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ON THE INVERSE LAPLACE-STIELTJES TRANSFORM OF A-STABLE RATIONAL FUNCTIONS

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Abstract. Let r be an A-stable rational approximation of the exponential function of order $q \geq 1$ and let $t > 0$. It is shown that the inverse Laplace-Stieltjes transforms $\alpha_n : s \to \alpha^{n*}(\frac{ns}{t})$ of $r_n(z) := r^n(\frac{tz}{n})$ converge in $L_p(\mathbb{R}_+)$ to the Heaviside function H_t with a rate of $t^{1/p}n^{-1/2p}(\ln(n+1))^{1-1/p}$. Moreover, for $0 \leq k \leq q$, the k-th antiderivatives of α_n converge in $L_p(\mathbb{R}_+)$ to the k-th antiderivative of the Heaviside function with a speed that increases with k. In particular, the q-th antiderivatives of α_n converge in $L_1(\mathbb{R}_+)$ to the q-th antiderivative of the Heaviside function H_t with the optimal rate of $t(\frac{t}{n})^q$. In addition to the L^p-estimates, bounds on the total variation and supremum norms of α_n are given. Via the Hille-Phillips functional calculus for operator semigroups, the results have immediate applications to the error analysis of rational time discretization methods for evolution equations.

1. Introduction

Let $r_n(z) := \int_0^\infty e^{zs}\, d\alpha_n(s)$ converge pointwise to $v(z) = \int_0^\infty e^{zs}\, d\alpha(s)$ $(z < 0)$, where α, α_n are functions of bounded total variation. Does this imply the convergence of α_n to α and, if yes, in what sense? Moreover, if it is known how fast r_n converges to v, what can be said about the speed of convergence of α_n to α in various norms? Motivated by applications to time discretization methods, of particular interest are cases where $r_n(z) := r^n(\frac{tz}{n}) \to e^{tz}$ for some rational function r with

(a) $r(z) = e^z + O(z^{q+1})$ as $z \to 0$ for some $q \in \mathbb{N}$, and
(b) $|r(z)| \leq 1$ for $\operatorname{Re} z \leq 0$.

Such functions r are called A-stable rational approximations of the exponential of order q. Each such r is the Laplace-Stieltjes transform of a function α with finite total variation. Moreover,

$$r^n(\frac{tz}{n}) = \int_0^\infty e^{zs}\, d\alpha_{n,t}(s) \to e^{tz} = \int_0^\infty e^{zs}\, dH_t(s)$$

$(z < 0, t > 0, n \to \infty)$, where $\alpha_n := \alpha_{n,t}$ is the n-th Stieltjes convolution power $s \to \alpha^{n*}(\frac{ns}{t})$ and H_t is the Heaviside function with jump at t. By translating technical arguments of [**4**] and [**8**] into a Laplace-Stieltjes transform setting, in Theorems 3.1 and 3.4 it is shown that the total variation of $\alpha_{n,t}$ may grow at most like $\sqrt{n}$. Hence, in general, the functions $\alpha_{n,t}$ will not converge towards H_t with respect to

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the total variation norm. In Theorem 3.6 it is established that the L_∞-norm of $\alpha_{n,t}$ cannot increase faster than $\ln(n+1)$. With this result one obtains convergence in $L_p(\mathbb{R}_+)$ once the convergence in $L_1(\mathbb{R}_+)$ is established. In Section 4, based on the complex inversion formula for the Laplace-Stieltjes transform and analytical techniques developed by P. Brenner and V. Thomée in [**4**], the convergence of $\alpha_{n,t}$ and its k-th antiderivatives $(0 \le k \le q)$ in $L_1(\mathbb{R}_+)$ is established together with convergence rate estimates that improve with increased k. In Theorem 4.5, combining the L_1–result with the logarithmic L_∞–growth bound, L_p-error estimates are given for $\alpha_{n,t} - H_t$ and its k-th antiderivatives $I^{(k)}(\alpha_{n,t} - H_t)$ when $1 \le p < \infty$.

Using the Hille–Phillips functional calculus, these estimates yield convergence estimates for rational approximation schemes for strongly continuous semigroups[1](see [**4**], [**9**], [**12**], [**14**]). Let X be a Banach space and let $A : X \supset \mathcal{D}(A) \to X$ generate a strongly continuous semigroup of linear operators $T(\cdot)$ bounded by $M \ge 1$ (for details, see [**1**]). For A-stable rational approximations r of the exponential of order q the operators

$$r^n(\frac{t}{n}A)x = \int_0^\infty T(s)x \, d\alpha_{n,t}(s)$$

are well defined (where $\alpha_{n,t}$ is as above; for details see, for example, [**10**, Chapter XV] and [**13**]). It is immediate from the definition that for any $\tau \ge 0$ we have $||f(\tau A)|| \le MV_\alpha(\infty)$ which gives the estimate $||r^n(\tau A)|| \le K\sqrt{n}$ by Theorem 3.1. For sufficiently smooth initial data one can integrate by parts k-times ($k = 1, 2, \ldots, q+1$) and obtain

$$\begin{aligned} &r^n(\frac{t}{n}A)x - T(t)x \\ &= \int_0^\infty T(s)x \, d[\alpha_{n,t}(s) - H_t(s)] = (-1)^k \int_0^\infty I^{(k-1)}(\alpha_{n,t} - H_t)(s) \frac{d^k T(s)x}{ds^k} \, ds. \end{aligned}$$

Hence, L_p-estimates of $I^{(k-1)}(\alpha_{n,t} - H_t)$ result in error estimates for $r^n(\frac{t}{n}A)x - T(t)x$ for those x with appropriately regular orbits $s \mapsto T(s)x$ (for details, see [**14**]).

2. Preliminaries and Basic Inequalities

A bounded variation function $\alpha : [0, R] \to \mathbb{C}$ is in NBV_R if it is *normalized*; i.e., $\alpha(0) = 0$ and $\alpha(u) = \frac{\alpha(u+)+\alpha(u-)}{2}$ $(u \in (0, R))$. The space $NBV_{loc} := \cap_{R>0} NBV_R$ is an algebra with multiplication defined by the *Stieltjes convolution* $(\alpha * \beta)(t) = \int_0^t \alpha(t-u) \, d\beta(u) = \int_0^t \beta(t-u) \, d\alpha(u)$ $(t \notin P_{\alpha+\beta})$, where $P_{\alpha+\beta} := \{t \in \mathbb{R} : \ t = t_\alpha + t_\beta, \ t_\alpha \in P_\alpha, \ t_\beta \in P_\beta\}$, and where P_α (and similarly P_β) denotes the countable set of discontinuity points of α. If P_α or P_β is empty, then $P_{\alpha+\beta}$ is defined to be the empty set. If $\alpha, \beta \in NBV_R$, then $\gamma := \alpha * \beta$ exists on $[0, R] \setminus P_{\alpha+\beta}$ and γ may be defined on $P_{\alpha+\beta}$ so that it becomes normalized (see [**16**, Thms 11.1 and 11.2a]). Let $V_\alpha(\infty)$ denote the total variation of $\alpha \in NBV_{loc}$ on $[0, \infty)$. Then $NBV := \{\alpha \in NBV_{loc} : \ V_\alpha(\infty) < +\infty\}$ is a Banach algebra with norm $||\alpha|| := V_\alpha(\infty)$. Let $\mathcal{G} := \{f_\alpha : \ f_\alpha(z) = \int_0^\infty e^{zt} \, d\alpha(t) \text{ if } \operatorname{Re} z \le 0, \ \alpha \in NBV\}$. Next, we show that A-stable rational functions belong to $\mathcal{G}$ (see, also, [**10**, p. 441]).

Proposition 2.1. *If a rational function r is bounded for* $\operatorname{Re} z \le 0$, *then* $r \in \mathcal{G}$.

[1]For convergence estimates for distribution or C-regularized semigroups, see [**11**]

Proof. Clearly, constant functions and the functions $z \to \frac{1}{a-z}$ belong to the algebra $\mathcal{G}$ for $\operatorname{Re} a > 0$. Developing r into partial fractions, we see that $r \in \mathcal{G}$. $\square$

The proof of the following inequality is a straightforward modification of the proof of [**9**, Lemma 5] and is provided for convenience[2].

Proposition 2.2 (Carlson's Inequality)**.** *Assume that $f \in L_2(\mathbb{R})$ and $s \mapsto sf(s) \in L_2(\mathbb{R})$. Then, $f \in L_1(\mathbb{R})$ and*

$$\int_{-\infty}^{\infty} |f(s)|\,ds \leq 2 \left(\int_{-\infty}^{\infty} |f(s)|^2\,ds \right)^{\frac{1}{4}} \left(\int_{-\infty}^{\infty} |sf(s)|^2\,ds \right)^{\frac{1}{4}}.$$

Proof. Let $f \neq 0$ and note that $\|f\|_1 = \int_{-\infty}^{-c^2} |s|^{-1}|sf(s)|\,ds + \int_{-c^2}^{0} 1 \cdot |f(s)|\,ds + \int_0^{c^2} 1 \cdot |f(s)|\,ds + \int_{c^2}^{\infty} s^{-1}|sf(s)|\,ds$. By the Cauchy-Schwartz inequality,

$$\begin{aligned} \|f\|_1 \quad &\leq \quad \left(\int_{-\infty}^{-c^2} |s|^{-2}\,ds \right)^{\frac{1}{2}} \left(\int_{-\infty}^{-c^2} |sf(s)|^2\,ds \right)^{\frac{1}{2}} + c \left(\int_{-c^2}^{0} |f(s)|^2\,ds \right)^{\frac{1}{2}} \\ &\qquad + c \left(\int_0^{c^2} |f(s)|^2\,ds \right)^{\frac{1}{2}} + \left(\int_{c^2}^{\infty} s^{-2}\,ds \right)^{\frac{1}{2}} \left(\int_{c^2}^{\infty} |sf(s)|^2\,ds \right)^{\frac{1}{2}} \\ &\leq \quad c \left(\int_{-\infty}^{\infty} |f(s)|^2\,ds \right)^{\frac{1}{2}} + c^{-1} \left(\int_{-\infty}^{\infty} |sf(s)|^2\,ds \right)^{\frac{1}{2}}. \end{aligned}$$

The choice $c := \left(\int_{-\infty}^{\infty} |f(s)|^2\,ds \right)^{-\frac{1}{4}} \left(\int_{-\infty}^{\infty} |sf(s)|^2\,ds \right)^{\frac{1}{4}}$ yields the desired result. $\square$

It is noted that the above inequality remains true if we replace the constant 2 by $\sqrt{\pi}$ as shown by Carlson in [**7**] with equality for $f(s) = \frac{1}{1+s^2}$.

Corollary 2.3. *Assume that $f, f' \in L_2(\mathbb{R})$. Then the Fourier transform $\mathcal{F}(f)$ is in $L_1(\mathbb{R})$ and $||\mathcal{F}(f)||_1 \leq 2||f||_2^{\frac{1}{2}}\,||f'||_2^{\frac{1}{2}}$.*

Proof. Parseval's identity yields $\|\mathcal{F}(f)\|_2 = \|f\|_2$ and $\int_{-\infty}^{\infty} |s\mathcal{F}(f)(s)|^2\,ds = \|f'\|_2^2$. Now the result follows immediately from Proposition 2.2. $\square$

Throughout the paper the following inversion formula for the Laplace-Stieltjes transform will be useful (see, for example, [**16**, Chapter II, Thm 7a]).

Proposition 2.4 (Complex Inversion Formula)**.** *Let $f(z) = \int_0^{\infty} e^{zs}\,d\alpha(s)$ for $\alpha \in NBV_{loc}$ and $Re\, z < \sigma$. Then, for $c > \max(-\sigma, 0)$,*

$$\frac{1}{2\pi i} \lim_{R\to\infty} \int_{c-iR}^{c+iR} \frac{f(-z)}{z} e^{zs}\,dz = \begin{cases} \alpha(s) & \text{if } s > 0 \\ \frac{\alpha(0+)}{2} & \text{if } s = 0 \\ 0 & \text{if } s < 0. \end{cases} \tag{2.1}$$

A consequence of the Complex Inversion Theorem is a crucial estimate of $V_\alpha(\infty)$ using information of the behavior of its Laplace-Stieltjes transform on the imaginary axis. A related statement with a different proof can be found in [**4**, Lemma 2].

[2] For a more general version, see [**1**, Lemma 8.2.1]

Theorem 2.5. *Let* $f(z) = \int_0^\infty e^{zs}\,d\alpha(s)$, $\operatorname{Re} z \le 0$, *where* $\alpha \in NBV_{loc}$ *with* $\alpha(0+) = 0$ *and define* $f_0(s) := f(is)$. *Assume that* f *has an analytic extension to a neighborhood of* $i\mathbb{R}$. *If* $f_0, f_0' \in L_2(\mathbb{R})$, *then* α *is absolutely continuous on* $\mathbb{R}_+$ *and*

$$V_\alpha(\infty) = \frac{1}{\sqrt{2\pi}}||\mathcal{F}(f_0)||_1 \le \sqrt{\frac{2}{\pi}}||f_0||_2^{\frac{1}{2}}\,||f_0'||_2^{\frac{1}{2}}. \tag{2.2}$$

Proof. The integral in (2.1) can be replaced by $\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\Gamma_\varepsilon^R\cup\gamma_\varepsilon}\frac{f(-z)}{z}e^{zs}\,dz$, where $c > \varepsilon > 0$, $\gamma_\varepsilon(u) = \varepsilon e^{iu},\ u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\Gamma_\varepsilon^R(u) = iu,\ u \in [-R, -\varepsilon] \cup [\varepsilon, R]$. This follows from Cauchy's theorem and the fact that

$$\left|\int_{\Gamma_{\pm R,c}}\frac{f(-z)}{z}e^{zs}\,dz\right| \le \frac{c}{R}e^{cs}\sup_{\operatorname{Re} z\ge 0}|f(-z)| \to 0 \text{ as } R \to \infty,$$

where $\Gamma_{\pm R,c}(u) = \pm iR + u,\ u \in [0, c]$. Fix $s_0 \ge 0$. Since $\alpha(0+) = 0$ and $z \mapsto f(-z)e^{zs}$ is analytic in a neighborhood of $i\mathbb{R}$, it follows from Proposition 2.4 that

$$\begin{aligned}
\alpha(s_0) &= \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\Gamma_\varepsilon^R\cup\gamma_\varepsilon}\frac{f(-z)}{z}(e^{zs_0} - 1)\,dz \\
&= \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\Gamma_\varepsilon^R\cup\gamma_\varepsilon}\int_0^{s_0} f(-z)e^{zs}\,ds\,dz = \frac{1}{2\pi i}\lim_{R\to\infty}\int_0^{s_0}\int_{\Gamma_\varepsilon^R\cup\gamma_\varepsilon} f(-z)e^{zs}\,dz\,ds \\
&= \frac{1}{2\pi i}\lim_{R\to\infty}\int_0^{s_0}\int_{-iR}^{iR} f(-z)e^{zs}\,dz\,ds = \frac{1}{2\pi i}\int_0^{s_0}\overset{(2)}{\lim_{R\to\infty}}\int_{-iR}^{iR} f(-z)e^{zs}\,dz\,ds \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{s_0}\frac{1}{\sqrt{2\pi}}\overset{(2)}{\lim_{R\to\infty}}\int_{-R}^{R} f_0(v)e^{-ivs}\,dv\,ds,
\end{aligned}$$

where $\overset{(2)}{\lim}$ denotes the limit in $L_2(\mathbb{R})$. To see that we can interchange the limit and the integral above, let $f_R(s) := \frac{1}{\sqrt{2\pi}}\int_{-R}^{R} f_0(v)e^{-ivs}\,dv$. Since $f_0 \in L_2(\mathbb{R})$ it follows that $\overset{(2)}{\lim}_{R\to\infty} f_R := \mathcal{F}(f_0)$ exists and defines a uniquely determined function in $L_2(\mathbb{R})$ (see, for example, [**6**, p.210]). Therefore, $f_R \to \mathcal{F}(f_0)$ weakly as $R \to \infty$. Let $\chi_{[0,s_0]}$ denote he characteristic function of $[0, s_0]$. Then,

$$\lim_{R\to\infty}\int_0^{s_0} f_R(s)\,ds = \lim_{R\to\infty}\langle f_R, \chi_{[0,s_0]}\rangle = \langle\mathcal{F}(f_0), \chi_{[0,s_0]}\rangle = \int_0^{s_0}\mathcal{F}(f_0)(s)\,ds.$$

This proves that we can interchange the limit and the integral above, that α is absolutely continuous since

$$\alpha(s_0) = \frac{1}{\sqrt{2\pi}}\int_0^{s_0}\mathcal{F}(f_0)(s)\,ds, \tag{2.3}$$

and that $V_\alpha(\infty) = \frac{1}{\sqrt{2\pi}}||\mathcal{F}(f_0)||_1$. Since $f_0, f_0' \in L_2(\mathbb{R})$, it follows from Corollary 2.3 that $\mathcal{F}(f_0) \in L_1(\mathbb{R})$ and $||\mathcal{F}(f_0)||_1 \le 2||f_0||_2^{\frac{1}{2}}||f_0'||_2^{\frac{1}{2}}$. Therefore,

$$V_\alpha(\infty) = \frac{1}{\sqrt{2\pi}}||\mathcal{F}(f_0)||_1 \le \sqrt{\frac{2}{\pi}}||f_0||_2^{\frac{1}{2}}\,||f_0'||_2^{\frac{1}{2}}.$$

□

Corollary 2.6. *Let* $f(z) = \int_0^\infty e^{zs}\, d\alpha(s)$ *for* $\operatorname{Re} z \leq 0$ *and* $\alpha \in NBV_{loc}$. *If* f *extends analytically to a neighborhood of* $i\mathbb{R}$ *and* $f_0 - f(-\infty), f_0' \in L_2(\mathbb{R})$, *then* $\alpha \in NBV$. *In particular, if* $f(-\infty) := \lim_{x\to-\infty} f(x)$, *then*

$$\begin{aligned} V_\alpha(\infty) &= |f(-\infty)| + \frac{1}{\sqrt{2\pi}} \int_0^\infty |\mathcal{F}(f_0 - f(-\infty))(s)|\, ds \\ &\leq |f(-\infty)| + \sqrt{\frac{2}{\pi}} ||f_0 - f(-\infty)||_2^{\frac{1}{2}}\, ||f_0'||_2^{\frac{1}{2}}. \end{aligned}$$

Proof. Since $f(-\infty) = \alpha(0+)$ exists for $\alpha \in NBV_{loc}$ (see [**16**, Cor. 1c]), define

$$f(z) - f(-\infty) := \int_0^\infty e^{zs}\, d[\alpha(s) - f(-\infty)H_0(s)].$$

Then $f - f(-\infty)$ and $\alpha - f(-\infty)H_0$ satisfy the conditions of Theorem 2.5 and $V_\alpha(\infty) = V_{f(-\infty)H_0}(\infty) + V_{\alpha - f(-\infty)H_0}(\infty) = |f(-\infty)| + \frac{1}{\sqrt{2\pi}}\|\mathcal{F}(f_0 - f(-\infty)\|_1 \leq |f(-\infty)| + \sqrt{\frac{2}{\pi}}||f_0 - f(-\infty)||_2^{\frac{1}{2}}\, ||f_0'||_2^{\frac{1}{2}}$. □

3. Bounds on the Convolution Powers of the Determining Function

3.1. NBV**-bounds.** By Proposition 2.1, an A-stable rational function r can be represented by $r(z) = \int_0^\infty e^{zs}\, d\alpha(s)$ ($\operatorname{Re} z \leq 0$) for some $\alpha \in NBV$. In this section, the total variation of the convolution powers α^{n*} will be estimated.

Employing techniques due to P. Brenner and V. Thomée ([**4**] [**5**, Ch. 2]), the following partition of unity is needed. Let $0 \leq \phi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}(\phi) \subset (-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)$ and $\sum_{j=1}^\infty \phi(2^{-j}s) = 1$ for $|s| > 2$. Define $\phi_j(s) := \phi(2^{-j}s)$ for $j > 0$ and $\phi_0 = 1 - \sum_{j=1}^\infty \phi_j$. Note that $\operatorname{supp}(\phi_j) \subset (-2^{j+1}, -2^{j-1}) \cup (2^{j-1}, 2^{j+1})$ for $j > 0$. The proof of the next theorem follows [**4**, Theorem 1].

Theorem 3.1. *Let* $r(z) = \int_0^\infty e^{zs}\, d\alpha(s)$, $\alpha \in NBV$, *be an A-stable rational function. Then there is a constant* $K > 0$ *such that*

$$V_{\alpha^{n*}}(\infty) \leq K\sqrt{n} \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Proof. Since r is an A-stable rational function it follows that $r(\infty) := \lim_{|z|\to\infty} r(z)$ exists. By Corollary 2.6,

$$\begin{aligned} V_{\alpha^{n*}}(\infty) &\leq |r^n(\infty)| + \frac{1}{\sqrt{2\pi}} \int_0^\infty |\mathcal{F}(r_0^n - r^n(\infty))(s)|\, ds \\ &= |r^n(\infty)| + \frac{1}{\sqrt{2\pi}} \int_0^\infty |\mathcal{F}[(r_0^n - r^n(\infty)) \cdot \sum_{k=0}^\infty \phi_k](s)|\, ds \\ &\leq |r^n(\infty)| + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \int_0^\infty |\mathcal{F}[\phi_k \cdot (r_0^n - r^n(\infty))](s)|\, ds \\ &\leq |r^n(\infty)| + \sqrt{\frac{2}{\pi}} \sum_{k=0}^\infty ||\phi_k \cdot (r_0^n - r^n(\infty))||_2^{\frac{1}{2}}\, ||[\phi_k \cdot (r_0^n - r^n(\infty))]'||_2^{\frac{1}{2}}, \end{aligned} \tag{3.2}$$

where we use Corollary 2.3 for the last inequality. Since r is A-stable and rational, there exist polynomials p, q with $\deg(p) < \deg(q)$ such that $r(z) - r(\infty) = \frac{p(z)}{q(z)}$.

Thus, by the binomial formula (and using C to denote a constant whose value may change from line to line),

$$|r^n(is) - r^n(\infty)| = |r(is) - r(\infty)| \left| \sum_{k=0}^{n-1} r^k(is) r^{n-k}(\infty) \right| \le C \frac{n}{1+|s|}, \ s \in \mathbb{R}.$$

The A-stability of r also implies that $|r^n(is) - r^n(\infty)| \le 2$ for $s \in \mathbb{R}$. Hence,

$$|r^n(is) - r^n(\infty)| \le C \min\left(1, \frac{n}{1+|s|}\right), \ s \in \mathbb{R}. \tag{3.3}$$

There are polynomials p_1, q_1 with $\deg(p_1) < \deg(q_1) - 1$ such that $r' = \frac{p_1}{q_1}$. Thus,

$$|\frac{d}{ds}(r^n(is) - r^n(\infty))| = |nr^{n-1}(is)r'(is)| \le C \frac{n}{1+|s|^2}, \ s \in \mathbb{R}. \tag{3.4}$$

By (3.3),

$$\begin{aligned} ||\phi_k \cdot (r_0^n - r^n(\infty))||_2^2 &= \int_{-\infty}^{\infty} |\phi_k(s)(r^n(is) - r^n(\infty))|^2 \, ds \\ &\le C \int_{2^{k-1}}^{2^{k+1}} \min\left(1, \frac{n^2}{(1+|s|)^2}\right) ds \le C \min(2^k, n^2 2^{-k}) \end{aligned} \tag{3.5}$$

if $k > 0$. Since $|r_0^n - r^n(\infty)| \le 2$ it follows that $||\phi_0 \cdot (r_0^n - r^n(\infty))||_2^2 \le C$. Therefore, (3.5) holds for $k \ge 0$. Notice that from the definition of ϕ_j it follows that

$$\left| \frac{d}{ds} \phi_k(s) \right| = |2^{-k} \phi'(2^{-k}s)| \le C 2^{-k} \text{ for } s \in \mathbb{R}.$$

Let $k > 0$. By the product rule and the inequality $(a+b)^2 \le 2(a^2 + b^2)$,

$$\begin{aligned} &\left| \frac{d}{ds}[\phi_k(s)(r^n(is) - r^n(\infty))] \right|^2 \\ &\qquad \le 2 \left(|2^{-k}\phi'(2^{-k}s)(r^n(is) - r^n(\infty))|^2 + \left| \phi_k(s) \frac{d}{ds}(r^n(is) - r^n(\infty)) \right|^2 \right). \end{aligned}$$

It follows from(3.3) and (3.4) that

$$\begin{aligned} ||[\phi_k \cdot (r_0^n - r^n(\infty))]'||_2^2 &= \int_{-\infty}^{\infty} \left| \frac{d}{ds}[\phi_k(s) r^n(is) - r^n(\infty)] \right|^2 ds \\ &\le C \left(\int_{2^{k-1}}^{2^{k+1}} 2^{-2k} \min\left(1, \frac{n^2}{(1+|s|)^2}\right) ds + \int_{2^{k-1}}^{2^{k+1}} \frac{n^2}{(1+|s|^2)^2} \, ds \right) \\ &\le C \min(2^{-k}, n^2 2^{-3k}) + C n^2 2^{-3k} \le C(2^{-k} + n^2 2^{-3k}). \end{aligned} \tag{3.6}$$

Note, that the final estimate in (3.6) holds also for $k = 0$ by (3.3) and (3.4). Finally, from (3.5) and (3.6) it follows that

$$||\phi_k \cdot (r_0^n - r^n(\infty))||_2^{\frac{1}{2}} \, ||[\phi_k \cdot (r_0^n - r^n(\infty))]'||_2^{\frac{1}{2}} \le C\sqrt{n} 2^{-\frac{k}{2}}.$$

Hence, by (3.2), the final estimate of $V_{\alpha^{n*}}(\infty)$ is

$$|r_0^n(\infty)| + \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} ||\phi_k \cdot (r_0^n - r^n(\infty))||_2^{\frac{1}{2}} \, ||[\phi_k \cdot (r_0^n - r^n(\infty))]'||_2^{\frac{1}{2}} \le K\sqrt{n}.$$

□

If r_0 satisfies additional conditions at ∞ and at 0, then the estimate (3.1) can be improved by an order up to $\frac{1}{2}$ (see [**4**]). For example, the inverse Laplace-Stieltjes transform α of $r(z) = \frac{1}{1-z}$ is monotonic on $(0, \infty)$ with $\alpha(0) = \alpha(0+) = 0$, $\alpha(\infty) = 1$, and hence $V_{\alpha^{n*}}(\infty) \leq [V_\alpha(\infty)]^n = 1$. However, in general, (3.1) is sharp as will be shown in Theorem 3.4. Although crucial technical details are adopted from [**8**] and [**3**], our approach does not use Fourier multipliers and operator semigroups. A few preliminary lemmas are needed.

Lemma 3.2. *Let $g \in L_1(\mathbb{R}) \cap C(\mathbb{R})$ with $\mathcal{F}(g) \in L_1(\mathbb{R})$. If $f(s) = \int_0^\infty e^{ist}\, d\alpha(t)$ for some $\alpha \in NBV$, then $||\mathcal{F}(gf)||_1 \leq ||\mathcal{F}(g)||_1 V_\alpha(\infty)$.*

Proof. The proof is straightforward using Fubini's theorem for the Riemann-Stieltjes integral [**16**, Theorem 15c, p. 25]. □

The next lemma is one of the basic tools when estimating oscillatory integrals (for the proof, see [**5**, Lemma 5.1, p. 24]).

Lemma 3.3 (Van der Corput)**.** *If $\phi \in C^2[a, b]$ is real with $|\phi''| \geq \delta > 0$ on $[a, b]$, then $|\int_a^b e^{i\phi(s)}\, ds| \leq 8\delta^{-\frac{1}{2}}$.*

The following result shows the sharpness of Theorem 3.1 when the A-stable rational function r satisfies $|r(is)| = |r_0(s)| = 1$.[3]

Theorem 3.4. *Let r be an A-stable rational function given by $r(z) = \int_0^\infty e^{zt}\, d\alpha(t)$, $\alpha \in NBV$, $Re\, z \leq 0$, with $|r(is)| = 1$ for all $s \in \mathbb{R}$. Then there is a constant $K > 0$ such that $V_{\alpha^{n*}}(\infty) \geq K\sqrt{n}$ for all $n \in \mathbb{N}$.*

Proof. Since $|r(is)| = 1$ for all $s \in \mathbb{R}$ it follows that $r(is) = e^{i\psi(s)}$ for some $\psi \in C^\infty(\mathbb{R})$. Since r is rational, ψ can not be linear; i.e., $\psi'' \not\equiv 0$. Hence, there is $\delta > 0$ and a C^∞-function g with compact support such that $|\psi''| \geq \delta > 0$ on $\text{supp}(g)$. By Parseval's identity, Hölder's inequality, and $|r_0(s)| = |r(is)| = 1$ it follows that

$$||g||_2^2 = ||gr_0^n||_2^2 = ||\mathcal{F}(gr_0^n)||_2^2 \leq ||\mathcal{F}(gr_0^n)||_1 ||\mathcal{F}(gr_0^n)||_\infty. \tag{3.7}$$

To see that the last two norms in (3.7) are finite, first observe that Lemma 3.2 yields

$$||\mathcal{F}(gr_0^n)||_1 \leq ||\mathcal{F}(g)||_1 V_{\alpha^{n*}}(\infty). \tag{3.8}$$

Using Lemma 3.3, an upper estimate for $||\mathcal{F}(gr_0^n)||_\infty$ can be obtained as follows.

$$\begin{aligned}\sqrt{2\pi}||\mathcal{F}(gr_0^n)||_\infty &= \sup_{s\in\mathbb{R}} \left| \int_{-\infty}^{\infty} g(t) e^{in\psi(t)-ist}\, dt \right| \\ &= \sup_{s\in\mathbb{R}} \left| \int_{-\infty}^{\infty} g'(t) \int_{t_0}^{t} e^{in\psi(r)-isr}\, dr\, dt \right| \leq ||g'||_1 8(\delta n)^{-\frac{1}{2}}. \end{aligned} \tag{3.9}$$

Therefore, by (3.7), (3.8), and (3.9), it follows that

$$V_{\alpha^{n*}}(\infty) \geq \frac{||g||_2^2}{||\mathcal{F}(g)||_1} \frac{\sqrt{2\pi} 8(\delta n)^{\frac{1}{2}}}{||g'||_1} = K\sqrt{n}.$$

□

[3] For example, the function $r(z) = \frac{2+z}{2-z}$ satisfies this property.

3.2. L_∞-bound. In this section it is shown that $\|\alpha^{n*}\|_\infty \le K\ln(n+1)$ (although there is numerical evidence that supports the conjecture that, in fact, $\|\alpha^{n*}\|_\infty \le K$). The L_∞-estimate shows that the possible $\sqrt{n}$-growth of $V_{\alpha^{n*}}(\infty)$ is generated by strengthening oscillations rather than from the growth in absolute value. The logarithmic growth bound is essential in Theorem 4.5 whose proof does not go through using a $\sqrt{n}$–growth bound of the L_∞–norm (this fact is an immediate consequence of the $\sqrt{n}$–growth bound on the total variation).

Lemma 3.5. *If a rational function r is A-stable, then there are positive constants $\varepsilon, m, \omega, L, C$ such that $|r(z)| \le e^{C|z|}$ for $|z| \le \varepsilon$ and $|r(z)| \le e^{L|z|^{-m}}$ for $|z| \ge \omega \ge 1$.*

For the proof we refer to [**15**, Lemmas 8.2 and 8.3].

Theorem 3.6. *If r is an A-stable rational function given by $r(z) = \int_0^\infty e^{zs}d\alpha(s)$, $\alpha \in NBV$, $Re\, z \le 0$, then $\|\alpha^{n*}\|_\infty \le K\ln(n+1)$ for some $K > 0$ and all $n \in \mathbb{N}$.*

Proof. It suffices to consider the case $s > 0$ since α^{n*} is normalized with $\alpha^{n*}(0) = 0$. It is not difficult to see that the path of integration in the complex inversion formula (Proposition 2.4) can be replaced by the contour integral, oriented counter-clockwise,

$$\alpha^{n*}(s) = \frac{1}{2\pi i}\int_{\Gamma^R_{\frac{\varepsilon}{n}} \cup \gamma(R) \cup \gamma(\frac{\varepsilon}{n})} \frac{r^n(z)}{z} e^{-zs}\, dz + 1 \tag{3.10}$$

where $\Gamma^R_{\frac{\varepsilon}{n}} := \{z \in \mathbb{C} : \frac{\varepsilon}{n} \le |\mathrm{Im}\, z| \le R,\ \mathrm{Re}\, z = 0\}$, $\gamma(R) := \{z \in \mathbb{C} : |z| = R,\ \mathrm{Re}\, z \ge 0\}$ and $\gamma(\frac{\varepsilon}{n}) := \{z \in \mathbb{C} : |z| = \frac{\varepsilon}{n},\ \mathrm{Re}\, z \ge 0\}$. Here, R and ε are chosen so that the singularities of the integrand lie inside the path of integration except the one at $z = 0$. Note that the additional constant 1 comes from the residue of the integrand at $z = 0$. For the purpose of this proof, $\Gamma^R_{\frac{\varepsilon}{n}}$ is defined by $R := \omega n^{\frac{1}{m}}$ where ω (large enough), ε (small enough), and m are as in Lemma 3.5. Then

$$\alpha^{n*}(s) - 1 = \left(\frac{1}{2\pi i}\int_{\Gamma^R_{\frac{\varepsilon}{n}}} + \frac{1}{2\pi i}\int_{\gamma(R)} + \frac{1}{2\pi i}\int_{\gamma(\frac{\varepsilon}{n})}\right)\frac{r^n(z)}{z} e^{-zs}\, dz := I_1 + I_2 + I_3.$$

By Lemma 3.5, $|I_1| \le \frac{1}{\pi}\ln\frac{n\omega n^{\frac{1}{m}}}{\varepsilon} = \frac{1}{\pi}(\frac{m+1}{m})\ln(\frac{\omega^{\frac{m}{m+1}} n}{\varepsilon})$, $|I_2| \le \frac{1}{2}e^{L/\omega^m}$, and $|I_3| \le \frac{1}{2}e^{C\varepsilon}$. Thus, $\|\alpha^{n*}\|_\infty \le K\ln(n+1)$ for some $K > 0$ and all $n \in \mathbb{N}$. □

4. Convergence of the Determining Functions Induced by the Convergence of Their Laplace-Stieltjes Transforms

If r is an A-stable rational function, then

$$r^n(\tfrac{t}{n}z) = \int_0^\infty e^{zs}\, d\alpha_n(s), \tag{4.1}$$

where $\alpha_n(s) := \alpha^{n*}(\frac{n}{t}s)$, $\alpha \in NBV$, $n \in \mathbb{N}$, $t > 0$, and $\mathrm{Re}\, z \le 0$. Note that in fact $\alpha_n = \alpha_{n,t}$ but the dependence on t will be surpressed in the notation for simplicity. If, in addition, r is a rational approximation of the exponential of order q (i.e., $r(z) = e^z + O(z^{q+1})$ as $z \to 0$), then, for $\mathrm{Re}\, z \le 0$,

$$|r^n(\frac{t}{n}z) - e^{tz}| = |r(\frac{t}{n}z) - e^{\frac{t}{n}z}|\left|\sum_{k=0}^{n-1} r(\frac{t}{n}z)^{n-1-k} e^{\frac{tk}{n}z}\right| \le Mt^{q+1}\frac{1}{n^q}|z^{q+1}|.$$

Since $r^n(\frac{t}{n}z) \to e^{tz} = \int_0^\infty e^{zs}\, dH_t(s)$ $(n \to \infty,\ \operatorname{Re} z \le 0)$, one may expect that α_n converges to H_t in some sense as $n \to \infty$. In Theorems 4.4 and 4.5 it will be shown, among others, that indeed α_n converges to H_t in $L_p(\mathbb{R}_+)$ for all $1 \le p < \infty$ with a rate proportional to $n^{-1/2p}(\ln(n+1))^{1-1/p}$. The proofs use a modified version of the complex inversion formula for the differences $\alpha_n - H_t$ and their k-th antiderivatives

$$I^{(k)}[\alpha_n - H_t](s) := \int_0^s \ldots \int_0^{s_3} \int_0^{s_2} (\alpha_n - H_t)(s_1)\, ds_1\, ds_2 \ldots ds_k,\ k \in \mathbb{N}. \tag{4.2}$$

Proposition 4.1. *Let r be an A-stable rational approximation of the exponential of order q and $t > 0$. Then, for all $n \in \mathbb{N}$,*

$$I^{(k)}[\alpha_n - H_t] = \left(\frac{-1}{i}\right)^{k+1} \frac{1}{\sqrt{2\pi}} \mathcal{F}\Big[\frac{r^n(i\frac{t}{n}(\cdot)) - e^{it(\cdot)}}{(\cdot)^{k+1}}\Big],\ k = 0, 1, \ldots, q$$

on $(0,\infty)$. For $k = 0$ the equality holds pointwise almost everywhere on $(0,\infty)$.

Proof. Let $k = 0$ and $t, s > 0$. By Proposition 2.4,

$$\alpha_n(s) - H_t(s) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{c-iR}^{c+iR} \frac{r^n(-\frac{t}{n}z) - e^{-zt}}{z} e^{zs}\, dz. \tag{4.3}$$

Since r is an A-stable rational approximation of the exponential of order q it follows that $z \mapsto \frac{r^n(-\frac{t}{n}z) - e^{-zt}}{z}$ is analytic at 0 and in a neighborhood of $i\mathbb{R}$. Moreover,

$$\left| \int_{\Gamma_{\pm R,c}} \frac{r^n(-\frac{t}{n}z) - e^{-zt}}{z} e^{zt}\, dz \right| \le \frac{2c}{R} 2e^{ct} \to 0 \text{ as } R \to \infty,$$

where $\Gamma_{\pm R,c} = \{z :\ z = \pm iR + s,\ s \in [0, c]\}$. Therefore, by Cauchy's theorem, one can integrate along the imaginary axis in (4.3) and obtain

$$\begin{aligned} \alpha_n(s) - H_t(s) &= \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-iR}^{iR} \frac{r^n(-\frac{t}{n}z) - e^{-zt}}{z} e^{zs}\, dz \\ &= -\frac{1}{2\pi i} \lim_{R\to\infty} \int_{-R}^{R} \frac{r^n(i\frac{t}{n}v) - e^{itv}}{v} e^{-ivs}\, dv = -\frac{1}{\sqrt{2\pi}\, i} \lim_{R\to\infty} f_R(s), \end{aligned} \tag{4.4}$$

where $f_R(s) := \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \frac{r^n(i\frac{t}{n}v) - e^{itv}}{v} e^{-ivs}\, dv$. Since $v \mapsto \frac{r^n(i\frac{t}{n}v) - e^{itv}}{v} \in L_2(\mathbb{R})$,

$$\overset{(2)}{\lim}_{R\to\infty} f_R = -\frac{1}{\sqrt{2\pi}\, i} \mathcal{F}\Big[\frac{r^n(i\frac{t}{n}(\cdot)) - e^{it(\cdot)}}{(\cdot)}\Big] \in L_2(\mathbb{R})$$

(see, for example, [**6**, p. 209]). By (4.4), f_R converges also pointwise and hence the pointwise limit is a.e. the same as the L_2-limit. Thus,

$$\alpha_n(s) - H_t(s) = -\frac{1}{\sqrt{2\pi}\, i} \mathcal{F}\Big[\frac{r^n(i\frac{t}{n}(\cdot)) - e^{it(\cdot)}}{(\cdot)}\Big](s).$$

This proves the claim for $k = 0$. Assume that the claim holds for $0 \le k < q$. Define

$$f_R^{[k]}(s) := \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \frac{r^n(i\frac{t}{n}v) - e^{itv}}{v^{k+1}} e^{-ivs}\, dv.$$

With the same weak convergence argument as in the proof of Theorem 2.5 one obtains

$$I^{(k+1)}[\alpha_n - H_t](s) = \int_0^s \left(\frac{-1}{i}\right)^{k+1} \frac{1}{\sqrt{2\pi}} \mathcal{F}\Big[\frac{r^n(i\frac{t}{n}(\cdot)) - e^{it(\cdot)}}{(\cdot)^{k+1}}\Big](\tau)\, d\tau$$
$$= \int_0^s \left(\frac{-1}{i}\right)^{k+1} \frac{1}{\sqrt{2\pi}} \Big[\overset{(2)}{\lim_{R\to\infty}} f_R^{[k]}\Big](\tau)\, d\tau = \left(\frac{-1}{i}\right)^{k+1} \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_0^s f_R^{[k]}(\tau)\, d\tau. \tag{4.5}$$

By Fubini's theorem,

$$\int_0^s f_R^{[k]}(\tau)\, d\tau = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \int_0^s \frac{r^n(i\frac{t}{n}v) - e^{itv}}{v^{k+1}} e^{-iv\tau}\, d\tau\, dv$$
$$= \frac{-1}{i} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \frac{r^n(i\frac{t}{n}v) - e^{itv}}{v^{k+2}} (e^{-ivr} - 1)\, dv. \tag{4.6}$$

Next, it will be shown that $\lim_{R\to\infty} \int_{-R}^{R} \frac{r^n(i\frac{t}{n}v) - e^{itv}}{v^{k+2}}\, dv = 0$. Since r is an A-stable rational approximation of the exponential of order q and $k+2 \le q+1$, it follows that $z \to \frac{r^n(\frac{t}{n}z) - e^{zt}}{z^{k+2}}\, dz$ is analytic in a neighborhood of $\{z : Re(z) \le 0\}$. By Cauchy's theorem and $|r^n(\frac{t}{n}z) - e^{zt}| \le 2$ for $Re(z) \le 0$,

$$\lim_{R\to\infty} \int_{-iR}^{iR} \frac{r^n(\frac{t}{n}z) - e^{zt}}{z^{k+2}}\, dz = \lim_{R\to\infty} \int_{\Gamma_R} \frac{r^n(\frac{t}{n}z) - e^{zt}}{z^{k+2}}\, dz = 0,$$

where $\Gamma_R = \{z \in \mathbb{C} : \ z = Re^{is},\ s \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}$. Thus, from (4.5) and (4.6) one obtains

$$I^{(k+1)}[\alpha_n - H_t](s) = \lim_{R\to\infty} \left(\frac{-1}{i}\right)^{k+2} \frac{1}{\sqrt{2\pi}} f_R^{[k+1]}(s).$$

Finally, since $v \to \frac{r^n(i\frac{t}{n}v) - e^{itv}}{v^{k+2}} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ it follows that

$$I^{(k+1)}[\alpha_n - H_t](s) = \left(\frac{-1}{i}\right)^{k+2} \frac{1}{\sqrt{2\pi}} \mathcal{F}\Big[\frac{r^n(i\frac{t}{n}(\cdot)) - e^{it(\cdot)}}{(\cdot)^{k+2}}\Big](s) \text{ for all } s > 0.$$

□

Corollary 4.2. *Let r be an A-stable rational approximation of the exponential of order q and $t > 0$. Then, for all $n \in \mathbb{N}$, $\lim_{s\to\infty} I^{(k)}[\alpha_n - H_t](s) = 0,\ k = 0, 1, \ldots, q$.*

Proof. First, let $k = 0$. Since $1 = r^n(0) = r^n(0-) = \alpha_n(\infty)$ it follows that $\lim_{s\to\infty} \alpha_n(s) - H_t(s) = 0,\ n \in \mathbb{N}$. If $k > 0$, then $v \to \frac{r^n(i\frac{t}{n}v) - e^{itv}}{v^{k+1}} \in L_1(\mathbb{R})$. Thus, by the Riemann-Lebesgue Lemma and by Proposition 4.1, the claim follows. □

For the main convergence result of this section another technical lemma is needed. Its elementary proof uses change of variables and is omitted.

Lemma 4.3. *Let $a \in \mathbb{R}$ and $b > 0$. If $f \in L_2(\mathbb{R})$ with $\mathcal{F}(f) \in L_1(\mathbb{R})$, then*

$$||\mathcal{F}(f)||_1 = ||\mathcal{F}(f(b\cdot))||_1 = ||\mathcal{F}(f(\cdot)e^{ia(\cdot)})||_1. \tag{4.7}$$

Combining analytical tools from the proofs of [**4**, Theorems 3 and 4] with Proposition 4.1, the main L_1-convergence result can now be proved. For $q \in \mathbb{N}$ define

$$\theta_q(k) := \begin{cases} k + \frac{1}{2} & \text{if } k < \frac{q-1}{2} \\ (k+1)\frac{q}{q+1} & \text{if } \frac{q-1}{2} \le k. \end{cases}$$

Theorem 4.4. *Let r be an A-stable rational approximation of the exponential of order q, $t > 0$, and $k = 0, 1, \ldots, q$. Then there is $K > 0$ such that, for all $n \in \mathbb{N}$,*

$$||I^{(k)}[\alpha_n - H_t]||_{L_1(\mathbb{R}_+)} \le \begin{cases} Kt^{k+1-\theta_q(k)} \left(\frac{t}{n}\right)^{\theta_q(k)} & \text{if } k \ne \frac{q-1}{2} \\ Kt^{k+1-\theta_q(k)} \left(\frac{t}{n}\right)^{\theta_q(k)} \ln(n+1) & \text{if } k = \frac{q-1}{2}. \end{cases}$$

Proof. Combining Lemma 4.3 with $a = t$ and $b = n^{-\frac{q}{q+1}}t$ with Proposition 4.1 yields

$$\begin{aligned} &||I^{(k)}[\alpha_n - H_t]||_{L_1(\mathbb{R}_+)} \le \frac{1}{\sqrt{2\pi}} \Big|\Big|\mathcal{F}\Big[\frac{r^n(i\frac{t}{n}(\cdot)) - e^{it(\cdot)}}{(\cdot)^{k+1}}\Big]\Big|\Big|_1 \\ &= \frac{1}{\sqrt{2\pi}} \Big|\Big|\mathcal{F}\Big[e^{ti(\cdot)} \frac{\left(e^{-n^{-\frac{1}{q+1}}n^{-\frac{q}{q+1}}ti(\cdot)} r(n^{-\frac{1}{q+1}}n^{-\frac{q}{q+1}}ti(\cdot))\right)^n - 1}{(n^{-\frac{q}{q+1}}t(\cdot))^{k+1}} (tn^{-\frac{q}{q+1}})^{k+1}\Big]\Big|\Big|_1 \\ &= \frac{1}{\sqrt{2\pi}} \Big|\Big|\mathcal{F}\Big[\frac{\left(e^{-n^{-\frac{1}{q+1}}i(\cdot)} r(n^{-\frac{1}{q+1}}i(\cdot))\right)^n - 1}{(\cdot)^{k+1}}\Big]\Big|\Big|_1 (tn^{-\frac{q}{q+1}})^{k+1} \\ &= \frac{1}{\sqrt{2\pi}} \Big|\Big|\mathcal{F}[g_k]\Big|\Big|_1 (tn^{-\frac{q}{q+1}})^{k+1}, \end{aligned} \tag{4.8}$$

where $g_k(s) := \left[(e^{-n^{-\frac{1}{q+1}}is} r(n^{-\frac{1}{q+1}}is))^n - 1\right]/s^{k+1}$. Using the partition of unity as in the estimate (3.2) and employing Corollary 2.3, one obtains

$$\|\mathcal{F}[g_k]\|_1 \le \sum_{j=0}^{\infty} \|\mathcal{F}[\phi_j g_k]\|_1 \le \sum_{j=0}^{\infty} \|\phi_j g_k\|_2^{\frac{1}{2}} \|[\phi_j g_k]'\|_2^{\frac{1}{2}}. \tag{4.9}$$

Define $h(s) := e^{-n^{-\frac{1}{q+1}}is} r(n^{-\frac{1}{q+1}}is)$. Then $|h(s)| \le 1$ and

$$|h(s)^n - 1| \le 2 \text{ for all } s \in \mathbb{R}. \tag{4.10}$$

Moreover, $e^{-z}r(z) - 1 = O(z^{q+1})$ as $z \to 0$ since $r(z) = e^z + O(z^{q+1})$. Thus,

$$h(s) - 1 = e^{-n^{-\frac{1}{q+1}}is} r(n^{-\frac{1}{q+1}}is) - 1 = O\left((n^{-\frac{1}{q+1}}s)^{q+1}\right) \text{ as } n^{-\frac{1}{q+1}}s \to 0.$$

By the binomial formula,

$$|h(s)^n - 1| = |h(s) - 1| \left|\sum_{j=0}^{n-1} h(s)^j\right| \le C|n^{-\frac{1}{q+1}}s|^{q+1}n = C|s|^{q+1} \tag{4.11}$$

for $|n^{-\frac{1}{q+1}}s|$ sufficiently small. Therefore, by (4.10) and (4.11), one obtains for $s \in \mathbb{R}$

$$|h(s)^n - 1| \le C\min(|s|^{q+1}, 1), \text{ and} \tag{4.12}$$

$$|g_k(s)| = \left|\frac{h(s)^n - 1}{s^{k+1}}\right| \le C\min(|s|^{q-k}, \frac{1}{|s|^{k+1}}). \tag{4.13}$$

To handle the derivatives in (4.9), observe that $h'(s) = n^{-\frac{1}{q+1}}\left[e^{i(\cdot)}r(i\cdot)\right]'(n^{-\frac{1}{q+1}}s)$. Since $r'(z) = e^z + O(z^q)$ it follows that

$$(e^{-z}r(z))' = r'(z)e^{-z} - r(z)e^{-z} = 1 + O(z^q) - (1 + O(z^{q+1})) = O(z^q) \text{ as } z \to 0. \tag{4.14}$$

Thus,

$$|h'(s)| = n^{-\frac{1}{q+1}}\left|\left[e^{i(\cdot)}r(i\cdot)\right]'(n^{-\frac{1}{q+1}}s)\right| \le Cn^{-\frac{1}{q+1}}|n^{-\frac{1}{q+1}}s|^q = C\frac{1}{n}|s|^q$$

for $|n^{-\frac{1}{q+1}}s|$ sufficiently small. For $\epsilon \le |n^{-\frac{1}{q+1}}s|$ the inequality holds since $[r(is)]'$ ($s \in \mathbb{R}$) is bounded (see (3.4)) and hence $\left|\left[e^{i(\cdot)}r(i\cdot)\right]'(n^{-\frac{1}{q+1}}is)\right| \le C\epsilon^q \le C|n^{-\frac{1}{q+1}}s|^q$. (Remember that C is a universal constant that can change from line to line). Thus,

$$\left|\frac{d}{ds}\left[h(s)^n - 1\right]\right| = \left|nh(s)^{n-1}h'(s)\right| \le C|s|^q, \text{ for } s \in \mathbb{R}. \tag{4.15}$$

By (4.12), (4.15), and the product rule it follows that

$$|g_k'(s)| = \left|\frac{d}{ds}\frac{h(s)^n - 1}{s^{k+1}}\right| \le C(|s|^{q-k-1} + \min\{|s|^{q-k-1}, \frac{1}{|s|^{k+2}}\}) \le C|s|^{q-k-1}. \tag{4.16}$$

These etimates will be useful if $0 \le k \le q-1$. The case $k = q$ requires an additional estimate. Since $w \mapsto \frac{e^{-iw}r(iw)-1}{w^{q+1}}$ is analytic at the origin and infinitely often differentiable on $i\mathbb{R}\setminus\{0\}$, it and its derivative are bounded on compact intervals containing the origin. Let $|s| \le 1$ and $w := n^{-\frac{1}{q+1}}s$. Then $|h'(s)| \le C\frac{1}{n}$ and

$$\begin{aligned}
\left|\frac{d}{ds}\left(\frac{h(s)^n - 1}{s^{q+1}}\right)\right| &\le \left|\frac{d}{ds}\left(\frac{h(s) - 1}{s^{q+1}}\right)\sum_{j=0}^{n-1}h(s)^j\right| + \left|\frac{h(s) - 1}{s^{q+1}}\frac{d}{ds}\Big[\sum_{j=0}^{n-1}h(s)^j\Big]\right| \\
&= \left|n^{-1}\frac{d}{ds}\left(\frac{h(s) - 1}{(sn^{-\frac{1}{q+1}})^{q+1}}\right)\sum_{j=0}^{n-1}h(s)^j\right| + \left|n^{-1}\frac{h(s) - 1}{(sn^{-\frac{1}{q+1}})^{q+1}}\sum_{j=1}^{n-1}jh(s)^{j-1}h'(s)\right| \\
&= \left|n^{-1}\frac{d}{ds}\left(\frac{e^{-iw}r(iw) - 1}{w^{q+1}}\right)\sum_{j=0}^{n-1}h(s)^j\right| + \left|n^{-1}\frac{e^{-iw}r(iw) - 1}{w^{q+1}}\sum_{j=1}^{n-1}jh(s)^{j-1}h'(s)\right| \\
&\le n^{-1}Cn^{-\frac{1}{q+1}}n + Cn^{-1}\frac{(n-1)n}{2}n^{-1} \le C.
\end{aligned}$$

Thus,

$$|g_q'(s)| = \left|\frac{d}{ds}\left(\frac{h(s)^n - 1}{s^{q+1}}\right)\right| \le C\min(1, \frac{1}{|s|}). \tag{4.17}$$

The estimate (4.16) shows that the use of a partition of unity is necessary if $k \le q-1$ since the function that bounds the derivative is not in $L_2(\mathbb{R})$. Since the estimates in (4.13), (4.16), and (4.17) are independent of n it follows that

$$\left|\left|\phi_0 g_k\right|\right|_2^{\frac{1}{2}} \le C \text{ and } \left|\left|\left[\phi_0 g_k\right]'\right|\right|_2^{\frac{1}{2}} \le C.$$

Let $j \geq 1$. Since $\operatorname{supp}(\phi_j) \subset (-2^{j+1}, -2^{j-1}) \cup (2^{j-1}, 2^{j+1})$, by (4.13) there exist constants C (depending on k but not on j) such that

$$||\phi_j g_k||_2^2 \leq C \int_{2^{j-1}}^{2^{j+1}} \frac{1}{s^{2(k+1)}}\, ds \leq C 2^{-2j(k+\frac{1}{2})}. \tag{4.18}$$

¿From the definition of ϕ_j it follows that $\left|\frac{d}{ds}\phi_j(s)\right| = |2^{-j}\phi'(2^{-j}s)| \leq C2^{-j}$ for $s \in \mathbb{R}$. Hence, by (4.13), (4.16), and the product rule,

$$\begin{aligned} ||[\phi_j g_k]'||_2^2 &\leq C\frac{1}{2^{2j}} \int_{2^{j-1}}^{2^{j+1}} \frac{1}{s^{2k+2}}\, ds + C \int_{2^{j-1}}^{2^{j+1}} s^{2(q-k-1)}\, ds && (4.19) \\ &\leq C 2^{-2j(k+\frac{3}{2})} + C 2^j 2^{2j(q-k-1)}. && (4.20) \end{aligned}$$

Combining (4.18) and (4.19) yields

$$||\phi_j g_k||_2^2 \cdot ||[\phi_j g_k]'||_2^2 \leq C 2^{-4j(k+1)} + C 2^{4j(\frac{q-1}{2}-k)} \leq C 2^{4j(\frac{q-1}{2}-k)}. \tag{4.21}$$

Therefore, if $k > \frac{q-1}{2}$, then we see from (4.9) and (4.21) that

$$||\mathcal{F}[g_k]||_1 \leq C,$$

which finishes the proof for this case in view of (4.8). If $k \leq \frac{q-1}{2}$, then we cannot sum the terms in (4.21) and we need different estimates. In the following we misuse notation by identifying $f(s)$ with the function f. If $j > 0$, then $0 \notin \operatorname{supp}(\phi_j)$. Thus

$$||\mathcal{F}[\phi_j g_k]||_1 \leq ||\mathcal{F}[\phi_j(s)\frac{h(s)^n}{s^{k+1}}]||_1 + ||\mathcal{F}[\frac{\phi_j(s)}{s^{k+1}}]||_1. \tag{4.22}$$

Recall that $r^n(z) = \int_0^\infty e^{zu}\, d\alpha^{n*}(u)$ with $\alpha \in NBV$. Thus,

$$\begin{aligned} h(s)^n = e^{n^{-\frac{q}{q+1}} is} r^n(n^{-\frac{1}{q+1}} is) &= \int_0^\infty e^{isu}\, dH_{n^{-\frac{q}{q+1}}}(u) \cdot \int_0^\infty e^{isu}\, d\alpha^{n*}(n^{\frac{1}{q+1}} u) \\ &= \int_0^\infty e^{isu}\, d[H_{n^{-\frac{q}{q+1}}}(\cdot) * \alpha^{n*}(n^{\frac{1}{q+1}}(\cdot))](u). \end{aligned}$$

Therefore, using Lemma 3.2,

$$\begin{aligned} ||\mathcal{F}[\phi_j(s)\frac{h(s)^n}{s^{k+1}}]||_1 &\leq ||\mathcal{F}[\frac{\phi_j(s)}{s^{k+1}}]||_1\, V_{H_{n^{-\frac{q}{q+1}}} * \alpha^{n*}(n^{\frac{1}{q+1}}(\cdot))}(\infty) \\ &\leq ||\mathcal{F}[\frac{\phi_j(s)}{s^{k+1}}]||_1\, V_{H_{n^{-\frac{q}{q+1}}}}(\infty)\, V_{\alpha^{n*}(n^{\frac{1}{q+1}}(\cdot))}(\infty). && (4.23) \end{aligned}$$

Since $V_{H_{n^{-\frac{q}{q+1}}}}(\infty) = 1$ and since $V_\alpha(\infty)$ is independent of positive scaling, Theorem 3.1 yields that $V_{\alpha^{n*}(n^{\frac{1}{q+1}}(\cdot))}(\infty) = V_{\alpha^{n*}}(\infty) \leq C\sqrt{n}$. Thus, by (4.22),

$$||\mathcal{F}[\phi_j g_k]||_1 \leq ||\mathcal{F}[\frac{\phi_j(s)}{s^{k+1}}]||_1 \left(C\sqrt{n} + 1\right) \leq C||\mathcal{F}[\frac{\phi_j(s)}{s^{k+1}}]||_1 \sqrt{n}. \tag{4.24}$$

¿From Corollary 2.3 it follows that

$$||\mathcal{F}[\frac{\phi_j(s)}{s^{k+1}}]||_1 \leq 2||\frac{\phi_j(s)}{s^{k+1}}||_2\, ||[\frac{\phi_j(s)}{s^{k+1}}]'||_2. \tag{4.25}$$

Since $\big|\big|\frac{\phi_j(s)}{s^{k+1}}\big|\big|_2^2 \le 2\int_{2^{j-1}}^{2^{j+1}} \frac{1}{s^{2(k+1)}}\,ds \le C2^{-2j(k+\frac{1}{2})}$ and

$$\big|\big|[\frac{\phi_j(s)}{s^{k+1}}]'\big|\big|_2^2 \le C2^{-2j}\int_{2^{j-1}}^{2^{j+1}} \frac{1}{s^{2(k+1)}}\,ds + C\int_{2^{j-1}}^{2^{j+1}} \frac{1}{s^{2(k+2)}}\,ds \le C2^{-2j(k+\frac{3}{2})}.$$

it follows that

$$\big|\big|\mathcal{F}[\phi_j g_k]\big|\big|_1 \le C2^{-j(k+1)}\sqrt{n}. \tag{4.26}$$

If n is large enough then choose $j_0 > 0$ such that $2^{j_0} \le n^{\frac{1}{q+1}} < 2^{j_0+1}$. Then, using (4.21) for $0 \le j \le j_0$ and (4.26) for $j_0 < j$ for $k < \frac{q-1}{2}$ one obtains

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\big|\big|\mathcal{F}[\phi_j g_k]\big|\big|_1 &\le C\sum_{j=0}^{j_0} 2^{j(\frac{q-1}{2}-k)} + Cn^{\frac{1}{2}}\sum_{j_0+1}^{\infty} 2^{-j(k+1)}\\ &\le C\left(2^{j_0(\frac{q-1}{2}-k)} + n^{\frac{1}{2}}2^{-(j_0+1)(k+1)}\right) \le Cn^{\frac{1}{2}-\frac{k+1}{q+1}}\end{aligned}$$

This proves the statement for $k < \frac{q-1}{2}$ in view of (4.8) and (4.9). If $k = \frac{q-1}{2}$ (or, equivalently, $\frac{k+1}{q+1} = \frac{1}{2}$), then similarly to the above one chooses j_0 with $2^{j_0} \le n^{\frac{1}{q+1}} < 2^{j_0+1}$. This implies that $j_0 \le C\ln n$ and hence

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\sum_{j=0}^{\infty}\big|\big|\mathcal{F}[\phi_j(g_k]\big|\big|_1 &\le \sum_{j=0}^{j_0} C + Cn^{\frac{1}{2}}\sum_{j_0+1}^{\infty} 2^{-j(k+1)}\\ &\le Cj_0 + Cn^{\frac{1}{2}}2^{-(j_0+1)(k+1)} \le C\ln n + C \le C\ln(n+1).\end{aligned}$$

Together with (4.8) and (4.9), this completes the proof of the statement. □

If r_0 satisfies additional conditions at ∞ and 0, then the estimate in Theorem 4.4 can be improved by an order up to $\frac{1}{2}$ for $k < \frac{q-1}{2}$. To do so one uses an improved estimate on $V_{\alpha^{n*}}(\infty)$. With the additional conditions, this estimate can be sharpened by an order up to $\frac{1}{2}$ as was already noted after Theorem 3.1. See [**3**] for details. Also note that the proof of the optimal convergence order of $I^{(q)}(\alpha_n - H_t)$ is relatively simple as it neither uses the partition of unity nor the estimate on $V_{\alpha_{n*}}(\infty)$.

Using the L_1–estimates and the L_∞–bounds yields the following L_p–convergence results for $\alpha_n - H_t$ for $1 \le p \le \infty$.

Theorem 4.5. *Let r be an A-stable rational approximation of the exponential of order q and $t > 0$. Then*[4]*, for $1 < q < \infty$, there is $K > 0$ such that*

$$||\alpha_n - H_t||_{L_p(\mathbb{R}_+)} \le Kt^{\frac{1}{p}}n^{-\frac{1}{2p}}(\ln(n+1))^{1-\frac{1}{p}}\ \ n \in \mathbb{N}. \tag{4.27}$$

If $k = 1, \ldots, q,\ k \ne \frac{q-1}{2}$, then[5] *there is a constant $K > 0$ such that*

$$||I^{(k)}[\alpha_n - H_t]||_{L_p(\mathbb{R}_+)} \le Kt^{\frac{k+1}{p}}n^{-\frac{\theta_q(k)}{p}}(tn^{-\frac{q}{q+1}})^{(1-\frac{1}{p})k},\ n \in \mathbb{N}. \tag{4.28}$$

[4]If $q = 1$, then a factor of $(\ln(n+1))^{\frac{1}{p}}$ has to be added to the estimate; see Theorem 4.4.

[5]If $k = \frac{q-1}{2}$, then a factor of $(\ln(n+1))^{\frac{1}{p}}$ has to be added in the estimate; see Theorem 4.4.

Proof. If $f \in L_1(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+)$, then $f \in L_p(\mathbb{R}_+)$ for all $1 \le p \le \infty$ and

$$||f||_{L_p(\mathbb{R}_+)} \le \left(||f||_{L_1(\mathbb{R}_+)}\right)^{\frac{1}{p}} \left(||f||_{L_\infty(\mathbb{R}_+)}\right)^{1-\frac{1}{p}}. \tag{4.29}$$

Thus, (4.27) follows immediately from Theorem 3.6, Theorem 4.4 and (4.29). To show (4.28) it suffices to demonstrate that for $k = 1, 2, \ldots, q$,

$$||I^{(k)}[\alpha_n - H_t]||_{L_\infty(\mathbb{R}_+)} \le K(tn^{-\frac{q}{q+1}})^k, \ n \in \mathbb{N},$$

in view of Theorem 4.4 and (4.29). By Proposition 4.1, (4.13) and a change of variables,

$$||I^{(k)}[\alpha_n - H_t]||_{L_\infty(\mathbb{R}_+)} \le \frac{1}{\sqrt{2\pi}} \left|\left| \frac{r^n(i\frac{t}{n}(\cdot)) - e^{it(\cdot)}}{(\cdot)^{k+1}} \right|\right|_{L_1(\mathbb{R})}$$

$$= \left|\left| e^{ti(\cdot)} \frac{\left(e^{-n^{-\frac{1}{q+1}} n^{-\frac{q}{q+1}} ti(\cdot)} r(n^{-\frac{1}{q+1}} n^{-\frac{q}{q+1}} ti(\cdot))\right)^n - 1}{(n^{-\frac{q}{q+1}} t(\cdot))^{k+1}} (tn^{-\frac{q}{q+1}})^{k+1} \right|\right|_{L_1(\mathbb{R}_+)}$$

$$= \left|\left| \frac{\left(e^{-n^{-\frac{1}{q+1}} i(\cdot)} r(n^{-\frac{1}{q+1}} i(\cdot))\right)^n - 1}{(\cdot)^{k+1}} (tn^{-\frac{q}{q+1}})^k \right|\right|_{L_1(\mathbb{R}_+)}$$

$$\le C(tn^{-\frac{q}{q+1}})^k \int_{-\infty}^{\infty} \min(|s|^{q-k}, \frac{1}{|s|^{k+1}})\, ds \le C(tn^{-\frac{q}{q+1}})^k, \ k = 1, \ldots, q, \ n \in \mathbb{N},$$

and the proof is complete. □

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