BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 62, No. 4, 2014
DOI: 10.2478/bpasts-2014-0097

Entropy generation minimization in transient heat conduction processes PART II – Transient heat conduction in solids

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Abstract. Formulation and solution of the initial boundary-value problem of heat conduction in solids have been presented when an entropy generation minimization principle is imposed as the arbitrary constraint. Using an entropy balance equation and the Euler-Lagrange variational approach a new form of the heat conduction equation (non-linear partial difference equation) is derived.

Key words: transient heat conduction, initial boundary value problem, entropy generation minimization.

Nomenclature

c – constants,
c_p – specific heat capacity [kJ/kg·K],
i – internal,
k – heat conduction coefficient [W/m·K],
L_0 – Wiedemann-Franz constant $[\text{W/A·K}]^2$,
$\dot{m}$ – mass flow rate [kg/h],
$\dot{q}$ – heat flux [W/m^2],
$\dot{q}_v$ – intensity of internal heat source [W/m^3],
$\dot{s}$ – entropy flux [W/m^2·K],
$\dot{S}_{gen}$ – entropy generation due to the process irreversibility [W/m^3·K],
T – absolute temperature [K],
x, y – Cartesian coordinates,
Θ – transformed temperature,
Ω – domain,
∇ – operator nabla,
τ - time,
ρ – density,
λ – Lagrange multiplier,
δ – difference.

Indexes

He – helium,
in – entering,
o – environment,
out – leaving,
t – total,
x,y – partial derivatives with respects to x and y.

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1. Introduction

The transient heat conduction equation assuming minimization of the entropy generation rate, $\dot{S}_{gen,\min}$, can be derived in two ways – from the Euler-Lagrange variational principle or direct minimization of the expression describing entropy generation rate.

Euler-Lagrange transient heat conduction equation.

The function to be minimized is

$$\dot{S}_{gen} = \int_{\Omega} \int_{\tau=0}^{\infty} \frac{k}{T^2} \nabla T \circ \nabla T d\tau,$$

where k is thermal conductivity coefficient (assumed constant), $T = T(x, y, z, \tau)$ – temperature field, τ denotes time and Ω is the domain model consideration, with additional condition

$$\int_{\Omega} \int_{\tau=0}^{\infty} \frac{\partial S}{\partial \tau} d\Omega d\tau = C, \tag{1}$$

where $S = S(\Omega, \tau)$ is entropy of the system and C is constant. Condition (1) represents entropy change of the system from initial to final equilibrium state. The problem can be dealt with by means of Lagrange's method of undetermined multipliers as follows

Find function $T(x, y, z, \tau)$ which satisfying required initial and boundary conditions minimizes simultaneously integral

$$\dot{S}_{gen} = \int_{\Omega} \int_{\tau=0}^{\infty} \frac{k}{T^2} \nabla T \circ \nabla T d\tau \Rightarrow \text{ minimum}$$

over the whole domain Ω provided that

$$\int_{\Omega} \int_{\tau=0}^{\infty} \frac{\partial S}{\partial \tau} d\Omega d\tau = C.$$

To solve the problem a new function [1]

$$K = \frac{k}{T^2}\nabla T \circ \nabla T + \lambda \frac{\partial S}{\partial \tau}$$

is introduced where λ is the Lagrange multiplier. The extremals must satisfy the Euler-Lagrange equation

$$\frac{\partial K}{\partial T} - \sum_{i=1}^{3} \frac{d}{dx_i}\left(\frac{\partial K}{\partial T_{x_i}}\right) = 0,$$

where T_{x_i} denotes gradient components $\partial T/\partial x_i$, ($x_i = x, y, z$). Introducing from thermodynamics

$$\frac{dS}{d\tau} = \frac{\rho c_p}{T}\frac{\partial T}{\partial \tau}$$

function K is

$$K = \frac{k}{T^2}\nabla T \circ \nabla T + \lambda \frac{\rho c_p}{T}\frac{\partial T}{\partial \tau} \tag{2}$$

Varational calculus requires the function (2) must satisfy the Euler-Lagrange equation

$$\frac{\partial K}{\partial T} - \frac{\partial}{\partial x}\left(\frac{\partial K}{\partial T_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial K}{\partial T_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial K}{\partial T_z}\right) = 0.$$

After calculations

$$k\nabla^2 T - \frac{k}{T}\nabla T \circ \nabla T + \frac{\lambda}{2}\rho c_p \frac{\partial T}{\partial \tau} = 0. \tag{3}$$

From the fact that the third component of Eq. (3) does not depend on the path of the process, value of λ can be chosen arbitrary. Assuming $\lambda = -2$, Eq. (3) becomes in the final form

$$a\nabla^2 T - \frac{a}{T}\nabla T \circ \nabla T = \frac{\partial T}{\partial \tau}, \tag{4}$$

where $a = k/\rho c_p$ is diffusion coefficient. Additional external heat source $\dot{q}_{v,ad}$ is described by the second term of Eq. (4) and is

$$\dot{q}_{v,a} = -\frac{a}{T}\nabla T \circ \nabla T.$$

2. Numerical example

Consider 1D classical initial-boundary problem given by: (in dimensionless form)

- governing equations

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial \tau}, \qquad T = T(x,\tau), \qquad x \in (-1,1)\,, \tag{5}$$

- boundary conditions

$$T\,(1,\tau) = T(1) = 1,$$

$$T\,(-1,\tau) = T(-1) = 1,$$

- initial conditions

$$T\,(x,0) = T(x) = e.$$

A general scheme is presented in Fig. 1. Analytical solution can be easily obtained with separation of variables method and is

$$T(x,\tau) = 1 + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp\left[-\frac{(2n+1)^2\pi^2}{4}\tau\right] \cos\,(2n+1)\,\frac{\pi}{2}x. \tag{6}$$

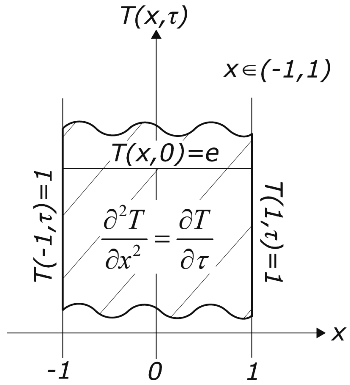

Fig. 1. Initial boundary value problem

When the entropy generation minimization approach is applied, the initial-boundary value problem takes the form: (dimensionless form)

- governing equations

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{T}\left(\frac{\partial T}{\partial x}\right)^2 = \frac{\partial T}{\partial \tau},$$

$$T = T(x,\tau), \qquad x \in (-1,1)\,,$$

- with boundary conditions

$$T\,(-1,\tau) = 1,$$
$$T\,(1,\tau) = 1,$$

- and initial conditions

$$T\,(x,0) = T(x) = e.$$

Introducing new variable

$$\theta\,(x,\tau) = \ln T(x,\tau)$$

initial-boundary value problem becomes

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial \tau}, \qquad \theta = \theta(x,\tau) \tag{7}$$

and

$$\theta\,(0,\tau) = 0,$$

$$\theta\,(1,\tau) = 0,$$

$$\theta\,(x,0) = 1.$$

Its analytical solution is [2]

$$\theta(x,\tau) = -\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp\left[-\frac{(2n+1)^2\pi^2}{4}\tau\right] \cos\,(2n+1)\,\frac{\pi}{2}x \tag{8}$$

and

$$T\,(x,\tau) = \exp\left(\theta(x,\tau)\right).$$

Solutions (6) and (8) are presented in Figs. 2 and 3 respectively. Difference

$$\delta T = T\,(x,\tau)_{clasical} - T\,(x,\tau)_{\dot{S}_{gen}} \tag{9}$$

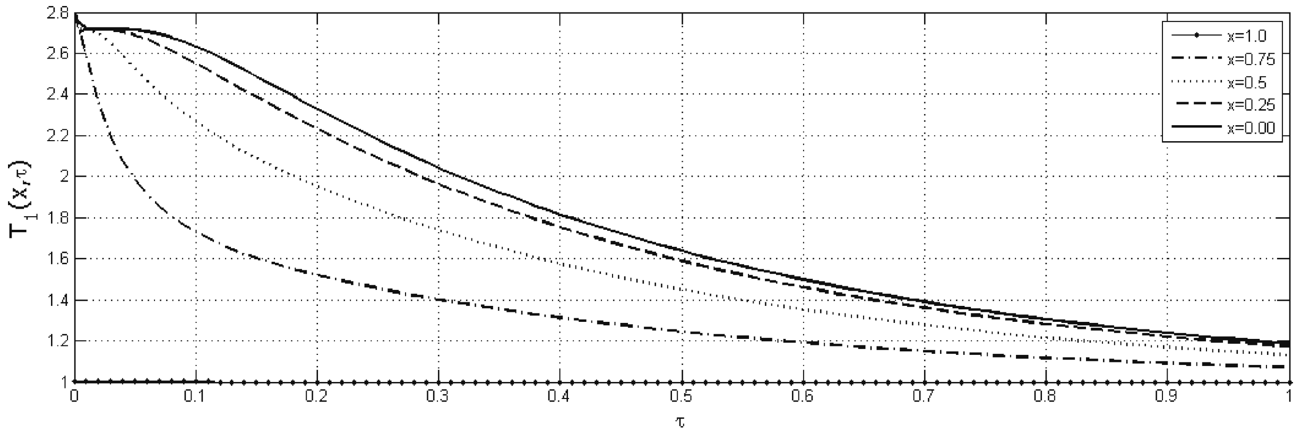

Fig. 2. Solution of classical problem

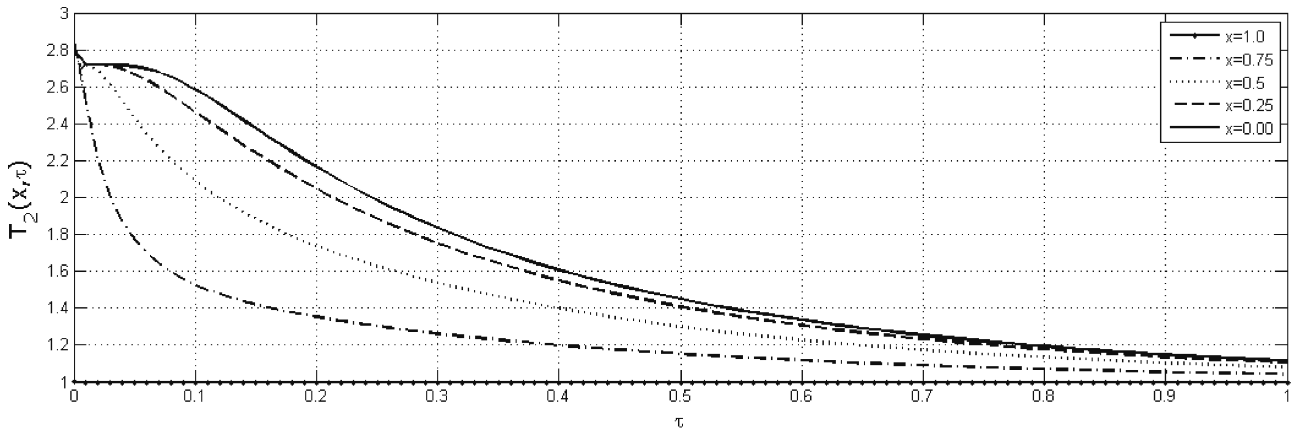

Fig. 3. Solution with entropy generation minimization condition

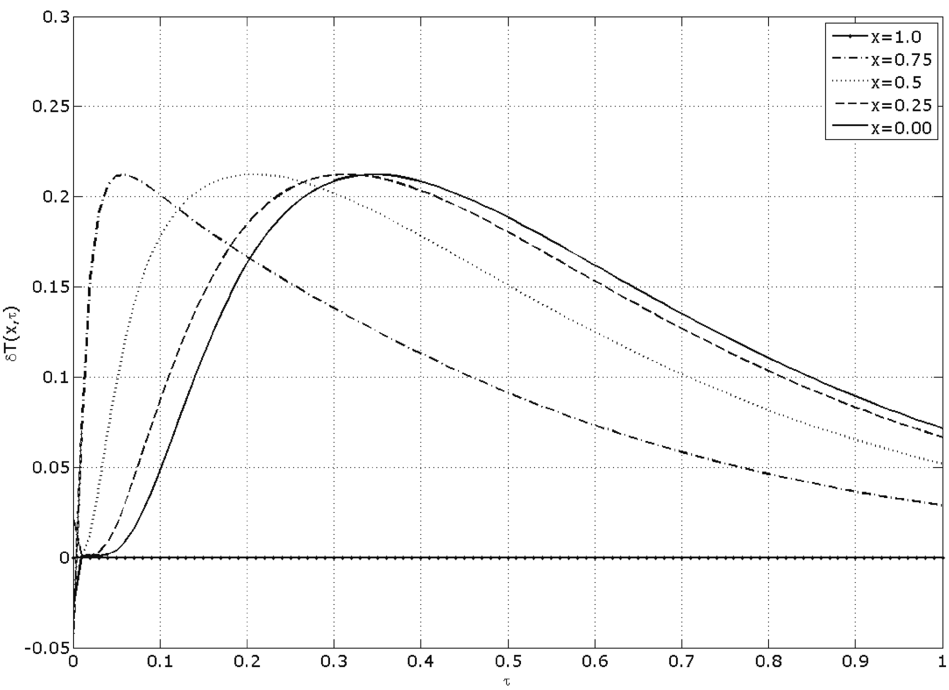

Fig. 4. Temperature difference $\delta T(x, \tau)$ according to Eq. (9)

is shown in Fig. 4 where $T(x,\tau)_{clasical}$ and $T(x,\tau)_{\dot{S}_{gen}}$ are solutions of classical and entropy generation minimization initial-boundary value problems (5) and (7). Solutions (6) and (8) allow to calculate entropy generation rates

$$\dot{S}_{gen,t} = 2\int\limits_0^1\int\limits_0^\infty \frac{1}{T^2}\left(\frac{\partial T(x,\tau)}{\partial x}\right)^2 dx d\tau$$

and

$$\dot{S}_{gen,t} = 2\int\limits_0^1\int\limits_0^\infty \left(\frac{\partial \theta(x,\tau)}{\partial x}\right)^2 dx d\tau.$$

Numerical calculations gives

- classical initial-boundary value problem

$$\dot{S}_{gen,cl} = 0.332 \ \ (\text{J/m}^3\text{K}),$$

- entropy generation minimization

$$\dot{S}_{gen,cl} = 0.267 \ \ (\text{J/m}^3\text{K}).$$

3. Example of general solution

3.1. Initial-boundary value problem when boundary conditions depends on time.

- governing equations (dimensionless form)

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{T}\left(\frac{\partial T}{\partial x}\right)^2 = \frac{\partial T}{\partial \tau},$$

$$T = T(x,\tau), \qquad x \in (0,1),$$

- boundary conditions

$$T(0,\tau) = \phi_1(\tau),$$

$$T(1,\tau) = \phi_2(\tau),$$

- initial conditions

$$T(x,0) = f(x).$$

After transformation

$$\theta(x,\tau) = \ln T(x,\tau)$$

the problem is

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial, \tau},$$

$$\theta(0,\tau) = \ln \phi_1(\tau),$$

$$\theta(1,\tau) = \ln \phi_2(\tau),$$

$$\theta(x,0) = \ln f(x)$$

and its solution is given with the use of Duhamel's theorem [2]:

$$\theta(x,\tau) = 2\sum_{n=0}^{\infty} \exp\left(-n^2\pi^2\tau\right)\sin(n\pi x)$$
$$\left[\int\limits_0^1 \ln f(x')\frac{\pi}{2}\sin(n\pi x')dx' + \right.$$
$$\left. + n\pi\int\limits_0^\tau \exp\left(n^2\pi^2\lambda\right)\left\{\ln\phi_1(x) - (-1)^n \ln\phi_2(x)\right\}\right]$$

and

$$T(x,\tau) = \exp(\theta(x,\tau)).$$

In such cases entropy generation rate calculation requires numerical calculation.

3.2. Consider 2D initial-boundary value problem.

- governing equations (dimensionless form)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{1}{T}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] = \frac{\partial T}{\partial \tau},$$

$$T = T(x,\tau), \qquad x, y \in (-1,1),$$

- boundary conditions

$$T(x,-1,\tau) = T(x,1,\tau)$$

$$= T(-1,y,\tau) = T(1,y,\tau) = 1,$$

- initial conditions

$$T(x,y,0) = T_0(x,y).$$

After introducing

$$\theta(x,y,\tau) = \ln T(x,y,\tau)$$

the problem takes the form

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{\partial \theta}{\partial \tau}$$

and

$$\theta(x,-1,\tau) = \theta(x,1,\tau) = \theta(-1,y,\tau) = \theta(1,y,\tau) = 0,$$

$$\theta(x,y,0) = \theta_0(x,y) = \ln T_0(x,y).$$

The solution for θ is [3]:

$$\theta(x,y,\tau) = \psi(x,1)\cdot\psi(y,1)\text{erf}\frac{x}{2\sqrt{\tau}},$$

where

$$\psi(x,1) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp\left[-\frac{(2n+1)^2\pi^2}{4}\tau\right]$$
$$\cos(2n+1)\frac{\pi}{2}x$$

and finally

$$T(x,y,\tau) = \exp(\theta(x,y,\tau)).$$

Solutions of many initial boundary-value problems can be found in heat transfer literature, [2] and in monographs on analytical solutions of non-linear partial differential equations [3].

4. Conclusions

The results illustrate a practical aspect of the Principle of Entropy Generation Minimization [4] that:

"The entropy of a system can be reduced only if it is made to interact with one or more auxiliary systems in a process which impacts to these at least a compensating amount of entropy".

In the case of heat conduction processes interaction with outside systems are realized by additional internal heat source which becomes as the component of Euler-Lagrange equation. In this way it is possible to explain formation processes of dissipative structures [5]. Additional heat source decreases irreversibility ratio.

Analysis of the data presented in Fig. 4 ($\delta T(x,\tau)$) points out that the temperature changes with time are more intensive (faster) when the source is included. It leads directly to more effective consumption of driving energy and minimization of natural resources. The method is directly connected with exergy optimization.

Acknowledgements. Authors are grateful for the support of the Ministry of Science and Higher Education (grant AGH no 11.11.210.198).

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