* 52, 1346–1349 (2007).

[15] R.S. Varga, *Matrix Interactive Analysis*, Springer-Verlag, Berlin, 2002.

[16] X.D. Zhao, L.X. Zhang, P. Shi, and M. Liu, "Stability of switched positive linear systems with average dwell time switching", *Automatica* 48, 1132–1137 (2012).