

A UNIFYING DEFINITION OF A SUBTEMPERATURE

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Abstract. We present a new definition of a subtemperature, and show that it unifies the potential theory of the heat operator from the outset, as the subcaloric functions and the subtemperatures defined earlier, arise as characterizations of the new subtemperatures almost simultaneously.

1. Introduction

The potential theory of the heat operator

$$\Theta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial t}$$

in \mathbb{R}^{n+1} , is now well developed, but it has been developed in two different ways. First there was the harmonic space approach of Bauer [2], [3], and second the heat ball approach of the present author [10], [11]. In [4], Bauer proved the equivalence of the two approaches. More precisely, he showed that the subsolutions in the two theories - the subcaloric functions and the subtemperatures - are the same. He used relatively sophisticated results from both theories. In this paper, we present a new approach using a new definition of subtemperature. This unifies the theory from the outset, as the two earlier definitions arise as characterizations of subtemperatures, almost simultaneously.

The first thing we need for our new approach is a bounded domain $D \subseteq \mathbb{R}^{n+1}$ for which it can be proved that, given any real valued, continuous function f on the appropriate part of the boundary ∂D , there is a unique function $u \in C(\overline{D})$ such that u is a temperature (solution of the heat equation) on D and $u = f$ where f is defined. For D , we could take a rectangle (an $(n+1)$ -dimensional interval), in view of the work of Hattner [8], but it fits in better with our overall approach to take for D a circular cylinder $B \times]a, b[$, where B is an open euclidean ball in \mathbb{R}^n . With this choice of D , the existence of u has been proved using the traditional method of double layer heat potentials, for example in [7]. This does not give such an explicit representation as was obtained for a rectangle in [8], but that is nowhere important.

We present several preliminary results in Section 2. In particular, we state a well-known theorem on the existence of solutions to the Dirichlet problem for a circular cylinder in space-time \mathbb{R}^{n+1} , reformulate it in terms of caloric measure, use the caloric measure to define integral mean values, and give some results about those mean values. In Section 3, we present the new definition of a subtemperature,

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which is given in terms of the mean values defined in Section 2. The remainder of Section 3, and all the subsequent sections, are devoted to developing the theory of the newly defined subtemperatures to the point where we can show that they are the same functions as the old subtemperatures and the subcaloric functions.

2. Notation and Preliminary Results

We denote by W the *Fundamental Temperature*, defined for all $(x, t) \in \mathbb{R}^{n+1}$ by

$$W(x, t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

For any point $p_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and any $c > 0$, the set

$$\Omega(p_0; c) = \Omega(x_0, t_0; c) = \{(y, s) \in \mathbb{R}^{n+1} : W(x_0 - y, t_0 - s) > (4\pi c)^{-\frac{n}{2}}\}$$

is called the *Heat Ball* with *centre* (x_0, t_0) and *radius* c . The boundary of the heat ball $\Omega(x_0, t_0; c)$ is called the *Heat Sphere* (with centre (x_0, t_0) and radius c).

Definition. The *fundamental mean value over heat spheres* is defined by

$$\mathcal{M}(u; x_0, t_0; c) = (4\pi c)^{-\frac{n}{2}} \int_{\partial\Omega(x_0, t_0; c)} Q(x_0 - x, t_0 - t) u(x, t) \, d\sigma$$

for any function u such that the integral exists. Here σ denotes surface area measure on $\partial\Omega(x_0, t_0; c)$, and

$$Q(x_0 - x, t_0 - t) = \frac{|x_0 - x|^2}{\left(4|x_0 - x|^2(t_0 - t)^2 + (|x_0 - x|^2 - 2n(t_0 - t))^2\right)^{1/2}}.$$

See [5] for details.

Definition. Given a function u on the heat ball $\Omega(x_0, t_0; c)$ for which the integral exists, we define the *volume mean value* of u by

$$\begin{aligned} \mathcal{V}(u; x_0, t_0; c) &= \frac{n}{2} c^{-\frac{n}{2}} \int_0^c l^{\frac{n}{2}-1} \mathcal{M}(u; x_0, t_0; l) \, dl \\ &= (4\pi c)^{-\frac{n}{2}} \int \int_{\Omega(x_0, t_0; c)} \frac{|x_0 - x|^2}{4(t_0 - t)^2} u(x, t) \, dx \, dt. \end{aligned} \tag{2.1}$$

See [10] for details.

Given an open set E and a point $p_0 = (x_0, t_0) \in E$, we denote by $\Lambda(p_0; E)$ (or $\Lambda(x_0, t_0; E)$) the set of points p that are lower than p_0 relative to E , in the sense that there is a polygonal path $\gamma \subseteq E$ joining p_0 to p , along which the temporal variable t is strictly decreasing. By a polygonal path, we mean a path which is a union of finitely many line segments.

Definition. A family \mathcal{F} of functions on E is said to be *upward-directed* if, for each pair $u, v \in \mathcal{F}$, there exists a $w \in \mathcal{F}$ such that $u \vee v \leq w$, where $u \vee v = \max\{u, v\}$. Similarly, \mathcal{F} is said to be *downward-directed* if $u, v \in \mathcal{F}$ implies that there is $w \in \mathcal{F}$

such that $u \wedge v \geq w$, where $u \wedge v = \min\{u, v\}$.

Theorem 1. *Let \mathcal{F} be an upward-directed family of temperatures on an open set E , and let $u = \sup \mathcal{F}$. If there is a point $p_0 \in E$ such that $u(p_0) < +\infty$, then u is a temperature on $\Lambda(p_0; E)$.*

Proof. This simple proof is adapted from that for the harmonic case given in [1]. Let K be any compact subset of $\Lambda(p_0; E)$. For each positive integer k , we can find a function $u_k \in \mathcal{F}$ such that

$$u(p_0) - u_k(p_0) < \frac{1}{k}.$$

Since \mathcal{F} is upward-directed, given any function $v \in \mathcal{F}$ and $k \in \mathbb{N}$, we can find a temperature $w_k \in \mathcal{F}$ such that $u_k \vee v \leq w_k$ on E . By the Harnack inequality for temperatures (see, for example, [13]), there is a positive constant κ , depending only on E , p_0 and K , such that

$$w_k(p) - u_k(p) \leq \kappa(w_k(p_0) - u_k(p_0))$$

for all $p \in K$ and all k . Hence

$$v(p) - u_k(p) \leq w_k(p) - u_k(p) \leq \kappa(w_k(p_0) - u_k(p_0)) \leq \kappa(u(p_0) - u_k(p_0)) < \frac{\kappa}{k}$$

for all $p \in K$. Therefore

$$u(p) - u_k(p) = \sup\{v(p) - u_k(p) : v \in \mathcal{F}\} \leq \frac{\kappa}{k}$$

for all $p \in K$, so that the sequence $\{u_k\}$ converges uniformly to u on K . It now follows that u is a temperature on $\Lambda(p_0; E)$ (see, for example, [10] Theorem 5 Corollary).

Given any set $S \in \mathbb{R}^{n+1}$, we denote by $C(S)$ the class of all continuous, real valued functions on S . We also denote by $C^{2,1}(S)$ the set of real valued functions u on S such that the partial derivatives $\partial^2 u / \partial x_i \partial x_j$ ($i, j \in \{1, \dots, n\}$) and $\partial u / \partial t$ all exist and are continuous on S .

We consider an open ball B in \mathbb{R}^n , and a bounded time interval $]a, b[$, and denote by D the circular cylinder $D = B \times]a, b[\subseteq \mathbb{R}^{n+1}$. We denote by $\partial_n D$ the *normal boundary* of D , which consists of the union of the *lateral surface* $\partial B \times]a, b[$ and the *initial surface* $\overline{B} \times \{a\}$. The *Dirichlet Problem* on D consists of showing that, for any function $f \in C(\partial_n D)$, there is a temperature u on D which has a continuous extension by f to $\partial_n D$. It transpires that the function u is actually a temperature on $\overline{D} \setminus \partial_n D$, which means that $u \in C^{2,1}(\overline{D} \setminus \partial_n D)$ and satisfies the heat equation there.

Theorem 2. *Let $f \in C(\partial_n D)$. Then there is a function $u_f \in C(\overline{D})$ such that u_f is a temperature on $\overline{D} \setminus \partial_n D$ and $u_f = f$ on $\partial_n D$.*

Proof. See, for example, [7].

Theorem 3. *Let $f \in C(\partial_n D)$, and let u_f be the temperature on $\overline{D} \setminus \partial_n D$ associated with f by Theorem 2. Then, given any point $p \in \overline{D} \setminus \partial_n D$, there is a unique*

positive Borel measure μ_p on $\partial_n D$ such that

$$u_f(p) = \int_{\partial_n D} f \, d\mu_p.$$

Proof. We show that the mapping $f \mapsto u_f(p)$ is a positive linear functional on the Banach space $C(\partial_n D)$ with the supremum norm.

If $f \geq 0$, then the boundary minimum principle shows that $u_f \geq 0$.

If $\alpha \in \mathbb{R}$, then αf is continuous on $\partial_n D$, and so there is a temperature $u_{\alpha f}$ associated with αf by Theorem 2. Furthermore,

$$\lim_{p \rightarrow q, p \in D} (u_{\alpha f}(p) - \alpha u_f(p)) = \lim_{p \rightarrow q, p \in D} u_{\alpha f}(p) - \alpha \lim_{p \rightarrow q, p \in D} u_f(p) = 0$$

for all $q \in \partial_n D$, so that the boundary uniqueness principle shows that $u_{\alpha f} = \alpha u_f$ on D .

If g is another continuous function on $\partial_n D$, then so is $f + g$, and hence there is a temperature u_{f+g} associated with $f + g$ by Theorem 2. Also,

$$\begin{aligned} \lim_{p \rightarrow q, p \in D} (u_{f+g}(p) - u_f(p) - u_g(p)) \\ &= \lim_{p \rightarrow q, p \in D} u_{f+g}(p) - \lim_{p \rightarrow q, p \in D} u_f(p) - \lim_{p \rightarrow q, p \in D} u_g(p) \\ &= 0 \end{aligned}$$

for all $q \in \partial_n D$, so that $u_{f+g} - u_f - u_g = 0$ on D , by the boundary uniqueness principle.

Thus, given any $p \in \overline{D} \setminus \partial_n D$, the mapping $f \mapsto u_f(p)$ is a positive linear functional on $C(\partial_n D)$. It now follows from the Riesz representation theorem that there is a unique positive Borel measure μ_p on $\partial_n D$ such that

$$u_f(p) = \int_{\partial_n D} f \, d\mu_p.$$

Definitions. The measure μ_p in Theorem 3 is called the *Caloric Measure at p for D* , and the integral is called the *Poisson Integral of f* .

Since temperatures are invariant under translation and parabolic dilation, the caloric measure has similar properties. To see this, let $f \in C(\partial_n D)$, and let u_f be the Poisson integral of f . Take a translation of the cylinder D to another cylinder $D_0 = D + \{p_0\}$, and define a function f_0 on $\partial_n D_0$ by putting $f_0(q) = f(q - p_0)$. If u_{f_0} is the Poisson integral of f_0 , then for $p \in \overline{D} \setminus \partial_n D$ we have

$$u_{f_0}(p+p_0) = \int_{\partial_n D_0} f_0(q) \, d\mu_{p+p_0}(q) = \int_{\partial_n D_0} f(q-p_0) \, d\mu_{p+p_0}(q) = \int_{\partial_n D} f(q) \, d\nu_p(q),$$

where ν_p is the translation of μ_{p+p_0} from $\partial_n D_0$ to $\partial_n D$. Putting $v_{f_0}(p) = u_{f_0}(p+p_0)$, we get a temperature v_{f_0} on D with continuous boundary values f on $\partial_n D$. So $v_{f_0}(p) = u_f(p)$ by the boundary uniqueness principle, and hence

$$u_f(p) = \int_{\partial_n D} f(q) \, d\nu_p(q),$$

for any $f \in C(\partial_n D)$. So, by the uniqueness of the measure in Theorem 3, $\mu_p = \nu_p$.

For the parabolic dilation, we can now take

$$D = \{(y, s) : |y| < \sqrt{c}, -b < s < 0\}$$

without loss of generality, and dilate it to another cylinder

$$D_1 = \{(y, s) : |y| < \sqrt{ac}, -ab < s < 0\}.$$

Let u_f be as before, and define a function f_1 on $\partial_n D_1$ by putting

$$f_1(y, s) = f\left(\frac{y}{\sqrt{a}}, \frac{s}{a}\right).$$

If u_{f_1} is the Poisson integral of f_1 , then for $(x, t) \in \overline{D} \setminus \partial_n D$ we have

$$\begin{aligned} u_{f_1}(x\sqrt{a}, ta) &= \int_{\partial_n D_1} f_1(y, s) d\mu_{(x\sqrt{a}, ta)}(y, s) \\ &= \int_{\partial_n D_1} f\left(\frac{y}{\sqrt{a}}, \frac{s}{a}\right) d\mu_{(x\sqrt{a}, ta)}(y, s) \\ &= \int_{\partial_n D} f(y, s) d\chi_{(x,t)}(y, s), \end{aligned}$$

where $\chi_{(x,t)}$ is the parabolic dilation of $\mu_{(x\sqrt{a}, ta)}$ from $\partial_n D_1$ to $\partial_n D$. Putting $v_{f_1}(x, t) = u_{f_1}(x\sqrt{a}, ta)$, we get a temperature v_{f_1} on D with continuous boundary values f on $\partial_n D$. So $v_{f_1} = u_f$, and hence

$$u_f(x, t) = \int_{\partial_n D} f(y, s) d\chi_{(x,t)}(y, s),$$

for any $f \in C(\partial_n D)$. So, by the uniqueness of the measure in Theorem 3, we have

$$\mu_{(x,t)} = \chi_{(x,t)}.$$

We need some information on the sets of caloric measure zero.

Lemma 1. *Let $p_0 = (x_0, t_0) \in \overline{D} \setminus \partial_n D$, and let μ_{p_0} be the caloric measure at p_0 for D . Then*

$$(a) \mu_{p_0}(\{(y, s) \in \partial_n D : s \geq t_0\}) = 0,$$

and

(b) if V is any relatively open subset of $\{(y, s) \in \partial_n D : s < t_0\}$, then $\mu_{p_0}(V) > 0$.

Proof. (a) Let $D = B \times]a, b[$, where B is an open ball in \mathbb{R}^n . We choose a number $b^* > b$, and put $D^* = B \times]a, b^*[$. We also choose a decreasing sequence $\{f_k\}$ of functions in $C(\partial_n D^*)$ such that $f_k(y, s) = 1$ if $s \geq t_0$, $f_k(y, s) = 0$ if $s \leq t_0 - \frac{1}{k}(t_0 - a)$, and $f_k(y, s) \rightarrow 0$ as $k \rightarrow \infty$ whenever $t_0 - \frac{1}{k}(t_0 - a) < s < t_0$. Let u_k be the function in $C(\overline{D})$ associated with the restriction of f_k to $\partial_n D$ by Theorem 2, and u_k^* be that in $C(\overline{D}^*)$ associated with f_k itself. Then $u_k = f_k = u_k^*$ on $\partial_n D$, so that $u_k = u_k^*$ on \overline{D} by the boundary uniqueness principle. Since $\{f_k\}$ is a decreasing sequence, so are $\{u_k\}$ and $\{u_k^*\}$. Consider the restriction of u_k^* to the set $\{(x, t) \in \overline{D}^* : t \leq t_0 - \frac{1}{k}(t_0 - a)\}$. Since $u_k^* = 0$ on the normal boundary, the boundary uniqueness principle shows that $u_k^* = 0$ throughout that cylinder. Put $u^* = \lim_{k \rightarrow \infty} u_k^*$ on D^* , and let $T = \{(y, s) \in \partial_n D : s \geq t_0\}$. The Harnack monotone convergence theorem (see, for example, [13]) can be applied

to the increasing sequence $\{u_1^* - u_k^*\}$ of nonnegative temperatures, to show that $u_1^* - u^*$ is a temperature on D^* . Hence u^* is a temperature also. Since $u_k^*(x, t) = 0$ whenever $t \leq t_0 - \frac{1}{k}(t_0 - a)$, we have $u^*(x, t) = 0$ whenever $t < t_0$ and so, by continuity, whenever $t \leq t_0$. Since $u^* = \lim_{k \rightarrow \infty} u_k$ on $\overline{D} \setminus \partial_n D$, it now follows from Lebesgue's monotone convergence theorem that

$$0 = \lim_{k \rightarrow \infty} u_k(x_0, t_0) = \int_{\partial_n D} \lim_{k \rightarrow \infty} f_k \, d\mu_{p_0} = \int_T d\mu_{p_0} = \mu_{p_0}(T).$$

(b) We choose a function $f \in C(\partial_n D)$ such that $f \geq 0$ on $\partial_n D$, $f = 0$ except on V , and $f(q_0) = 1$ for some point $q_0 \in V$. Let u be the function in $C(\overline{D})$ associated with f by Theorem 2, so that $u = f$ on $\partial_n D$. Then $u \geq 0$ by the minimum principle, and $u(p) \rightarrow 1$ as $p \rightarrow q_0$. If we had $\mu_{p_0}(V) = 0$, it would follow from Theorem 3 that

$$u(p_0) = \int_{\partial_n D} f \, d\mu_{p_0} = \int_V f \, d\mu_{p_0} = 0,$$

which implies that $u(x, t) = 0$ whenever $t < t_0$ by the strong minimum principle, contrary to the fact that $u(p) \rightarrow 1$ as $p \rightarrow q_0$.

We now characterize temperatures in terms of the Poisson Integral.

Theorem 4. *If u is a temperature on an open set E , and D is a circular cylinder with $\overline{D} \subseteq E$, then u has the representation*

$$u(p) = \int_{\partial_n D} u \, d\mu_p$$

for all $p \in \overline{D} \setminus \partial_n D$, where μ_p is the caloric measure at p for D .

Conversely, suppose that $u \in C(E)$ and that, for each point $p_0 \in E$, there is some circular cylinder D containing p_0 such that $\overline{D} \subseteq E$ and u has the representation

$$u(p) = \int_{\partial_n D} u \, d\mu_p$$

for all $p \in D$, where μ_p is the caloric measure at p for D . Then u is a temperature on E .

Proof. Suppose that u is a temperature on E , and that D is a circular cylinder with $\overline{D} \subseteq E$. Let f be the restriction of u to $\partial_n D$. By Theorem 2, there is a function $u_f \in C(\overline{D})$ such that u_f is a temperature on $\overline{D} \setminus \partial_n D$ and $u_f = f$ on $\partial_n D$. By Theorem 3, u_f has the representation

$$u_f(p) = \int_{\partial_n D} f \, d\mu_p$$

for all $p \in \overline{D} \setminus \partial_n D$. The functions u and u_f belong to $C(\overline{D} \setminus \partial_n D)$, are temperatures on D , and are equal on $\partial_n D$. Therefore $u = u_f$ on D and hence, by continuity, on \overline{D} . So u has the required representation.

To prove the converse, take any $p_0 \in E$. Then there is some circular cylinder D containing p_0 , such that $\overline{D} \subseteq E$ and

$$u(p) = \int_{\partial_n D} u \, d\mu_p$$

for all $p \in D$. Let f be the restriction of u to $\partial_n D$. Then, by Theorems 2 and 3, there is a function $u_f \in C(\overline{D})$ such that u_f is a temperature on D , and

$$u_f(p) = \int_{\partial_n D} f \, d\mu_p$$

for all $p \in D$. So $u = u_f$ on D . Hence u is a temperature on a neighbourhood of the arbitrary point p_0 , and therefore on E .

The Poisson integral representation also gives a mean value characterization of temperatures, as follows.

Definitions. For each $(x, t) \in \mathbb{R}^{n+1}$ and $c > 0$, we put

$$\Delta(x, t; c) = B(x, \sqrt{c}) \times]t - c, t[,$$

where $B(x, \sqrt{c})$ denotes the open ball in \mathbb{R}^n with centre x and radius \sqrt{c} . We call $\Delta(x, t; c)$ the *Heat Cylinder* with centre (x, t) and radius c .

The *mean value over normal boundary of the heat cylinder* is defined, for any function u such that the integral exists, by

$$\mathcal{L}(u; x, t; c) = \int_{\partial_n \Delta(x, t; c)} u \, d\mu_{(x, t)},$$

where $\mu_{(x, t)}$ is the caloric measure at (x, t) for $\Delta(x, t; c)$. Since the caloric measure is invariant under translation and parabolic dilation, the mean $\mathcal{L}(u; x, t; c)$ depends only on u , (x, t) and c .

Note that, by taking $u = 1$ in Theorem 4, we obtain $\mathcal{L}(1; x, t; c) = 1$ for all (x, t) and c .

The proof of our characterization of temperatures in terms of the means $\mathcal{L}(u; x, t; c)$, depends on showing that a continuous function which satisfies a weak mean value property also satisfies the maximum principle.

Theorem 5. *Let $D = B \times]a, b[$ be an arbitrary circular cylinder in \mathbb{R}^{n+1} , and let $u \in C(D \cup \partial_n D)$. If, given any point $(x, t) \in D$ and $\epsilon > 0$, we can find a positive number $c < \epsilon$ such that $u(x, t) \leq \mathcal{L}(u; x, t; c)$ holds, then u satisfies the maximum principle. That is, if there is a point $(x_0, t_0) \in D$ such that $u(x_0, t_0) \geq u(x, t)$ whenever $(x, t) \in D$ and $t < t_0$, then $u(x_0, t_0) = u(x, t)$ for all such points (x, t) ; consequently*

$$\sup_{D \cup \partial_n D} u = \max_{\partial_n D} u.$$

Proof. Suppose that there is a point $(x_0, t_0) \in D$ such that $u(x_0, t_0) \geq u(x, t)$ whenever $(x, t) \in D$ and $t < t_0$. Put $M = u(x_0, t_0)$, and let (x_1, t_1) be any point of D such that $t_1 < t_0$. Join (x_0, t_0) and (x_1, t_1) with a closed line segment γ , and put

$$S = \{s : \text{there is a point } (y, s) \in \gamma \text{ with } u(y, s) = M\}.$$

Then $S \neq \emptyset$ because $t_0 \in S$, and S is lower bounded by t_1 . Put $s^* = \inf S$. If $\overline{\Delta}(x_0, t_0; c) \subseteq E$, then because Lemma 1 shows that the caloric measure at (x_0, t_0) of $\partial_n \Delta(x_0, t_0; c) \setminus \Lambda(x_0, t_0; E)$ is zero, we have $u \leq M$ almost everywhere

on $\partial_n \Delta(x_0, t_0; c)$ with respect to that measure. So, by our hypothesis, there is a number $c < t_0 - t_1$ such that

$$M = u(x_0, t_0) \leq \mathcal{L}(u; x_0, t_0; c) \leq \mathcal{L}(M; x_0, t_0; c) = M,$$

and hence $u = M$ on a dense subset of $\partial_n \Delta(x_0, t_0; c)$, by Lemma 1. The continuity of u now implies that $u \equiv M$ on $\partial_n \Delta(x_0, t_0; c)$. Since $c < t_0 - t_1$, the intersection $\gamma \cap \partial_n \Delta(x_0, t_0; c) \neq \emptyset$, so that we can find a point $(y_1, s_1) \in \gamma$ such that $s_1 < t_0$ and $u(y_1, s_1) = M$. Therefore $s^* < t_0$.

Suppose that $t_1 < s^* < t_0$. Then there is a sequence of points $\{(z_k, r_k)\}$ on γ such that $u(z_k, r_k) = M$ for all k and $r_k \rightarrow s^*$ as $k \rightarrow \infty$. The continuity of u now implies that there is a point (y^*, s^*) on γ such that $u(y^*, s^*) = M$. By our hypothesis and an argument similar to the one above, we can find $c < s^* - t_1$ such that $u \equiv M$ on $\partial_n \Delta(y^*, s^*; c)$, and therefore a point $s_2 \in S$ such that $s_2 < s^*$, so we have a contradiction. Hence $s^* = t_1$, and $u(x_1, t_1) = M$.

For the last part, given any α such that $a < \alpha < b$, we put $D_\alpha = B \times]a, \alpha[$ and $M_\alpha = \max\{u(p) : p \in \overline{D}_\alpha\}$. Choose a point $(x', t') \in \overline{D}_\alpha$ such that $u(x', t') = M_\alpha$. If $(x', t') \in D$, then the first part of the theorem shows that $u(x, t) = M_\alpha$ for all $(x, t) \in \overline{D}_\alpha$ such that $t < t'$, so we can assume that $(x', t') \in \overline{D}_\alpha \setminus D$. It follows that

$$\sup_{D \cup \partial_n D} u = \sup_{\alpha \in]a, b[} M_\alpha$$

is attained at some point of $\partial_n D$.

We can now give our mean value characterization of temperatures.

Theorem 6. *If u is a temperature on an open set E and $(x, t) \in E$, then the equality $u(x, t) = \mathcal{L}(u; x, t; c)$ holds whenever $\overline{\Delta}(x, t; c) \subseteq E$.*

Conversely, if $u \in C(E)$ and, given any point $(x, t) \in E$ and $\epsilon > 0$, we can find a positive number $c < \epsilon$ such that $u(x, t) = \mathcal{L}(u; x, t; c)$ holds, then u is a temperature on E .

Proof. If u is a temperature on E , then the first part follows from Theorem 4.

Conversely, suppose that $u \in C(E)$, and let D be an arbitrary circular cylinder such that $\overline{D} \subseteq E$. Let f denote the restriction of u to $\partial_n D$. By Theorem 2, there is a function $u_f \in C(\overline{D})$ which is a temperature on D and satisfies $u_f = f$ on $\partial_n D$. By Theorem 4, whenever $\Delta(x, t; c)$ is a heat cylinder such that $\overline{\Delta}(x, t; c) \subseteq D$, the equality

$$u_f(x, t) = \mathcal{L}(u_f; x, t; c)$$

holds. Therefore, if $v = u - u_f$ on \overline{D} , then v satisfies the same condition that u satisfies on E . Hence, by applying Theorem 5 to both v and $-v$, we obtain

$$0 = \min_{\partial_n D} v = \inf_{D \cup \partial_n D} v \leq \sup_{D \cup \partial_n D} v = \max_{\partial_n D} v = 0,$$

so that $u = u_f$ on D . Thus u is a temperature on any circular cylinder D such that $\overline{D} \subseteq E$, and hence on E .

3. The New Definition of Subtemperature

We define subtemperatures in terms of the means \mathcal{L} .

Definition. Let w be an extended-real valued function on an open set E . We call w a *subtemperature* on E if it satisfies the following four conditions.

- (δ_1) $-\infty \leq w(p) < +\infty$ for all $p \in E$.
- (δ_2) w is upper semicontinuous on E .
- (δ_3) w is finite on a dense subset of E .
- (δ_4) Given any point $p \in E$ and $\epsilon > 0$, there is a positive number $c < \epsilon$ such that $\overline{\Delta}(p; c) \subseteq E$ and $w(p) \leq \mathcal{L}(w; p; c)$.

If w is a subtemperature on E , and V is an open subset of E , then w is a subtemperature on V .

We call a function v a *supertemperature* on E if $-v$ is a subtemperature on E . Note that v is a temperature on E if and only if it is both a subtemperature and a supertemperature on E , in view of Theorem 6.

We refer to any function w that satisfies condition (δ_1) as *upper finite*. If w is both upper finite and upper semicontinuous, then w is locally upper bounded, so that the means $\mathcal{L}(w; p; c)$ (for $\overline{\Delta}(p; c) \subseteq E$) always exist and are never $+\infty$.

If w is a subtemperature on E , and a is a positive number, then aw is a subtemperature on E .

Note that, if v and w are both subtemperatures on E , we cannot in general conclude that $v + w$ is also a subtemperature. The conditions (δ_3) and (δ_4) are both too weak for that. If either v or w is a temperature, then $v + w$ is a subtemperature, in view of Theorem 6. The general case is given in Theorem 21 Corollary 1 below.

If u is a temperature on E , and ϕ is a convex function defined on an interval containing $u(E)$, then $\phi \circ u$ is a subtemperature on E . This follows from the fact that $\phi \circ u \in C(E)$ and Jensen's inequality.

Theorem 7. (The Strong Maximum Principle.) *Let w be a subtemperature on an open set E . If there is a point $(x_0, t_0) \in E$ such that $w(x_0, t_0) \geq w(x, t)$ for all $(x, t) \in \Lambda(x_0, t_0; E)$, then $w(x_0, t_0) = w(x, t)$ for all such points (x, t) .*

Proof. Put $M = w(x_0, t_0)$, and let (x_1, t_1) be an arbitrary point of $\Lambda(x_0, t_0; E)$. Let γ be a polygonal path in E that connects (x_0, t_0) to (x_1, t_1) , along which the temporal variable is strictly decreasing. Put

$$S = \{s : \text{there is a point } (y, s) \in \gamma \text{ with } w(y, s) = M\}.$$

Then $S \neq \emptyset$ because $t_0 \in S$, and S is lower bounded by t_1 . Put $s^* = \inf S$. If $\overline{\Delta}(x_0, t_0; c) \subseteq E$, then because Lemma 1 shows that the caloric measure at (x_0, t_0) of $\partial_n \Delta(x_0, t_0; c) \setminus \Lambda(x_0, t_0; E)$ is zero, we have $w \leq M$ almost everywhere on $\partial_n \Delta(x_0, t_0; c)$ with respect to that measure. So, since w satisfies condition (δ_4), there is a number $c < t_0 - t_1$ such that

$$M = w(x_0, t_0) \leq \mathcal{L}(w; x_0, t_0; c) \leq \mathcal{L}(M; x_0, t_0; c) = M,$$

and hence $w = M$ on a dense subset of $\partial_n \Delta(x_0, t_0; c)$, by Lemma 1. Therefore, for any point $(y, s) \in \partial_n \Delta(x_0, t_0; c)$ such that $s < t_0$,

$$M = \limsup_{(x,t) \rightarrow (y,s)} w(x, t) \leq w(y, s) \leq M.$$

Since $c < t_0 - t_1$, the set $\gamma \cap \partial_n \Delta(x_0, t_0; c) \neq \emptyset$, so that we can find a point $(y_1, s_1) \in \gamma$ such that $s_1 < t_0$ and $w(y_1, s_1) = M$. Therefore $s^* < t_0$.

Suppose that $t_1 < s^* < t_0$. Then there is a sequence of points $\{(z_k, r_k)\}$ on γ such that $w(z_k, r_k) = M$ for all k , and $r_k \rightarrow s^*$ as $k \rightarrow \infty$. This implies first that there is a point $(y^*, s^*) \in \gamma$, and second that, since w is upper semicontinuous,

$$M = \lim_{k \rightarrow \infty} w(z_k, r_k) \leq w(y^*, s^*) \leq M.$$

Hence $s^* \in S$. Therefore, since w satisfies condition (δ_4) , there is $c < s^* - t_1$ such that

$$M = w(y^*, s^*) \leq \mathcal{L}(w; y^*, s^*; c) \leq \mathcal{L}(M; y^*, s^*; c) = M,$$

so that $w = M$ on a dense subset of $\partial_n \Delta(y^*, s^*; c)$ which, as before, implies that there is a point $(y_2, s_2) \in \gamma \cap \partial_n \Delta(y^*, s^*; c)$ such that $s_2 < s^*$ and $w(y_2, s_2) = M$. This contradicts the definition of s^* , so it is not possible to have $t_1 < s^*$. Hence $t_1 = s^*$, and $w(x_1, t_1) = M$.

Corollary. *Let w be a subtemperature on E . Given any point $(x_0, t_0) \in E$, there is a point $(x_1, t_1) \in \Lambda(x_0, t_0; E)$ such that $w(x_0, t_0) \leq w(x_1, t_1)$.*

Proof. If $w(x_0, t_0) \geq w(x, t)$ for all $(x, t) \in \Lambda(x_0, t_0; E)$, then $w(x_0, t_0) = w(x, t)$ for all such points (x, t) . The only other possibility is that there is a point $(x_1, t_1) \in \Lambda(x_0, t_0; E)$ such that $w(x_0, t_0) < w(x_1, t_1)$.

We shall prove a boundary maximum principle for subtemperatures on an arbitrary open set, using the *Hausdorff Maximality Theorem* [9], as in [12].

Theorem 8. *Let w be a subtemperature on an open set E , and suppose that*

$$\limsup_{k \rightarrow \infty} w(p_k) \leq A$$

for every sequence $\{p_k\}$ in E that satisfies $p_{k+1} \in \Lambda(p_k; E)$ for all k , and which tends either to a boundary point of E or to the point at infinity. Then $w(p) \leq A$ for all $p \in E$.

Proof. Given any number $\alpha > A$, we put $S_\alpha = \{p \in E : w(p) \geq \alpha\}$. If $S_\alpha = \emptyset$ for all α , there is nothing to prove. If $S_\alpha \neq \emptyset$ for some α , we define a partial order \prec on S_α by putting $p \prec q$ if $p \in \Lambda(q; E) \cup \{q\}$. By the Hausdorff Maximality Theorem, S_α contains a maximal totally ordered subset T_α . We put $t^* = \inf\{t : \text{there is a point } (x, t) \in T_\alpha\}$. Since T_α is totally ordered, there is a sequence $\{p_i\} = \{(x_i, t_i)\}$ of points of T_α such that $p_{i+1} \in \Lambda(p_i; E) \cup \{p_i\}$ for all i , and $t_i \rightarrow t^*$ as $i \rightarrow \infty$.

If the sequence $\{p_i\}$ has a cluster point in ∂E , or is unbounded, then it contains infinitely many points. It therefore has a subsequence $\{p_{i_k}\}$ that converges to a point of ∂E , or tends to the point at infinity, such that $p_{i_{k+1}} \in \Lambda(p_{i_k}; E)$ for all k . Hence, by hypothesis,

$$\alpha \leq \limsup_{k \rightarrow \infty} w(p_{i_k}) \leq A < \alpha,$$

a contradiction. Therefore $\{p_i\}$ is contained in some compact subset of E . Hence $t^* > -\infty$, and $\{p_i\}$ has a subsequence $\{p_{i_j}\}$ that converges to a point $p^* = (x^*, t^*)$ in $E \cap \bar{T}_\alpha$. Put $q_j = (y_j, s_j) = p_{i_j}$ for all j . Then $q_{j+1} \in \Lambda(q_j; E) \cup \{q_j\}$ for all j . Since $p^* \in E \cap \bar{T}_\alpha$ and $w \geq \alpha$ on T_α , the upper semicontinuity of w implies that $w(p^*) \geq \alpha$, so that $p^* \in S_\alpha$. Furthermore, p^* is the centre of some euclidean ball $B(p^*, \delta) \subseteq E$, and there exists some number N such that $q_j \in B(p^*, \delta)$ for all $j \geq N$. It follows that $p^* \prec q_j$ for all $j \geq N$. Since T_α is totally ordered and $q_j \rightarrow p^* = (x^*, t^*)$, for each point $p \in T_\alpha$ there is some $j \geq N$ such that $q_j \prec p$. Hence $p^* \prec p$ for all $p \in T_\alpha$, so that $T_\alpha \cup \{p^*\}$ is totally ordered. Since T_α is maximal, this shows that $p^* \in T_\alpha$. By Theorem 7 Corollary, there is some point $p' \in \Lambda(p^*; E)$ such that $w(p') \geq w(p^*) \geq \alpha$. This implies first that $p' \in S_\alpha$, then that $p' \in T_\alpha$. Now we have another contradiction, because $t^* = \inf\{t : (x, t) \in T_\alpha\}$ and $p' \in \Lambda(x^*, t^*; E)$.

Thus if $S_\alpha \neq \emptyset$, we obtain a contradiction in every possible situation. We conclude that $S_\alpha = \emptyset$ for all α , so that $w(p) \leq A$ for all $p \in E$.

For the case of a circular cylinder, Theorem 8 gives a predictable result, as follows.

Corollary. *Let w be a subtemperature on a circular cylinder D . If*

$$\limsup_{p \rightarrow q} w(p) \leq A \quad \text{for all } q \in \partial_n D,$$

then $w(p) \leq A$ for all $p \in D$.

Proof. If $\{p_k\}$ is a sequence in D that satisfies $p_{k+1} \in \Lambda(p_k; D)$ for all k , and tends to a point $q \in \partial D$, then $q \in \partial_n D$. Hence

$$\limsup_{k \rightarrow \infty} w(p_k) \leq \limsup_{p \rightarrow q} w(p) \leq A.$$

Our next theorem characterizes subtemperatures in terms of being majorized by temperatures on circular cylinders, and strengthens condition (δ_4) . To prove it, we need a lemma that refines the condition of upper semicontinuity.

Lemma 2. *Let w be a subtemperature on an open set E , and let (y, s) be a point in E . Then*

$$w(y, s) = \limsup_{(x,t) \rightarrow (y,s-)} w(x, t).$$

Proof. We put $q = (y, s)$ and $l = \limsup_{(x,t) \rightarrow (y,s-)} w(x, t)$. Since w is upper semicontinuous and upper finite, we have $l \leq w(q) < +\infty$. Given any number $L > l$, we can find a heat cylinder $\Delta(q; c_0)$ such that $w(p) \leq L$ for all $p \in \Delta(q; c_0)$. Now condition (δ_4) shows that there is a positive number $c < c_0$ such that

$$w(q) \leq \mathcal{L}(w; q; c) \leq \mathcal{L}(L; q; c) = L$$

since, in view of Lemma 1, $w \leq L$ almost everywhere on $\partial_n \Delta(q; c_0)$ with respect to the caloric measure at q . Thus $w(q) \leq L$ whenever $l < L$, so that $w(q) \leq l$. Hence $w(q) = l$.

Theorem 9. *Let w be an extended-real valued function on an open set E , that satisfies conditions (δ_1) , (δ_2) and (δ_3) of the definition of a subtemperature. Consider the following property: Whenever D is a circular cylinder such that $\overline{D} \subseteq E$, and v is a function in $C(\overline{D})$ that is a temperature on D and satisfies $v \geq w$ on $\partial_n D$, then $v \geq w$ on \overline{D} .*

If w is a subtemperature on E , then the stated property holds.

Conversely, if the stated property holds then w is a subtemperature on E . Moreover, given any point $p \in E$, the inequality $w(p) \leq \mathcal{L}(w; p; c)$ holds whenever the closed heat cylinder $\overline{\Delta}(p; c) \subseteq E$.

Proof. Suppose that w is a subtemperature on E , that D is a circular cylinder such that $\overline{D} \subseteq E$, and that $v \in C(\overline{D})$, is a temperature on D , and satisfies $v \geq w$ on $\partial_n D$. Then $w - v$ is a subtemperature on D , in view of Theorem 6. Furthermore, whenever $q \in \partial_n D$ we have

$$\limsup_{p \rightarrow q, p \in D} (w(p) - v(p)) \leq w(q) - v(q) \leq 0,$$

so that $w(p) \leq v(p)$ for all $p \in D$, by Theorem 8 Corollary. Finally, if $q \in \overline{D}$ but $q \notin D \cup \partial_n D$, Lemma 2 shows that

$$w(q) - v(q) = \limsup_{p \rightarrow q, p \in D} w(p) - \lim_{p \rightarrow q, p \in D} v(p) = \limsup_{p \rightarrow q, p \in D} (w(p) - v(p)) \leq 0.$$

Conversely, suppose that w has the stated property, and let $\Delta(p; c)$ be a heat cylinder such that $\overline{\Delta}(p; c) \subseteq E$. The restriction of w to $\partial_n \Delta(p; c)$ is upper semi-continuous and upper finite, and hence upper bounded. Therefore we can find a sequence $\{f_k\}$ in $C(\partial_n \Delta(p; c))$ that decreases to w on $\partial_n \Delta(p; c)$. For each k , let v_k be the Poisson integral of f_k on $\overline{\Delta}(p; c) \setminus \partial_n \Delta(p; c)$, and let $v_k = f_k$ on $\partial_n \Delta(p; c)$. Then $v_k \in C(\overline{\Delta}(p; c))$, v_k is a temperature on $\Delta(p; c)$, and $v_k \geq w$ on $\partial_n \Delta(p; c)$. So our hypothesis implies that $v_k \geq w$ on $\overline{\Delta}(p; c)$. In particular,

$$w(p) \leq \lim_{k \rightarrow \infty} v_k(p) = \mathcal{L}(\lim_{k \rightarrow \infty} f_k; p; c) = \mathcal{L}(w; p; c)$$

by Lebesgue's monotone convergence theorem.

Corollary 1. *If v and w are subtemperatures on E , then so is $w \vee v$.*

Proof. Conditions (δ_1) , (δ_2) and (δ_3) obviously hold for $w \vee v$, and (δ_4) holds because

$$(w \vee v)(p) \leq \mathcal{L}(w; p; c) \vee \mathcal{L}(v; p; c) \leq \mathcal{L}(w \vee v; p; c)$$

for all values of c such that $\overline{\Delta}(p; c) \subseteq E$.

Corollary 2. *If v and w are subtemperatures on E , and either one is real valued, then $v + w$ is a subtemperature on E .*

Proof. Conditions (δ_1) , (δ_2) and (δ_3) obviously hold for $v + w$, and (δ_4) follows from Theorem 9.

Theorem 10. *Let w be a subtemperature on an open set E , and let D be a circular cylinder such that $\overline{D} \subseteq E$. Then the Poisson integral u of the restriction of w to $\partial_n D$ exists, and the function $\pi_D w$, defined on E by putting*

$$\pi_D w = \begin{cases} u & \text{on } \overline{D} \setminus \partial_n D, \\ w & \text{on } E \setminus (\overline{D} \setminus \partial_n D), \end{cases}$$

has the following properties:

- (a) $\pi_D w$ is a subtemperature on E ,
- (b) $\pi_D w \geq w$ on E ,
- (c) $\pi_D w$ is a temperature on $\overline{D} \setminus \partial_n D$,
- (d) $\pi_D w = w$ on $\partial_n D \cup (E \setminus \overline{D})$,
- (e) if $v \in C(\overline{D})$, $v \geq w$ on \overline{D} , and v is a temperature on D , then $v \geq \pi_D w$ on \overline{D} .

Proof. Let $D = B \times]a, b[$, where B is an open ball in \mathbb{R}^n and $]a, b[$ is a bounded interval in \mathbb{R} . Choose a number $b^* > b$ such that the cylinder $D^* = B \times]a, b^*[$ also has its closure contained in E . Since w is upper semicontinuous and upper finite on the compact set $\partial_n D^*$, it is upper bounded on $\partial_n D^*$. Therefore we can find a decreasing sequence $\{f_k\}$ of functions in $C(\partial_n D^*)$ such that $f_k \rightarrow w$ on $\partial_n D^*$ as $k \rightarrow \infty$. For each k , we put u_k equal to the Poisson integral of f_k on $\overline{D^*} \setminus \partial_n D^*$, and u_k equal to f_k on $\partial_n D^*$. Then $u_k \in C(\overline{D^*})$ and u_k is a temperature on D^* . Since $\{f_k\}$ is a decreasing sequence, so is $\{u_k\}$. We put $u = \lim_{k \rightarrow \infty} u_k$. By Theorem 9, $w \leq u_k$ on $\overline{D^*}$ for all k , and hence $w \leq u$. Since u is the limit of a decreasing sequence of continuous functions, it is upper semicontinuous on $\overline{D^*}$; and since $f_k \rightarrow w$ on $\partial_n D^*$, $u = w$ there. Lebesgue's monotone convergence theorem now shows that u is the Poisson integral of the restriction of w to $\partial_n D^*$. Furthermore, u is the limit of the decreasing sequence $\{u_k\}$ of nonnegative temperatures on D^* , so that the Harnack monotone convergence theorem for temperatures shows that u is a temperature on D^* . Hence, in particular, the restriction of u to \overline{D} is a temperature on $\overline{D} \setminus \partial_n D$, is the Poisson integral of the restriction of w to $\partial_n D$ on $\overline{D} \setminus \partial_n D$, in view of Lemma 1, and $u(p) = \mathcal{L}(u; p; c)$ whenever $\overline{\Delta}(p; c) \subseteq \overline{D} \setminus \partial_n D$, by Theorem 6.

We now define the function $\pi_D w$ as in the statement of the theorem, and show that $\pi_D w$ is a subtemperature on E . Since $w < +\infty$ on E , and u is the limit of a decreasing sequence of functions in $C(\overline{D})$, $\pi_D w$ is upper finite on E . Since $u \geq w$ on \overline{D} , and w satisfies condition (δ_3) , $\pi_D w$ also satisfies that condition, and $\pi_D w \geq w$ on E . Furthermore, $\pi_D w$ is certainly upper semicontinuous at points outside $B \times \{b\}$; and if $q \in B \times \{b\}$, then

$$\lim_{p \rightarrow q, p \in D} u(p) = u(q) \geq w(q) \geq \limsup_{p \rightarrow q, p \notin D} w(p),$$

which implies that $\pi_D w$ is upper semicontinuous at q . It remains to prove that $\pi_D w$ satisfies condition (δ_4) . If $p \in E$ but $p \notin \overline{D} \setminus \partial_n D$, then whenever the closed heat cylinder $\overline{\Delta}(p; c) \subseteq E$, we have

$$\pi_D w(p) = w(p) \leq \mathcal{L}(w; p; c) \leq \mathcal{L}(\pi_D w; p; c).$$

On the other hand, we have already shown that $u(p) = \mathcal{L}(u; p; c)$ whenever $\overline{\Delta}(p; c) \subseteq \overline{D} \setminus \partial_n D$, so that $\pi_D w(p) = \mathcal{L}(\pi_D w; p; c)$ for such values of p and c . Hence $\pi_D w$ is a subtemperature on E .

It only remains to prove part (e). Suppose that $v \in C(\overline{D})$, $v \geq w$ on \overline{D} , and v

is a temperature on D . Given any $\epsilon > 0$, the sequence $\{f_k\}$ decreases to the limit $w < v + \epsilon$ on $\partial_n D$. Therefore the sequence of sets $\{S_k\}$, defined by

$$S_k = \{q \in \partial_n D : f_k(q) < v(q) + \epsilon\}$$

is expanding to the union $\partial_n D$. Both f_k and v are continuous on $\partial_n D$, so that each set S_k is relatively open. Therefore, since $\partial_n D$ is compact, there is a number κ such that $S_k = \partial_n D$ whenever $k > \kappa$. Thus $f_k(q) < v(q) + \epsilon$ for all $q \in \partial_n D$ if $k > \kappa$. This implies, using the maximum principle, that $u_k(q) < v(q) + \epsilon$ for all $q \in D$ if $k > \kappa$. Therefore $u < v + \epsilon$ on D for any $\epsilon > 0$, and so $u \leq v$.

4. The Dirichlet Problem on Convex Domains of Revolution

We need to discuss the Dirichlet problem on the heat ball and some approximating domains. They are all of the following form, as are circular cylinders.

Let $x_0 \in \mathbb{R}$ and $a, b \in \mathbb{R}^n$. A *Convex Domain of Revolution* is any open set that has the form

$$R = R(x_0; \rho; a, b) = \{(x, t) \in \mathbb{R}^{n+1} : |x - x_0| < \rho(t), a < t < b\}$$

for some continuous concave function $\rho : [a, b] \rightarrow [0, +\infty[$.

Corresponding to the normal boundary of a circular cylinder, we define the *normal boundary* of a convex domain of revolution R to be

$$\partial_n R = \partial R \setminus \{(x, b) : |x - x_0| < \rho(b)\}.$$

Note that $\partial_n R$ is compact.

The *Dirichlet Problem* on a convex domain of revolution R consists of showing that, for an arbitrary function $f \in C(\partial_n R)$, there is a function $u_f \in C(R \cup \partial_n R)$ that is a temperature on R and coincides with f on $\partial_n R$.

We show that this problem has a solution, except when the left hand derivative $\rho'_-(b) = -\infty$. We use the *Perron-Wiener-Brelot*, or PWB, method.

Definition. A non-empty family \mathcal{F} of supertemperatures on an open set E , is called a *saturated family* if it satisfies the two conditions:

- (a) if $v, w \in \mathcal{F}$, then $v \wedge w \in \mathcal{F}$;
- (b) if $w \in \mathcal{F}$, D is a circular cylinder such that $\overline{D} \subseteq E$, and $\pi_D w$ is the function defined in Theorem 10, then $\pi_D w \in \mathcal{F}$.

Theorem 11. *If \mathcal{F} is a saturated family of supertemperatures on an open set E , and the function $u = \inf \mathcal{F}$ satisfies $u(p_0) > -\infty$ at some point $p_0 \in E$, then u is a temperature on $\Lambda(p_0; E)$.*

Proof. Let q_0 be any point of E such that $u(q_0) > -\infty$. Let D be any circular cylinder such that $q_0 \in D$ and $\overline{D} \subseteq E$. For each supertemperature $w \in \mathcal{F}$, we let $\pi_D w$ be the function defined in Theorem 10, so that $\pi_D w$ is a supertemperature on E , and $\pi_D w \leq w$ on E . Since \mathcal{F} saturated, $\pi_D w \in \mathcal{F}$. Therefore, on D , we have $u = \inf\{\pi_D w : w \in \mathcal{F}\}$. If $v, w \in \mathcal{F}$, then $v \wedge w \in \mathcal{F}$ because \mathcal{F} is saturated, and so the family \mathcal{F} is downward-directed. Furthermore, an application of the minimum principle on D shows that $\pi_D(v \wedge w) \leq \pi_D v \wedge \pi_D w$, and therefore the family $\{\pi_D w : w \in \mathcal{F}\}$ is also downward-directed. Since $\pi_D w$ is a temperature on D for all $w \in \mathcal{F}$, it follows from Theorem 1 that u is a temperature on $\Lambda(q_0; D)$.

Now let p_* be any point of $\Lambda(p_0; E)$, and let γ be a polygonal path in E that connects p_0 to p_* , along which the temporal variable is strictly decreasing. For each point $p = (x, t) \in \gamma$ and positive number c , we put

$$D(p; c) = B(x, c) \times]t - c, t + c[\quad \text{and} \quad \Lambda(p; c) = B(x, c) \times]t - c, t[= \Lambda(p; D(p; c)).$$

Since γ is a compact subset of the open set E , we can find $c_0 > 0$ such that $\overline{D(p; c_0)} \subseteq E$ for all $p \in \gamma$. We now let m be the integer such that the length of γ lies in the interval $]mc_0/2, (m + 1)c_0/2]$. Since $u(p_0) > -\infty$, we know that u is a temperature on $\Lambda(p_0; c_0)$. The length of that portion of γ which is contained in $\Lambda(p_0; c_0)$ is at least c_0 , and so there is a point $p_1 \in \gamma \cap \Lambda(p_0; c_0)$ such that the length of that portion of γ which lies between p_0 and p_1 is $c_0/2$. Since $u(p_1) > -\infty$, u is a temperature on $\Lambda(p_1; c_0)$. The length of γ contained in $\Lambda(p_1; c_0)$ is at least c_0 , and so there is a point $p_2 \in \gamma \cap \Lambda(p_1; c_0)$ such that the length of γ between p_1 and p_2 is $c_0/2$. Repeating this argument m times, we find that there is a point $p_m \in \gamma$ such that u is a temperature on $\Lambda(p_m; c_0)$ and $p_* \in \Lambda(p_m; c_0)$. Thus u is a temperature on a neighbourhood of p_* , and hence on $\Lambda(p_0; E)$.

We note that the boundary maximum principle for subtemperatures on a convex domain of revolution, takes the same form as it does on a circular cylinder (Theorem 8 Corollary), with a similar proof.

Definition. Let R be a convex domain of revolution, and let $f \in C(\partial_n R)$. The *Upper Class* \mathfrak{U}_f , determined by f , consists of all upper bounded supertemperatures v on R that satisfy

$$\liminf_{p \rightarrow q} v(p) \geq f(q)$$

for all $q \in \partial_n R$.

Note that, by the boundary minimum principle, $v \geq \min f$ on R . Note also that, because $v \wedge (\max f)$ is also a supertemperature, by Theorem 9 Corollary 1, the condition that v is upper bounded is no real restriction.

The *Lower Class* \mathfrak{L}_f , determined by f , consists of all lower bounded subtemperatures u on R that satisfy

$$\limsup_{p \rightarrow q} u(p) \leq f(q)$$

for all $q \in \partial_n R$.

Note that neither class is empty, because \mathfrak{U}_f contains the constant function $\max f$, and \mathfrak{L}_f contains $\min f$.

The *Upper PWB Solution* for f on R is the function U_f given by

$$U_f(p) = \inf\{v(p) : v \in \mathfrak{U}_f\},$$

and the *Lower PWB Solution* is given by

$$L_f(p) = \sup\{u(p) : u \in \mathfrak{L}_f\}.$$

Both functions are bounded.

If $U_f = L_f$, and is a temperature on R , then we put $S_f = U_f$ and call it the *PWB Solution* for f on R .

We shall show that every $f \in C(\partial_n R)$ has a PWB solution on R , then investigate the boundary values of S_f . First we show that, if the Dirichlet problem for f has

a solution, then it is given by S_f .

Lemma 3. *Let R be a convex domain of revolution, and let $f \in C(\partial_n R)$. If $u \in \mathfrak{L}_f$ and $v \in \mathfrak{U}_f$, then $u \leq v$ on R . Consequently $L_f \leq U_f$ on R .*

Proof. Since u is a bounded subtemperature on R , and v is a bounded supertemperature, the difference $u - v$ is a subtemperature, by Theorem 9 Corollary 2. Furthermore, whenever $q \in \partial_n R$ we have

$$\limsup_{p \rightarrow q} (u - v)(p) \leq \limsup_{p \rightarrow q} u(p) - \liminf_{p \rightarrow q} v(p) \leq 0,$$

and so it follows from the boundary maximum principle that $u \leq v$ on R . Thus any function $u \in \mathfrak{L}_f$ satisfies $u \leq U_f$, and therefore $L_f \leq U_f$.

Theorem 12. *Let R be a convex domain of revolution, and let $f \in C(\partial_n R)$. If there is a temperature u_f on R such that*

$$\lim_{p \rightarrow q} u_f(p) = f(q)$$

for all $q \in \partial_n R$, then f has a PWB-solution and it is u_f .

Proof. It follows from the boundary maximum principle that $\min f \leq u_f \leq \max f$ on R . Therefore $u_f \in \mathfrak{L}_f \cap \mathfrak{U}_f$, and so $U_f \leq u_f \leq L_f$. Since $L_f \leq U_f$ by Lemma 3, we deduce that $U_f = u_f = L_f$. Since u_f is a temperature on R , the PWB solution for f on R exists and is equal to u_f .

Lemma 4. *Let R be a convex domain of revolution, and let $f \in C(\partial_n R)$. Then both L_f and U_f are temperatures on R .*

Proof. Let $v, w \in \mathfrak{U}_f$. Then $v \wedge w$ is an upper bounded supertemperature on R , by Theorem 9 Corollary 1, and

$$\liminf_{p \rightarrow q} (v \wedge w)(p) = (\liminf_{p \rightarrow q} v(p)) \wedge (\liminf_{p \rightarrow q} w(p)) \geq f(q)$$

for all $q \in \partial_n R$. Therefore $v \wedge w \in \mathfrak{U}_f$. Next, if $v \in \mathfrak{U}_f$ and D is a circular cylinder such that $\bar{D} \subseteq R$, then the function $\pi_D v$ of Theorem 10, is a supertemperature on R , is upper bounded on R , and satisfies

$$\liminf_{p \rightarrow q} \pi_D v(p) = \liminf_{p \rightarrow q} v(p) \geq f(q)$$

for all $q \in \partial_n R$. Therefore $\pi_D v \in \mathfrak{U}_f$. Thus \mathfrak{U}_f is a saturated family of supertemperatures on R . Furthermore, since $v \geq \min f$ for every $v \in \mathfrak{U}_f$, it follows from Theorem 11 that U_f is a temperature on R . Dually, L_f is also a temperature.

Definition. Let R be a convex domain of revolution, and let $f \in C(\partial_n R)$. If f has a PWB solution on R , we say that f is *resolutive*.

Lemma 5. *Let R be a convex domain of revolution, let $f, g \in C(\partial_n R)$, and let $\alpha \in \mathbb{R}$.*

(a) *The constant function α is resolutive, and $S_\alpha = \alpha$ on R .*

(b) *$U_{f+\alpha} = U_f + \alpha$ and $L_{f+\alpha} = L_f + \alpha$. If f is resolutive, then $f + \alpha$ is resolutive and $S_{f+\alpha} = S_f + \alpha$.*

(c) *If $\alpha > 0$, then $U_{\alpha f} = \alpha U_f$ and $L_{\alpha f} = \alpha L_f$. If f is resolutive, then αf is resolutive and $S_{\alpha f} = \alpha S_f$.*

- (d) If $f \leq g$, then $U_f \leq U_g$ and $L_f \leq L_g$.
- (e) $U_{-f} = -L_f$. If f is resolvable, then $-f$ is resolvable and $S_{-f} = -S_f$.
- (f) $U_{f+g} \leq U_f + U_g$ and $L_{f+g} \geq L_f + L_g$. If f and g are resolvable, then $f + g$ is resolvable and $S_{f+g} = S_f + S_g$.

Proof. (a) This is a special case of Theorem 12.

(b) If $v \in \mathfrak{U}_f$ then $v + \alpha \in \mathfrak{U}_{f+\alpha}$, and conversely. So $U_{f+\alpha} = U_f + \alpha$. Similarly, $L_{f+\alpha} = L_f + \alpha$. If f is resolvable, then $L_f = U_f$ and is a temperature, so that $U_{f+\alpha} = U_f + \alpha = L_f + \alpha = L_{f+\alpha}$ and is also a temperature.

(c) If $v \in \mathfrak{U}_f$ then $\alpha v \in \mathfrak{U}_{\alpha f}$, and conversely. So $U_{\alpha f} = \alpha U_f$. Similarly, $L_{\alpha f} = \alpha L_f$. If f is resolvable, then $L_f = U_f$ and is a temperature, so that $U_{\alpha f} = \alpha U_f = \alpha L_f = L_{\alpha f}$ and is also a temperature.

(d) If $v \in \mathfrak{U}_g$, then $v \in \mathfrak{U}_f$, so that \mathfrak{U}_f is the infimum over a more inclusive class of functions, and so $U_f \leq U_g$. Similarly, if $u \in \mathfrak{L}_f$ then $u \in \mathfrak{L}_g$, so that $L_f \leq L_g$.

(e) If $v \in \mathfrak{U}_f$ then $-v \in \mathfrak{L}_{-f}$, and conversely. So $U_f = -L_{-f}$. Similarly, $L_f = -U_{-f}$. If f is resolvable, then $L_f = U_f$ and is a temperature, so that $-U_{-f} = -L_{-f}$ and is also a temperature.

(f) If $v \in \mathfrak{U}_f$ and $w \in \mathfrak{U}_g$, then Theorem 9 Corollary 2 implies that $v + w \in \mathfrak{U}_{f+g}$. So for each function $w \in \mathfrak{U}_g$ we have $U_f + w \geq U_{f+g}$. Therefore $U_f + U_g \geq U_{f+g}$. Now the inequality $L_{f+g} \geq L_f + L_g$ follows from part (e). If f and g are resolvable, then (using Lemma 3)

$$S_f + S_g = L_f + L_g \leq L_{f+g} \leq U_{f+g} \leq U_f + U_g = S_f + S_g,$$

which shows that $L_{f+g} = U_{f+g} = S_f + S_g$.

In order to show that every function in $C(\partial_n R)$ is resolvable, we first obtain a class of resolvable functions such that every real continuous function can be obtained as the limit of a uniformly convergent sequence in that class, using the Stone-Weierstrass theorem ([6], Theorem 7.29). Then we show that the limit of a uniformly convergent sequence of resolvable functions is itself resolvable.

Lemma 6. *If R is a convex domain of revolution, and w is a function in $C(R \cup \partial_n R)$ that is a subtemperature on R , then the restriction of w to $\partial_n R$ is resolvable.*

Proof. Let f denote the restriction of w to $\partial_n R$. By Lemma 4, the lower PWB solution L_f is a temperature on R . Furthermore $u \in \mathfrak{L}_f$, so that $u \leq L_f$ on R . Therefore

$$\liminf_{p \rightarrow q} L_f(p) \geq \lim_{p \rightarrow q} u(p) = f(q)$$

for all $q \in \partial_n R$, so that $L_f \in \mathfrak{U}_f$, and hence $L_f \geq U_f$. But we always have $L_f \leq U_f$, by Lemma 3, and so f is resolvable, in view of Lemma 4.

Lemma 7. *Let R be a convex domain of revolution, and let $\{f_j\}$ be a sequence of resolvable functions in $C(\partial_n R)$. If $\{f_j\}$ converges uniformly on $\partial_n R$ to a function f , then f is resolvable and the sequence $\{S_{f_j}\}$ converges uniformly on R to S_f .*

Proof. Note that $f \in C(\partial_n R)$, so that U_f and L_f are temperatures on R , by Lemma 4. Given $\epsilon > 0$, we can find a number N such that $f_j - \epsilon < f < f_j + \epsilon$ on $\partial_n R$ whenever $j > N$. Therefore, by Lemma 5, $U_{f_j} - \epsilon = U_{f_j - \epsilon} \leq U_f \leq U_{f_j + \epsilon} = U_{f_j} + \epsilon$ on R whenever $j > N$. Hence the sequence $\{S_{f_j}\} = \{U_{f_j}\}$ converges uniformly

on R to U_f . A similar argument with the lower solutions shows that the sequence $\{S_{f_j}\} = \{L_{f_j}\}$ converges uniformly on R to L_f . Hence $U_f = L_f$, and the lemma is established.

Theorem 13. *If R is a convex domain of revolution, then every function in $C(\partial_n R)$ is resolvable.*

Proof. Let \mathcal{G} denote the class of functions in $C(R \cup \partial_n R)$ that are supertemperatures on R , let \mathcal{D} denote the class of differences $u - v$ of functions in \mathcal{G} , and let \mathcal{F} denote the class of restrictions to $\partial_n R$ of the functions in \mathcal{D} . Then \mathcal{F} is a linear subspace of $C(\partial_n R)$ that contains the constant functions. By Lemmas 6 and 5, the restrictions to $\partial_n R$ of the functions in \mathcal{G} are resolvable, and the functions in \mathcal{F} are all resolvable. Furthermore, for any point (x_0, t_0) such that $\bar{R} \subseteq \mathbb{R}^n \times]t_0, +\infty[$, the class \mathcal{D} contains the function $(x, t) \mapsto W(x - x_0, t - t_0)$, and so separates points. Finally, if $u, v \in \mathcal{G}$ then the Theorem 9 Corollaries imply that $u \wedge v, u + v \in \mathcal{G}$, so that if $u_1, u_2, v_1, v_2 \in \mathcal{G}$ the function

$$(u_1 - v_1) \vee (u_2 - v_2) = u_1 + u_2 - (u_2 + v_1) \wedge (u_1 + v_2) \in \mathcal{D}.$$

Thus $f \vee g \in \mathcal{F}$ whenever $f, g \in \mathcal{F}$. It now follows from the Stone-Weierstrass theorem that every function in $C(\partial_n R)$ can be expressed as the uniform limit of a sequence in \mathcal{F} . Since every function in \mathcal{F} is resolvable, it follows from Lemma 7 that every function in $C(\partial_n R)$ is resolvable.

5. Boundary Behaviour of the PWB Solution

We now show that, if R is a convex domain of revolution satisfying an auxiliary condition, then for any function $f \in C(\partial_n R)$, the PWB solution S_f solves the Dirichlet problem for f on R . The extra condition cannot be omitted altogether, although it can be weakened.

Theorem 14. *Let $R = \{(x, t) \in \mathbb{R}^{n+1} : |x - x_0| < \rho(t), a < t < b\}$ be a convex domain of revolution such that $\rho'_-(b) > -\infty$, and let $f \in C(\partial_n R)$. Then the PWB solution S_f for f on R satisfies*

$$\lim_{p \rightarrow q} S_f(p) = f(q)$$

for all $q \in \partial_n R$.

Proof. Because ρ is concave, given any point $(y_0, s_0) \in \partial_n R$ we can find a hyperplane H such that $(y_0, s_0) \in H$ and $R \cap H = \emptyset$. On the opposite side of H to R , we position a reflected heat ball

$$\Omega^*(\eta_0, \sigma_0; c_0) = \{(x, t) : W(x - \eta_0, t - \sigma_0) > (4\pi c_0)^{-\frac{n}{2}}\},$$

with $\sigma_0 < s_0$, so that it is tangential to H at (y_0, s_0) . This is possible unless $s_0 = b$ and $H = \mathbb{R}^n \times \{b\}$. Our condition that $\rho'_-(b) > -\infty$ implies that, if $(y_0, s_0) \in \partial_n R$ and $s_0 = b$, we can find an H that is not equal to $\mathbb{R}^n \times \{b\}$. The function w , defined on R by

$$w(x, t) = (4\pi c_0)^{-\frac{n}{2}} - W(x - \eta_0, t - \sigma_0),$$

is a positive temperature on R such that

$$\lim_{(x,t) \rightarrow (y_0, s_0)} w(x, t) = 0,$$

and for any neighbourhood N of (y_0, s_0) ,

$$\inf_{R \setminus N} w > 0.$$

Given $\epsilon > 0$, we put $A = f(y_0, s_0) + \epsilon$. Since f is continuous at (y_0, s_0) , we can find a neighbourhood N of (y_0, s_0) such that $f < A$ on $N \cap \partial_n R$. Since $\inf_{R \setminus N} w > 0$, we can choose $\alpha > 0$ such that $\alpha \inf_{R \setminus N} w > \max f - A$. We put $u = A + \alpha w$ on R , and note that u is a lower bounded temperature on R . Whenever $(y, s) \in (\partial_n R) \setminus N$ we have

$$\liminf_{(x,t) \rightarrow (y,s)} u(x, t) \geq A + \alpha \inf_{R \setminus N} w > \max f \geq f(y, s);$$

and whenever $(y, s) \in (\partial_n R) \cap N$ we have

$$\liminf_{(x,t) \rightarrow (y,s)} u(x, t) \geq A > f(y, s).$$

Therefore the function $v = u \wedge (\max f)$, which is a supertemperature on R by Theorem 9 Corollary 1, belongs to the upper class \mathfrak{U}_f . Hence the upper PWB solution $U_f \leq v$ on R , which implies that

$$\limsup_{(x,t) \rightarrow (y_0, s_0)} U_f(x, t) \leq \limsup_{(x,t) \rightarrow (y_0, s_0)} u(x, t) = A + \alpha \lim_{(x,t) \rightarrow (y_0, s_0)} w(x, t) = A.$$

Hence, since f is resolutive by Theorem 13,

$$\limsup_{(x,t) \rightarrow (y_0, s_0)} S_f(x, t) \leq f(y_0, s_0).$$

A similar inequality holds with f replaced by $-f$, and so it follows from Lemma 5 that

$$\liminf_{(x,t) \rightarrow (y_0, s_0)} S_f(x, t) = - \limsup_{(x,t) \rightarrow (y_0, s_0)} S_{-f}(x, t) \geq f(y_0, s_0).$$

Hence $S_f(x, t) \rightarrow f(y_0, s_0)$ as $(x, t) \rightarrow (y_0, s_0)$.

Remark. Theorem 14 shows that, if $\kappa \in]0, +\infty[$ and R is the cone with vertex (x_0, b) given by $\{(x, t) : |x - x_0| < \kappa(b - t), a < t < b\}$, then the Dirichlet problem is solvable on R for any function $f \in C(\partial_n R)$, even though $\partial_n R = \partial R$. It follows that the class of temperatures satisfies the Base Axiom of a harmonic space [2], [3].

Corollary. Let $R = \{(x, t) \in \mathbb{R}^{n+1} : |x - x_0| < \rho(t), a < t < b\}$ be any convex domain of revolution, let $a < c < b$, and let $C = \{(x, t) : |x - x_0| < \rho(t), a < t < c\}$. If $f \in C(\partial_n R)$, then the PWB solution S_f for f on R satisfies

$$\lim_{p \rightarrow q} S_f(p) = f(q)$$

for all $q \in \partial_n C$. Furthermore, the restriction to C of S_f is the PWB solution on C for the restriction of f to $\partial_n C$.

Proof. We choose d such that $c < d < b$, and let D denote the convex domain of revolution $\{(x, t) : |x - x_0| < \rho(t), a < t < d\}$. Since ρ is a concave function on

a neighbourhood of d , we have $\rho'_-(d) > -\infty$. Therefore, if S_f^D denotes the PWB solution on D for the restriction of f to $\partial_n D$, Theorem 14 shows that

$$\lim_{p \rightarrow q, p \in D} S_f^D(p) = f(q)$$

for all $q \in \partial_n D$. Furthermore, $\min f \leq S_f^D \leq \max f$ on D . Now we define functions u and v on R by putting $u(p) = v(p) = S_f^D(p)$ for all $p \in R \cap \overline{C}$, and $u(p) = \min f$, $v(p) = \max f$ for all $p \in R \setminus \overline{C}$. Then u is a bounded subtemperature on R that satisfies $\limsup_{p \rightarrow q} u(p) \leq f(q)$ for all $q \in \partial_n R$, so that $u \in \mathfrak{L}_f$. Similarly $v \in \mathfrak{U}_f$. Therefore $u \leq S_f \leq v$ on R , which implies that

$$\lim_{p \rightarrow q, p \in \overline{C}} S_f(p) = f(q)$$

for all $q \in \partial_n C$. Since c is arbitrary, it follows that $\lim_{p \rightarrow q} S_f(p) = f(q)$ for all $q \in \partial_n C$. So the restriction to C of S_f solves the Dirichlet problem on C , and hence is the PWB solution on C for the restriction of f to $\partial_n C$, by Theorem 12.

6. Characterizations of Subtemperatures

In this final section, we give several characterizations of subtemperatures. In particular, we show that these subtemperatures are the same as the subcaloric functions of harmonic space theory [2], and also the same as the subtemperatures on [10]. Our characterizations are based on the following variant of Theorem 9, in which circular cylinders are replaced by convex domains of revolution.

Theorem 15. *Let w be an extended-real valued function on an open set E , that satisfies conditions (δ_1) , (δ_2) and (δ_3) of the definition of a subtemperature. Consider the following property: Whenever R is a convex domain of revolution such that $\overline{R} \subseteq E$, and v is a function in $C(\overline{R})$ that is a temperature on R and satisfies $v \geq w$ on $\partial_n R$, then $v \geq w$ on \overline{R} .*

The stated property holds if and only if w is a subtemperature on E .

Proof. The proof of one part is similar to that of the first part of Theorem 9. The converse follows from Theorem 9.

Another crucial part of our approach is that functions which satisfy the definition of a subtemperature with \mathcal{L} replaced by \mathcal{M} or \mathcal{V} , also satisfy the maximum principle.

Theorem 16. *Let $R = \{(x, t) : |x - x_0| < \rho(t), a < t < b\}$ be a convex domain of revolution, and let w be an extended-real valued function that satisfies conditions (δ_1) , (δ_2) and (δ_3) on R . If, given any point $(x, t) \in R$ and $\epsilon > 0$, we can find a positive number $c < \epsilon$ such that either*

$$(a) \ w(x, t) \leq \mathcal{M}(w; x, t; c)$$

or

$$(b) \ w(x, t) \leq \mathcal{V}(w; x, t; c)$$

holds, then w satisfies the maximum principle on R . That is, if there is a point $(x_0, t_0) \in R$ such that $w(x_0, t_0) \geq w(x, t)$ whenever $(x, t) \in R$ and $t < t_0$, then

$w(x_0, t_0) = w(x, t)$ for all such points (x, t) . Consequently, if

$$\limsup_{(x,t) \rightarrow (y,s)} w(x, t) \leq A$$

for all $(y, s) \in \partial_n R$, then $w(x, t) \leq A$ for all $(x, t) \in R$.

Proof. Suppose that there is a point $(x_0, t_0) \in R$ such that $w(x_0, t_0) \geq w(x, t)$ whenever $(x, t) \in R$ and $t < t_0$. Put $M = w(x_0, t_0)$, and let (x_1, t_1) be any point of R such that $t_1 < t_0$. Join (x_0, t_0) and (x_1, t_1) with a closed line segment γ , and put

$$S = \{s : \text{there is a point } (y, s) \in \gamma \text{ with } w(y, s) = M\}.$$

Then $S \neq \emptyset$ because $t_0 \in S$, and S is lower bounded by t_1 . Put $s^* = \inf S$. If condition (a) holds, we can find a number $c < t_0 - t_1$ such that

$$M = w(x_0, t_0) \leq \mathcal{M}(w; x_0, t_0; c) \leq \mathcal{M}(M; x_0, t_0; c) = M.$$

This implies that $w = M$ almost everywhere on $\partial\Omega(x_0, t_0; c)$, and so the upper semicontinuity of w shows that $w \equiv M$ on $\partial\Omega(x_0, t_0; c)$. Since $c < t_0 - t_1$, the set $\gamma \cap (\partial\Omega(x_0, t_0; c)) \neq \emptyset$, so that there is a point $s_1 \in S$ such that $s_1 < t_0$. A similar argument is valid if condition (b) is satisfied. Hence $s^* < t_0$.

Suppose that $t_1 < s^* < t_0$. There is a sequence of points $\{(z_k, r_k)\}$ on γ such that $w(z_k, r_k) = M$ for all k , and $r_k \rightarrow s^*$ as $k \rightarrow \infty$. The upper semicontinuity of w now implies that there is a point (y^*, s^*) on γ such that $w(y^*, s^*) = M$. If condition (a) holds, we can find $c < s^* - t_1$ such that $w \equiv M$ on $\partial\Omega(y^*, s^*; c)$, and therefore a point $s_2 \in S$ such that $s_2 < s^*$. A similar argument is valid if condition (b) is satisfied, so we have a contradiction. Hence $s^* = t_1$, and $w(x_1, t_1) = M$ by similar reasoning to that at the beginning of this paragraph. This proves the first part of the theorem.

For the second part, we extend w to $R \cup \partial_n R$ by putting

$$w(y, s) = \limsup_{(x,t) \rightarrow (y,s)} w(x, t) \leq A$$

for all $(y, s) \in \partial_n R$. Given any α such that $a < \alpha < b$, we let R_α denote the set $\{(x, t) : |x - x_0| < \rho(t), a < t < \alpha\}$. Then w is upper semicontinuous and upper finite on \overline{R}_α , and so has a maximum value M_α . We choose a point $(x', t') \in \overline{R}_\alpha$ such that $w(x', t') = M_\alpha$. If $(x', t') \in R$, then the first part of the theorem shows that $w(x, t) = M_\alpha$ for all $(x, t) \in R$ such that $t < t'$. So there is no loss of generality in assuming that $(x', t') \in \partial_n R$, which implies that $M_\alpha \leq A$. Since this holds for all α , we have $w \leq A$ on R , as required.

We now come to our characterization of subtemperatures using the fundamental means \mathcal{M} . A similar characterization using the volume means \mathcal{V} follows. We extract part of the proof as a lemma.

Lemma 8. *Let w be an extended-real valued function on an open set E , that satisfies conditions (δ_1) , (δ_2) and (δ_3) of the definition of a subtemperature. Let \mathfrak{R} denote the class of convex domains of revolution R , for which both $\partial_n R = \partial R$ and the Dirichlet problem on R has a solution for every $f \in C(\partial R)$. Consider the following property: Whenever $R \in \mathfrak{R}$ is such that $\overline{R} \subseteq E$, and v is a function in*

$C(\overline{R})$ that is a temperature on R and satisfies $v \geq w$ on ∂R , then $v \geq w$ on \overline{R} .

If the stated property holds, then the inequalities

$$w(p) \leq \mathcal{M}(w; p; d) \leq \mathcal{M}(w; p; c)$$

hold whenever $0 < d \leq c$ and $\overline{\Omega}(p; c) \subseteq E$.

Proof. Let $\Omega(x_0, t_0; c)$ be any heat ball whose closure is contained in E . Then

$$\Omega(x_0, t_0; c) = \{(x, t) : |x - x_0| < \phi(t), t_0 - c < t < t_0\}$$

is a convex domain of revolution with

$$\phi(t) = \sqrt{2n(t_0 - t) \log \frac{c}{t_0 - t}}.$$

We note that

$$\max\{\phi(t) : t_0 - c < t < t_0\} = \phi\left(t_0 - \frac{c}{e}\right).$$

Let k be a positive integer such that $1/k < c/e$, let $r = \lambda_k(t)$ be the equation of the tangent line to the curve $r = \phi(t)$ at the point $t = t_0 - \frac{1}{k}$, and let b_k denote the zero of λ_k . We put

$$\rho_k(t) = \begin{cases} \phi(t) & \text{if } t_0 - c \leq t \leq t_0 - \frac{1}{k}, \\ \lambda_k(t) & \text{if } t_0 - \frac{1}{k} \leq t \leq b_k, \end{cases}$$

and let

$$R_k = \{(x, t) : |x - x_0| < \rho_k(t), t_0 - c < t < b_k\}.$$

Since a concave curve lies below its tangent, each domain R_k contains $\Omega(x_0, t_0; c)$. Furthermore, for each k we have $\rho'_k(b_k) = \phi'(t_0 - \frac{1}{k}) > -\infty$, so that $R_k \in \mathfrak{R}$ in view of Theorem 14. Note that, if $t_0 - c \leq t \leq t_0 - \frac{1}{k}$, the point (x, t) belongs to $\partial\Omega(x_0, t_0; c)$ if and only if it belongs to ∂R_k .

The closures \overline{R}_k form a contracting sequence of sets with intersection $\overline{\Omega}(x_0, t_0; c)$, and so there is a number k_0 such that $\overline{R}_k \subseteq E$ for all $k > k_0$. For each $k > k_0$, the function w is upper semicontinuous and upper bounded on ∂R_k , and hence we can find a decreasing sequence $\psi_j^{(k)} \in C(\partial R_k)$ which tends pointwise to w on ∂R_k . For each j , we put $u_j^{(k)}$ equal to the PWB solution for $\psi_j^{(k)}$ on R_k , and $u_j^{(k)} = \psi_j^{(k)}$ on ∂R_k . Then each function $u_j^{(k)} \in C(\overline{R}_k)$, by Theorem 14, and is a temperature on R_k . In particular, each function $u_j^{(k)} \in C(\overline{\Omega}(x_0, t_0; c))$ and is a temperature on $\Omega(x_0, t_0; c)$. Therefore, by [10] (Lemma 7),

$$u_j^{(k)}(x_0, t_0) = \mathcal{M}(u_j^{(k)}; x_0, t_0; d)$$

whenever $0 < d \leq c$. Furthermore, by the stated property, $w \leq u_j^{(k)}$ on R_k for all j and k . Since the sequence $\{\psi_j^{(k)}\}$ is decreasing on ∂R_k , the maximum principle shows that the sequence $\{u_j^{(k)}\}$ is also decreasing. Put $v_k = \lim_{j \rightarrow \infty} u_j^{(k)} \geq w$ on \overline{R}_k , for each k . Then, whenever $0 < d \leq c$, we have

$$v_k(x_0, t_0) = \lim_{j \rightarrow \infty} \mathcal{M}(u_j^{(k)}; x_0, t_0; d) = \mathcal{M}(v_k; x_0, t_0; d),$$

by Lebesgue's monotone convergence theorem.

We need to show that the sequence $\{v_k\}$ is decreasing on $\overline{\Omega}(x_0, t_0; c)$, in order to apply the monotone convergence theorem again. Let $k > k_0$. Each function

$u_j^{(k)}$ belongs to $C(\overline{R}_k) \subseteq C(\overline{R}_{k+1})$, satisfies $u_j^{(k)} \geq w$ on $\overline{R}_k \supseteq \overline{R}_{k+1}$, and is a temperature on $R_k \supseteq R_{k+1}$. Given any $\epsilon > 0$ and positive integer J , the sequence $\{\psi_j^{(k+1)}\}$ decreases to the limit $w < u_J^{(k)} + \epsilon$ on ∂R_{k+1} . Therefore the sequence of sets $\{S_j\}$, defined by

$$S_j = \{q \in \partial R_{k+1} : \psi_j^{(k+1)}(q) < u_J^{(k)}(q) + \epsilon\}$$

is expanding to the union ∂R_{k+1} . Both $\psi_j^{(k+1)}$ and $u_J^{(k)}$ are continuous on ∂R_{k+1} , so that each set S_j is relatively open. Therefore, since ∂R_{k+1} is compact, there is a number j_0 such that $S_j = \partial R_{k+1}$ whenever $j > j_0$. Thus $\psi_j^{(k+1)}(q) < u_J^{(k)}(q) + \epsilon$ for all $q \in \partial R_{k+1}$ if $j > j_0$. This implies, using the maximum principle, that $u_j^{(k+1)}(q) < u_J^{(k)}(q) + \epsilon$ for all $q \in \overline{R}_{k+1}$ if $j > j_0$. Therefore $v_{k+1} < u_J^{(k)} + \epsilon$ on \overline{R}_{k+1} for any $\epsilon > 0$ and positive integer J , and so $v_{k+1} \leq v_k$. Hence the sequence $\{v_k\}$ is decreasing on $\overline{\Omega}(x_0, t_0; c)$. Put $v = \lim_{k \rightarrow \infty} v_k \geq w$ on $\overline{\Omega}(x_0, t_0; c)$. Whenever $(x, t) \in \partial\Omega(x_0, t_0; c)$ and $t \leq t_0 - \frac{1}{k}$, we have

$$v_k(x, t) = \lim_{j \rightarrow \infty} \psi_j^{(k)}(x, t) = w(x, t),$$

so that $v(x, t) = w(x, t)$ for all $(x, t) \in \partial\Omega(x_0, t_0; c) \setminus \{(x_0, t_0)\}$. Hence the monotone convergence theorem shows that

$$w(x_0, t_0) \leq v(x_0, t_0) = \lim_{k \rightarrow \infty} \mathcal{M}(v_k; x_0, t_0; d) = \mathcal{M}(v; x_0, t_0; d)$$

whenever $0 < d \leq c$. It follows that

$$\mathcal{M}(w; x_0, t_0; d) \leq \mathcal{M}(v; x_0, t_0; d) = v(x_0, t_0) = \mathcal{M}(v; x_0, t_0; c) = \mathcal{M}(w; x_0, t_0; c)$$

whenever $0 < d \leq c$. This proves the lemma.

Theorem 17. *Let w be an extended-real valued function on an open set E , that satisfies conditions (δ_1) , (δ_2) and (δ_3) of the definition of a subtemperature. Suppose that, given any point $p \in E$ and $\epsilon > 0$, we can find a positive number $c < \epsilon$ such that the inequality $w(p) \leq \mathcal{M}(w; p; c)$ holds. Then w is a subtemperature on E .*

Conversely, if w is a subtemperature on E and $p \in E$, then the inequality $w(p) \leq \mathcal{M}(w; p; c)$ holds for all $c > 0$ such that $\overline{\Omega}(p; c) \subseteq E$.

Proof. Suppose that, given any point $p \in E$ and $\epsilon > 0$, we can find a positive number $c < \epsilon$ such that $w(p) \leq \mathcal{M}(w; p; c)$. Let R be a convex domain of revolution such that $\overline{R} \subseteq E$. Then w satisfies the same conditions on R as it does on E . We use Theorem 15. Let $v \in C(\overline{R})$, be a temperature on R , and satisfy $v \geq w$ on $\partial_n R$. Then $w - v$ satisfies the same conditions on R as does w , in view of [13] (Theorem 2 Corollary). Therefore $w - v$ satisfies the maximum principle of Theorem 16. Furthermore, whenever $q \in \partial_n R$ we have

$$\limsup_{p \rightarrow q, p \in R} (w(p) - v(p)) \leq w(q) - v(q) \leq 0,$$

so that $w(p) \leq v(p)$ for all $p \in R$. Hence w is a subtemperature on E , by Theorem 15.

Now suppose, conversely, that w is a subtemperature on E . Then, by Theorem 15, w satisfies the hypotheses of Lemma 8, and the result follows.

Corollary. *Let $w \in C^{2,1}(E)$. Then w is a subtemperature on E if and only if $\Theta w \geq 0$ on E .*

Proof. If w is a subtemperature on E and $p \in E$, then the inequality $w(p) \leq \mathcal{M}(w; p; c)$ holds for all $c > 0$ such that $\bar{\Omega}(p; c) \subseteq E$, by Theorem 17. So $\Theta w \geq 0$ on E , by [13] (Theorem 2).

Conversely, if $\Theta w \geq 0$ then $w(p) \leq \mathcal{M}(w; p; c)$ holds whenever $\bar{\Omega}(p; c) \subseteq E$, by [13] (Theorem 2). Therefore w is a subtemperature on E , by Theorem 17.

Our next theorem shows that subtemperatures can be characterized in terms of the class \mathfrak{R} of Lemma 8.

Theorem 18. *Let w be an extended-real valued function on an open set E , that satisfies conditions (δ_1) , (δ_2) and (δ_3) of the definition of a subtemperature. Let \mathfrak{R} denote the class of convex domains of revolution R , for which both $\partial_n R = \partial R$ and the Dirichlet problem on R has a solution for every $f \in C(\partial R)$. Consider the following property: Whenever $R \in \mathfrak{R}$ is such that $\bar{R} \subseteq E$, and v is a function in $C(\bar{R})$ that is a temperature on R and satisfies $v \geq w$ on ∂R , then $v \geq w$ on \bar{R} .*

The stated property holds if and only if w is a subtemperature on E .

Proof. If w is a subtemperature on E , then the stated property follows from Theorem 15.

Conversely, if the stated property holds then, by Lemma 8, the inequality $w(p) \leq \mathcal{M}(w; p; c)$ holds whenever $\bar{\Omega}(p; c) \subseteq E$. So w is a subtemperature on E , by Theorem 17.

We can now show that a function is subcaloric if and only if it is a subtemperature. Recall that a bounded open set $V \subseteq \mathbb{R}^{n+1}$ is called *regular* (in harmonic space theory) if, for every function $f \in C(\partial V)$, there is a function $u_f \in C(\bar{V})$ which is a temperature on V and equal to f on ∂V . A function w on E is called *subcaloric* if it satisfies conditions (δ_1) , (δ_2) and (δ_3) of the definition of a subtemperature, and in addition has the following property: Whenever V is a regular, bounded open set such that $\bar{V} \subseteq E$, and $f \in C(\partial V)$ with $f \geq w$ on ∂V , then $u_f \geq w$ on V .

Theorem 19. *Let w be an extended-real valued function on an open set E . Then w is a subtemperature if and only if it is a subcaloric function.*

Proof. Suppose that w is a subtemperature on E , that V is a regular, bounded open set such that $\bar{V} \subseteq E$, and that f is a function in $C(\partial V)$ with $f \geq w$ on ∂V . Then the function $w - u_f$ satisfies the maximum principle of Theorem 8 on V , and

$$\limsup_{p \rightarrow q} (w - u_f)(p) = \limsup_{p \rightarrow q} w(p) - \lim_{p \rightarrow q} u_f(p) \leq w(q) - f(q) \leq 0$$

whenever $p \in V$ tends to a boundary point q of V . Thus $w \leq u_f$ on V , and hence w is subcaloric.

The converse follows from Theorem 18.

Now we come to our characterization of subtemperatures in terms of the volume means \mathcal{V} .

Theorem 20. *Let w be an extended-real valued function on an open set E , that satisfies conditions (δ_1) , (δ_2) and (δ_3) of the definition of a subtemperature. Suppose that, given any point $p \in E$ and $\epsilon > 0$, we can find a positive number $c < \epsilon$ such that the inequality $w(p) \leq \mathcal{V}(w; p; c)$ holds. Then w is a subtemperature on E .*

Conversely, if w is a subtemperature on E and $p \in E$, then the inequality $w(p) \leq \mathcal{V}(w; p; c)$ holds for all $c > 0$ such that $\bar{\Omega}(p; c) \subseteq E$.

Proof. The proof of the first part is similar to that of the first part of Theorem 17, but uses Theorem 6 Corollary of [13].

Conversely, if w is a subtemperature on E and $p \in E$ then, by Theorem 17, the inequality $w(p) \leq \mathcal{M}(w; p; l)$ holds for all $l > 0$ such that $\bar{\Omega}(p; l) \subseteq E$. It therefore follows from (2.1) that

$$\mathcal{V}(w; p; c) = \frac{n}{2}c^{-\frac{n}{2}} \int_0^c l^{\frac{n}{2}-1} \mathcal{M}(w; p; l) dl \geq \frac{n}{2}c^{-\frac{n}{2}} \int_0^c l^{\frac{n}{2}-1} w(p) dl = w(p)$$

whenever $\bar{\Omega}(p; c) \subseteq E$.

If we use either the fundamental means \mathcal{M} or the volume means \mathcal{V} , we can weaken the finiteness condition (δ_3) in the definition of a subtemperature, as the next theorem and its second corollary show. The bulk of the proof is contained in the following lemma.

Lemma 9. *Let w be a locally upper bounded, extended-real valued function on an open set E , and let $(x_0, t_0) \in E$. If $w(x_0, t_0) > -\infty$, and the inequality $w(y, s) \leq \mathcal{V}(w; y, s; c)$ holds whenever $\bar{\Omega}(y, s; c) \subseteq \Lambda(x_0, t_0; E) \cup \{(x_0, t_0)\}$, then w is locally integrable on $\Lambda(x_0, t_0; E)$.*

Proof. We prove the contrapositive. If w is not locally integrable on $\Lambda(x_0, t_0; E)$, then we can find a point $(x_1, t_1) \in \Lambda(x_0, t_0; E)$ such that w is not integrable on any neighbourhood of (x_1, t_1) . Join (x_0, t_0) to (x_1, t_1) by a polygonal path γ in $\Lambda(x_0, t_0; E) \cup \{(x_0, t_0)\}$ along which the temporal variable is strictly decreasing. Since γ is compact, its distance from $\mathbb{R}^{n+1} \setminus E$ is positive, and so we can find $c_0 > 0$ such that $\bar{\Omega}(x, t; c_0) \subseteq E$ for all $(x, t) \in \gamma$. Given $(x, t) \in \gamma$, we put

$$P(x, t) = \{(y, s) : |y - x|^2 < 2n(s - t), s - t < c_0/e\}.$$

The set $P(x, t)$ is a truncated paraboloid with vertex (x, t) , and if $(y, s) \in P(x, t)$ then

$$|y - x|^2 < 2n(s - t) < 2n(s - t) \log \left(\frac{c_0}{s - t} \right),$$

so that $(x, t) \in \Omega(y, s; c_0)$.

Observe that, because γ is a union of finitely many line segments, there is a positive number $c_1 < c_0/e$, independent of (x, t) , such that if $(x, t), (y, t + c_1) \in \gamma$ then $(y, t + c_1) \in P(x, t)$. Choose points $(x_2, t_2), \dots, (x_l, t_l)$ inductively, such that $t_j = t_1 + (j-1)c_1$ and $(x_j, t_j) \in \gamma$, for all $j \in \{2, \dots, l\}$, and such that $t_l < t_0 \leq t_l + c_1$. Note that $(x_j, t_j) \in P(x_{j-1}, t_{j-1})$ for all $j \in \{2, \dots, l\}$. Since $(x_1, t_1) \in \Lambda(x_0, t_0; E)$, we have $(x_1, t_1) \in \Omega(y, s; c_0)$ for all $(y, s) \in P(x_1, t_1)$. Therefore

$$w(y, s) \leq (4\pi c_0)^{-\frac{n}{2}} \iint_{\Omega(y, s; c_0)} \frac{|y - z|^2}{4(s - r)^2} w(z, r) dz dr = -\infty$$

for all $(y, s) \in P(x_1, t_1)$ such that $y \neq x_1$. In particular, w is not integrable on any neighbourhood of (x_2, t_2) . Proceeding stepwise along γ , we deduce successively that w is not integrable on any neighbourhood of $(x_2, t_2), \dots, (x_l, t_l)$. Since $t_l < t_0 \leq t_l + c_1$, we have $(x_l, t_l) \in \bar{\Omega}(x_0, t_0; c_0)$. Now $w(y, s) = -\infty$ for all $(y, s) \in P(x_l, t_l)$ such that $y \neq x_l$, so that $w(x_0, t_0) \leq \mathcal{V}(w; x_0, t_0; c_0) = -\infty$.

Theorem 21. *Let w be an extended-real valued function on an open set E . Then w is a subtemperature on E if and only if it satisfies the following four conditions:*

- (a) $-\infty \leq w(p) < \infty$ for all $p \in E$;
- (b) w is upper semicontinuous on E ;
- (c) given any point $p \in E$, we can find a point $q \in E$ such that $p \in \Lambda(q; E)$ and $w(q) > -\infty$;
- (d) the inequality $w(p) \leq \mathcal{V}(w; p; c)$ holds whenever $\bar{\Omega}(p; c) \subseteq E$.

Furthermore, every subtemperature on E is locally integrable on E .

Proof. Theorem 20 shows that any subtemperature on E satisfies the four conditions.

To prove the remainder of the theorem, it suffices to show that any function w which satisfies the four conditions is locally integrable on E , in view of Theorem 20. Given condition (c), this follows from Lemma 9.

Corollary 1. *If v and w are subtemperatures on the open set E , then $v + w$ is also a subtemperature on E .*

Proof. Conditions (δ_1) and (δ_2) are obviously satisfied by $v + w$, and (δ_4) follows from Theorem 9. For (δ_3) , Theorem 21 shows that each of v and w is finite outside a set of full measure, so that $v + w$ is too, and hence $v + w$ is finite on a dense subset of E .

Corollary 2. *Let w be an extended-real valued function on an open set E . Then w is a subtemperature on E if and only if it satisfies the following four conditions:*

- (a) $-\infty \leq w(p) < \infty$ for all $p \in E$;
- (b) w is upper semicontinuous on E ;
- (c) given any point $p \in E$, we can find a point $q \in E$ such that $p \in \Lambda(q; E)$ and $w(q) > -\infty$;
- (d) the inequality $w(p) \leq \mathcal{M}(w; p; c)$ holds whenever $\bar{\Omega}(p; c) \subseteq E$.

Proof. If w satisfies the conditions in the corollary, then it also satisfies those in the theorem, because of formula (2.1). So w is a subtemperature on E .

Conversely, if w is a subtemperature on E , then Theorem 17 shows that it satisfies the conditions of the present corollary.

Theorem 21 Corollary 2 shows that w is a subtemperature by our current definition if and only if it is a subtemperature by the definition in [10].

References

- [1] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory*, Springer-Verlag, London, 2001.

- [2] H. Bauer, *Harmonische Räume und ihre Potentialtheorie*, Lecture Notes in Math. 22, Springer-Verlag, Berlin, 1966.
- [3] H. Bauer, "Harmonic Spaces - a survey", *Conf. Semin. Mat. Univ. Bari*, 197(1984)34p.
- [4] H. Bauer, "Heat balls and Fulks measures", *Ann. Acad. Sci. Fenn., Ser. A, I. Math.*, 10(1985)67-82.
- [5] W. Fulks, "A mean value theorem for the heat equation", *Proc. Amer. Math. Soc.*, 17(1966)6-11.
- [6] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York, 1965.
- [7] E. M. Landis, *Second Order Equations of Elliptic and Parabolic Type*, Translations of Math. Monographs 171, American Math. Soc. (1998).
- [8] J. R. Hattemer, "Boundary behavior of temperatures I", *Studia Math.*, 25(1964)111-155.
- [9] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
- [10] N. A. Watson, "A theory of subtemperatures in several variables", *Proc. London Math. Soc.*, 26(1973)385-417.
- [11] N. A. Watson, "Green functions, potentials, and the Dirichlet problem for the heat equation", *Proc. London Math. Soc.*, 33(1976)251-298.
- [12] N. A. Watson, "The weak maximum principle for parabolic differential inequalities", *Rend. Circ. Mat. Palermo*, 33(1984)421-425.
- [13] N. A. Watson, "Elementary proofs of some basic subtemperature theorems", *Colloq. Math.*, 94(2002)111-140.

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